MATH 5110 – Applied Linear Algebra and Matrix Analysis

❖ Further Studies relating to linear algebra

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For future (Each topic is a advance graduate course):

1. Multilinear algebra
2. Metric space/Hilbert Spaces
3. Wavelet transform
4. Application or differential equations
5. Algebraic graph theory
6. Topological vector space
7. Topological data analysis (homology)
8. Module-Vector space over ring
9. Quantum Mechanics/Quantum computing
10. Random Matrix Theory
11. Optimization-Linear Programming
12. Statistical and Machine learning
13. More real-world applications – e.g., 2d/3d images
14. …

1. Multi-linear algebra

We have talked about several vector spaces:

- Subspaces
- Intersection
- Sum and Direct Sum
- Quotient space
- Space of linear transformations
- Dual Space

Now, let us look at another two classes of vector spaces:

- Exterior product
- Tensor product

❑ Tensor product

Let V and W be vector spaces over a field $F\,(\mathrm{e.g.}, \mathbb{R})$

Definition: The **tensor product** $V \otimes W$ is defined as quotient of

$$\mathrm{Span}\{\vec{v} \otimes \vec{w} \,|\, \vec{v} \in V \ and\ \vec{w} \in W\}$$

such that

$$1)\ \ \vec{v} \otimes (\vec{w}_1 + \vec{w}_2) = \vec{v} \otimes \vec{w}_1 + \vec{v} \otimes \vec{w}_2$$

$$2)\ \ (\vec{v}_1 + \vec{v}_2) \otimes \vec{w} = \vec{v}_1 \otimes \vec{w} + \vec{v}_2 \otimes \vec{w}$$

$$3)\ \ a(\vec{v} \otimes \vec{w}) = a\vec{v} \otimes \vec{w} = \vec{v} \otimes a\vec{w}$$

for all $\vec{v} \in V$, $\vec{w} \in W$ and $a \in \mathbb{R}$

Basis and dimension:

Let $\mathcal{A} = \{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m\}$ be a basis of V.

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ be a basis of W.

Theorem: A basis for $V \otimes W$ is given by

$$\{\vec{a}_i \otimes \vec{b}_j \mid 1 \le i \le m, 1 \le j \le n\}$$

Proposition:

$$\dim(V \otimes W) = mn$$

Example $\mathbb{R}^m \otimes \mathbb{R}^n \cong \mathbb{R}^{m\times n}$

For any $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{R}^m \; and \; \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$, define the isomorphism map

$$\mathbb{R}^m \otimes \mathbb{R}^n \longrightarrow \mathbb{R}^{m\times n}$$

$$\vec{v} \otimes \vec{w} \longrightarrow \begin{bmatrix} v_1 w_1 & \cdots & v_1 w_n \\ \vdots & \ddots & \vdots \\ v_m w_1 & \cdots & v_m w_n \end{bmatrix} \in \mathbb{R}^{m\times n}$$

❑ Exterior product

Let V be a vector space over a field $F\,(\mathrm{e.g.},\mathbb{R})$

Definition: The **2ed wedge product (exterior power)** $\wedge^2 V \coloneqq V \wedge V$ is a vector space as quotient of

$$\mathrm{Span}\{\vec{v} \wedge \vec{w}\,|\vec{v}, \vec{w} \in V\}$$

such that

$$1)\ \ \vec{v} \wedge \vec{v} = 0$$

$$2)\ \ \vec{v} \wedge \vec{w} = \vec{w} \wedge \vec{v}$$

$$3)\ \ (a\vec{v} + b\vec{w}) \wedge \vec{u} = a\vec{v} \wedge \vec{u} + b\vec{w} \wedge \vec{u}$$

for all $\vec{v} \in V$ and $a, b \in \mathbb{R}$

Basis and dimension:

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ be a basis of V.

