MATH 5110 – Applied Linear Algebra and Matrix Analysis

❖ Class Work-Matrix Algebra

Instructor: He Wang
Department of Mathematics
Northeastern University

Example. Show $(kI + AB)^{-1}A = A(kI + BA)^{-1}$

Proof: Compute

$$(kI + AB)^{-1}A(kI + BA) =$$

Example: Woodbury matrix identity (Easier version)

Show $(I + UV)^{-1} = I - U(I + VU)^{-1}V$

Proof: Compute

$$(I + UV)[I - U(I + VU)^{-1}V] =$$

Special case for numbers u, v: $$\frac{1}{1+uv} = 1 - \frac{uv}{1+uv}$$

Example: Woodbury matrix identity(matrix inversion lemma)

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$$

Proof: Direct verification.

$$(A + UCV)[A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}] =$$

$$\ldots = I$$

Application:

- Computation of inverses $(A + UCV)^{-1}$.
- Find an approximation of $(A + B)^{-1}$ where is approximated by SVD.
- Fast calculation of solutions to linear equations.
- In numerical linear algebra and numerical partial differential equations as the capacitance matrix.

Special cases:

Inverse of a sum

$$(A+B)^{-1} = A^{-1} - A^{-1}(B^{-1}+A^{-1})^{-1}A^{-1}$$

$$= A^{-1} - A^{-1}(AB^{-1}+I)^{-1}$$

$$= A^{-1} - (AB^{-1}A+A)^{-1}$$

Hua's identity

$$(A-B)^{-1} = A^{-1} + A^{-1}B(A-B)^{-1}$$

For any non-invertible B

$$(A-B)^{-1} = \sum_{k=0}^{\infty}(A^{-1}B)^k A^{-1}$$

Converge if the spectral radius of $A^{-1}B$ is less than 1.

Special case: Sherman-Morrison formula

$A + \vec{u}\,\vec{v}^T$ is invertible if and only if $1 + \vec{v}^T A^{-1} \vec{u} \neq 0$

Further more,

$$(A + \vec{u}\,\vec{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\vec{u}\,\vec{v}^T A^{-1}}{1 + \vec{v}^T A^{-1}\vec{u}}$$

Application:

- Fast calculation of $\left(A + \vec{u}\,\vec{v}^T\right)^{-1}$
- application in theoretical physics, i.e., in quantum field theory

Later, we will look at the determinant of $A + \vec{u}\,\vec{v}^T$ by determinant of A.

❑ Block matrix inverse and Schur complement

Let M be an $n \times n$ matrix written a as 2×2 block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Question: Solve linear system:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix} = \begin{bmatrix} \vec{c} \\ \vec{d} \end{bmatrix}$$

We want to find the inverse of $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

If D is invertible, by **block Gaussian** elimination, we have a LDU decomposition

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_p & BD^{-1} \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I_p & 0 \\ D^{-1}C & I_q \end{bmatrix}$$

Denote **Schur complement** of the block invertible *D* of the matrix *M as*

$$M/D \coloneqq A - BD^{-1}C$$

Then,

$$M^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (M/D)^{-1} & -(M/D)^{-1}BD^{-1} \\ -D^{-1}C(M/D)^{-1} & D^{-1} + D^{-1}C(M/D)^{-1}BD^{-1} \end{bmatrix}$$

Similarly, *if A is invertible*, we have a LDU decomposition

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ CA^{-1} & I_q \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_p & A^{-1}B \\ 0 & I_q \end{bmatrix},$$

The **Schur complement** of the block invertible *A* of the matrix *M* is

$$M/A := D - CA^{-1}B$$

and

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(M/A)^{-1}CA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{bmatrix}$$

The Schur complement arises naturally in solving

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix} = \begin{bmatrix} \vec{c} \\ \vec{d} \end{bmatrix}$$

equivalently

$$A\,\vec{x} + B\,\vec{y} = \vec{c}$$
$$C\,\vec{x} + D\,\vec{y} = \vec{d}$$

Suppose A is invertible, the first equation is $\vec{x} = A^{-1}(\vec{c} - B\vec{y})$. Together with the second equation, we have

$$(D - CA^{-1}B)\vec{y} = \vec{d} - CA^{-1}\vec{c}$$

So, we denote $M/A \coloneqq D - CA^{-1}B$

We can also assume

$M^{-1} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$ and compute $MM^{-1} = I$ to solve W, Y, X, Z.

Applications of Schur complement **:**

The Schur complement is first used to prove Schur's lemma, which is most fundamental lemma for representation theory of groups and algebras. https://en.wikipedia.org/wiki/Schur%27s_lemma

The Schur complement is a key tool in the fields of numerical analysis, Convex Optimization, statistics, and matrix analysis. For example, it provides another proof of the Woodbury matrix identity**.**

In electrical engineering, solving linear system using Schur complement is referred to as node elimination or Kron reduction.

In probability theory and statistics, we use it to compute the conditional covariance and expected value from a multivariant normal distribution with covariance

$$\Sigma = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix}$$

References

Schur Complement

https://www.cis.upenn.edu/~jean/schur-comp.pdf

Woodbury Matrix

https://en.wikipedia.org/wiki/Woodbury_matrix_identity

http://www.math.nagoya-u.ac.jp/~richard/teaching/s2022/Vic1.pdf

http://www0.cs.ucl.ac.uk/staff/g.ridgway/mil/mil.pdf

https://cran.r-project.org/web/packages/WoodburyMatrix/vignettes/WoodburyMatrix.html