Theorem: A basis for $\Lambda^2 V$ is given by

$$\{\vec{b}_i \wedge \vec{b}_j \mid 1 \leq i < j \leq n\}$$

Proposition:

$$\dim(\Lambda^2 V) = \binom{n}{2}$$

❑ Direct product vector spaces:

For any two **sets** $V \ and \ W$, the **Cartesian product** is defined by ordered pairs

$$V \times W \coloneqq \{(v, w) | v \in V, w \in W\}$$

Definition: Let $V \ and \ W$ be vector spaces over a field $F(\mathrm{e.g.}, \mathbb{R})$. The **direct product** $V \times W$ is the Cartesian product as set and satisfying linear properties:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$

and

$$k(v, w) = (kv, kw)$$

Remark:

- If $V \ and \ W$ are subspaces of some vector space U, then direct product is the same (isomorphic) as our direct sum.
- Definition of direct product works on infinite many vector spaces $\prod V_i$, which contains all infinite tuples.
- Generalization of direct sum to infinite many vector spaces $\coprod V_i$,which contains all finite tuples.

❑ k-linear map

Let V and W be vector spaces.

Let V^k be the direct product power with all k-tuples.

Definition: A function $T: V^k \to W$ is ***k*-linear** if it is linear in the i-th variable for each i, that is,

when we fix vectors $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k$, the map $T_i: V \to W$

$$T_i(\vec{v}) = T(\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}, \vec{v}_{i+1}, \dots, \vec{v}_k)$$

is linear.

For example, a multilinear map of one variable ($k = 1$) is a linear map, and of two variables ($k = 2$) is a bilinear map.

Examples:

1. Any inner product $\langle\ ,\rangle: \mathbb{R}\times\mathbb{R} \rightarrow \mathbb{R}$ on a vector space is a multilinear map.

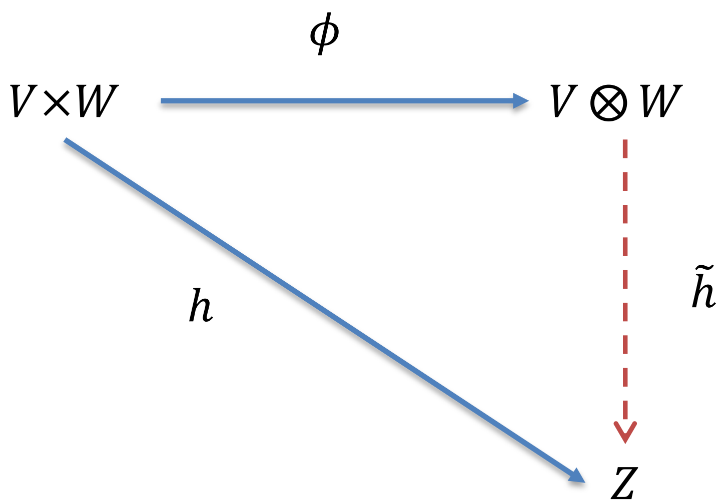
2. Determinant function $det: \mathbb{R}^{n\times n} \rightarrow \mathbb{R}$ is a (alternating) multilinear function of the columns (or rows) of a square matrix.

A map $f: V^k \rightarrow W$ is called **alternating** if

$$f(\vec{v}_1, \dots, \vec{v}_k) = 0 \text{ when ever } \vec{v}_i = \vec{v}_j \text{ for some } i \neq j.$$

Universal properties of tensor product

The universal definition (without basis or element) of tensor by **universal properties**:



The tensor product of two vector spaces V *and* W is a vector space, denoted by $V \otimes W$, together with a bilinear map $\phi(\vec{v}, \vec{w}) = \vec{v} \otimes \vec{w}$ such that

For every bilinear map h, there **exists** a **unique** linear map $\tilde{h}$ that makes the diagram commutative, i.e., $\tilde{h} \circ \phi = h$

There is a natural one-to-one correspondence between multilinear maps

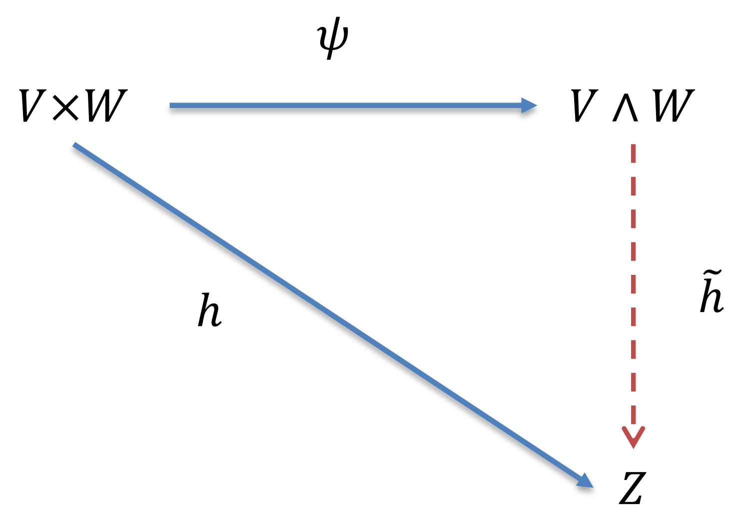
$$h\colon V \times W \to Z \text{ and } \tilde{h}\colon V \otimes W \to Z$$

by formula

$$h(\vec{v}, \vec{w}) = \tilde{h}(\vec{v} \otimes \vec{w})$$

Universal properties of exterior product

The universal definition (without basis or element) of tensor by **universal properties**:



The tensor product of two vector spaces $V\ and\ W$ is a vector space, denoted by $V \wedge W$, together with a **alternating bilinear** map $\psi(\vec{v}, \vec{w}) = \vec{v} \wedge \vec{w}$ such that

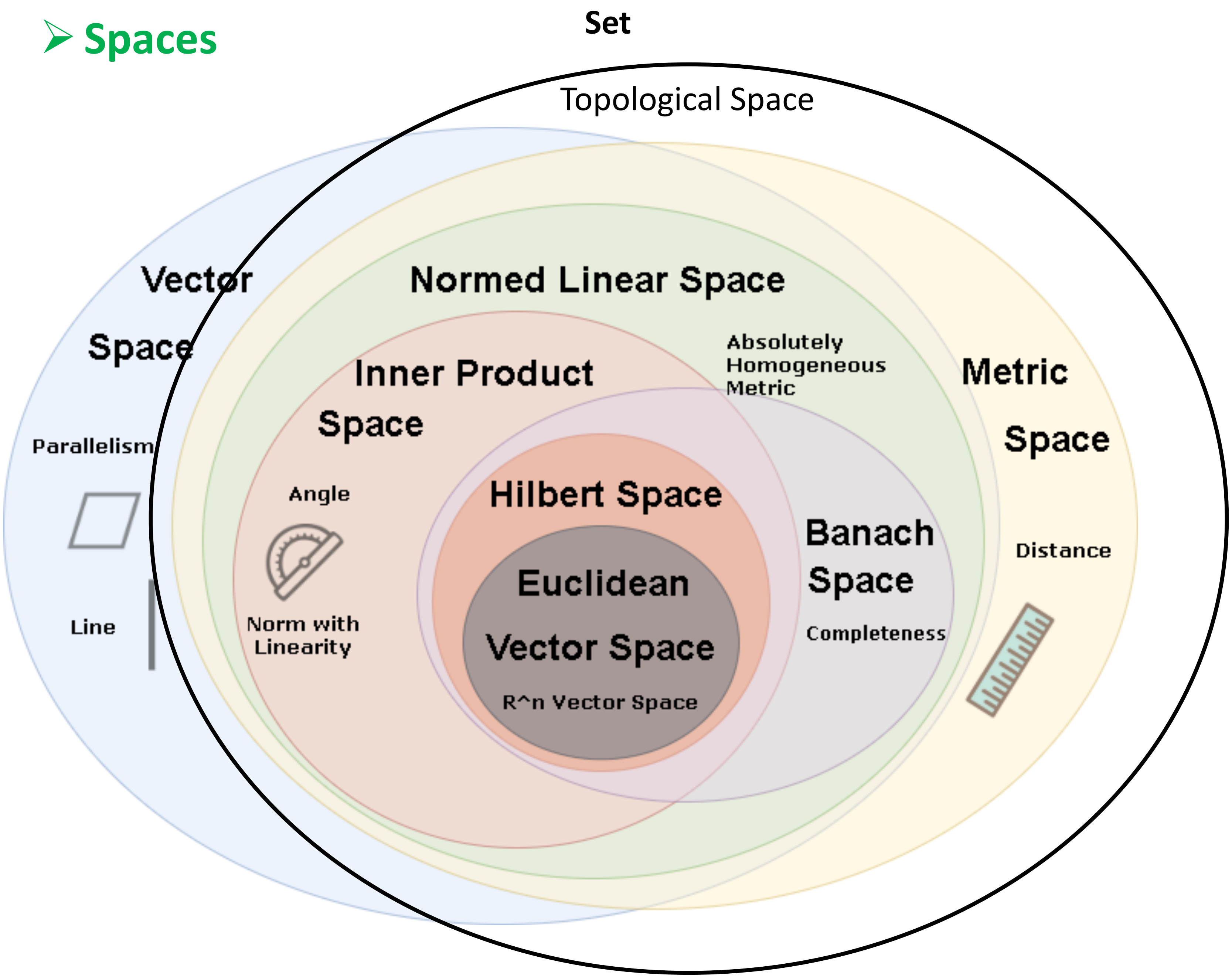
For every **alternating bilinear** map h, there **exists** a **unique linear** map $\tilde{h}$ that makes the diagram commutative, i.e., $\tilde{h} \circ \psi = h$.

Application:

Multilinear algebra is a foundational mathematical tool in engineering, machine learning, physics, and mathematics.

https://en.wikipedia.org/wiki/Multilinear_algebra

➢ Spaces



Set
Topological Space
Vector Space
Parallelism
Line
Normed Linear Space
Inner Product Space
Angle
Norm with Linearity
Absolutely Homogeneous Metric
Metric Space
Distance
Hilbert Space
Banach Space
Completeness
Euclidean Vector Space
R^n Vector Space

2. Metric Space and Hilbert Spaces

This is short introduction in real/functional analysis.

Definition (Metric). Let S be a **set**. A **metric**(distance) on S is a binary function

$$d: S \times S \longrightarrow \mathbb{R}$$

such that for vectors $\vec{u}, \vec{v}, \vec{w} \in S$ and a scalar $c \in \mathbb{R}$, the following hold:

(1.) $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$

(2.) $d(\vec{u}, \vec{v}) = 0$ if and only if $\vec{u} = \vec{v}$

(3.) $d(\vec{u}, \vec{w}) \leq d(\vec{u}, \vec{v}) + d(\vec{v}, \vec{w})$

We call S a metric space with metric function d.

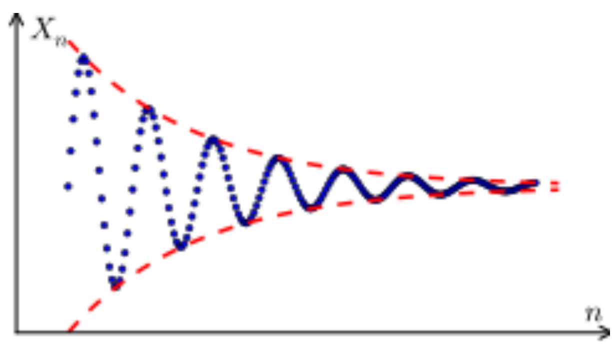
Examples:

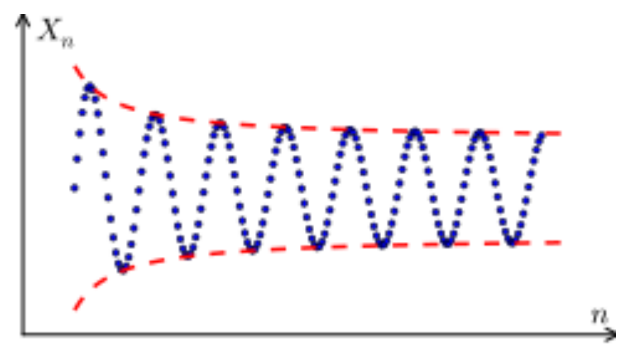
1. The **discrete metric on $\boldsymbol{S}$**, where $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise.
2. The **positive real numbers** with distance function $d(x, y) = |\log(y/x)|$ is a metric space.
3. If S is a vector space, metric is equivalent to norm.

❑ Cauchy sequence

Let $x_1, x_2, \ldots, x_n, \ldots$ be a sequence of elements in a metric space S.

Definition: The sequence $\{x_n\}_{n=1}^{\infty}$ is called **Cauchy sequence,** if for every $\epsilon \in \mathbb{R}$, there is a positive integer N such that $d(x_m, x_n) < \epsilon$ all natural numbers $m, n > N$.





Informally, the Cauchy sequence x_n are getting closer and closer.

https://en.wikipedia.org/wiki/Cauchy_sequence

❑ Complete metric space

Any convergent sequence is a Cauchy sequence, but a Cauchy sequence is not necessarily convergent to an element in S.

Definition: The metric space S is called **complete** if the limit of every Cauchy sequences $\{x_n\}$ is in the space S, i.e., $\lim_{n\to\infty} x_n = x \in S$.

Examples:

1. Real numbers $\mathbb{R}$ is complete under the metric induced by the usual absolute value.

2. Rational numbers $\mathbb{Q}$ is not complete. (e.g., $a_n = \left(1 + \frac{1}{n}\right)^n$ converge to e).

3. Open interval $(0,1)$ is not complete. (e.g., $a_n = \frac{1}{n}$ converge to 0).

❑ Normed Space

Definition (Norm). Let V be a real vector space. A **norm** on V is a function

$$\|-\|\colon V \longrightarrow \mathbb{R}$$

such that for vectors $\vec{u}, \vec{v} \in V$ and a scalar $c \in \mathbb{R}$, the following hold:

(1.) $\|\vec{u}\| \geq 0$

(2.) $\|\vec{u}\| = 0$ if and only if $\vec{u} = \vec{0}$

(3.) $\|c\vec{u}\| = c\ \|\vec{u}\|$

(4.) The triangle inequality $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

We call V an normed space with norm $\|-\|$.

Example: l_p-norm on $\mathbb{R}^n$

Definition: A complete normed vector space is called **Banach space.**

❑ Inner product Space

Let V be a **real vector space (**finite or **infinite** dimensional**)**.

Definition (Inner Product). An **inner product** on V is a binary function

$$\langle -,-\rangle : V\times V \longrightarrow \mathbb{R}$$

such that for vectors $\vec{u},\vec{v},\vec{w} \in V$ and a scalar $c \in \mathbb{R}$, the following hold:

(1.) $\langle \vec{u},\vec{v}\rangle = \langle \vec{v},\vec{u}\rangle$

(2.) $\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \langle \vec{u},\vec{w}\rangle + \langle \vec{v},\vec{w}\rangle$

(3.) $\langle c\vec{u},\vec{v}\rangle = c\langle \vec{v},\vec{u}\rangle$

(4.) $\langle \vec{u},\vec{u}\rangle \geq 0$

(5.) $\langle \vec{u},\vec{u}\rangle = 0$ if and only if $\vec{u} = \vec{0}$

We call V an **inner product space** with inner product $\langle -,-\rangle$.

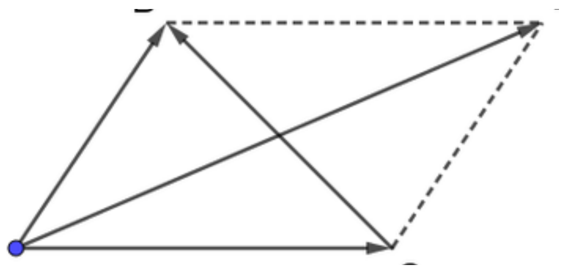
More examples:

1. $\mathbb{R}^{n\times n}$ with inner product $\langle A,B\rangle \coloneqq tr(AB^T)$

2. $\{$Random Variables $X \mid E(X^2) < \infty\}$ with inner product $\langle X,Y\rangle \coloneqq E(XY)$

Theorem: An normed induced by an inner product if and only if

$$\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2=2\|\vec{u}\|^2+2\|\vec{v}\|^2$$



❑ Hilbert Space

Definition: A complete inner product space is called **Hilbert space**.

Examples:

1. Euclidean inner product space $\mathbb{R}^n$ (with dot product) is a Hilbert space.

2. Sequence space ($\mathbb{R}^\infty$ or $\mathbb{C}^\infty$)

$$\{\text{all infinite sequences } z = (z_1, \ldots, z_n, \ldots) \mid \sum_{n=1}^{\infty} |z_n|^2 < \infty\}$$

The inner product is defined by

$$\langle z, w \rangle := \sum_{n=1}^{\infty} z_n \overline{w_n}$$

Q: Orthonormal basis?

❑ Orthogonal projection and least squares

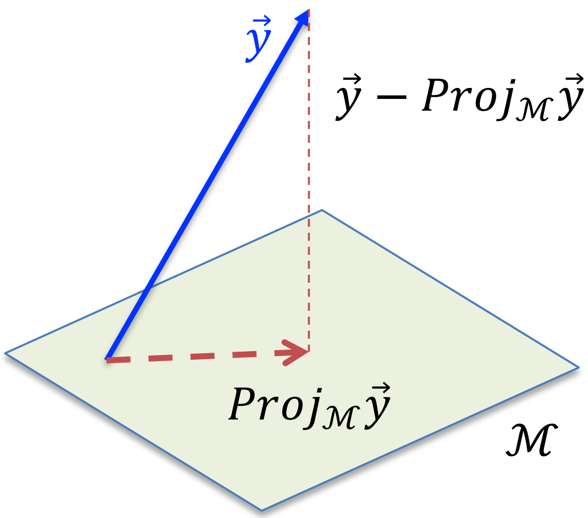
Theorem (Orthogonal Projection Theorem)

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ be a *closed* linear subspace of $\mathcal{H}$, and $\vec{y} \in \mathcal{H}$.

There exist an **unique** point $Proj_{\mathcal{M}}\vec{y} \in \mathcal{M}$ such that

1. $||\vec{y} - Proj_{\mathcal{M}}\vec{y}|| \leq ||\vec{w} - \vec{y}||$ for any $\vec{w} \in \mathcal{M}$

2. $\langle \vec{y} - Proj_{\mathcal{M}}\vec{y}, \vec{w} \rangle = 0$ for any $\vec{w} \in \mathcal{M}$



Infinite dimension remarks

Notation: in infinite dimensions, we don't use the arrow notation.

Definitions of span, linear independence, basis same as finite dimension linear algebra except we now allow infinite sums.

Infinite linear combinations:

$$\sum_{i=1}^{\infty} c_i v_i = u \qquad \text{iff} \qquad \lim_{n\to\infty} \left\| \sum_{i=1}^{\infty} c_i v_i - u \right\| = 0$$

Many properties from linear algebra are true for infinite dimension Hilbert Space. Details can be found in a real/functional analysis textbook.

Theorem: If a collection of vectors $\{v_1, v_2, \dots\}$ is orthonormal then it is automatically linearly independent.

❑ Incomplete Inner product space example.

Functional space P ={all **polynomials** on [0,1]} with inner product

$$\langle f, g \rangle := \int_0^1 f(t) g(t)\, dt$$

is **not** a Hilbert space.

Proof:

Consider the vectors v_m defined by the partial sums of the Taylor series of e^x

$$v_m = \sum_{k=0}^{m} \frac{x^k}{k!}$$

Claim 1. $\{v_1, v_2, \dots\}$ is a Cauchy sequence.

$$\|v_n - v_m\| = \left\| \sum_{k=m+1}^{n} \frac{x^k}{k!} \right\| \leq \sum_{k=m+1}^{n} \left\| \frac{x^k}{k!} \right\|$$

Since

$$\left\| \frac{x^k}{k!} \right\| = \frac{1}{k!} \|x^k\| = \frac{1}{k!} \left(\int_0^1 x^{2k}\, dx \right)^{1/2} = \frac{1}{k!} \left(\frac{1}{2k+1} \right)^{1/2}$$

So, $$\sum_{k=1}^{\infty} \left\| \frac{x^k}{k!} \right\| < \infty$$

So, $\|v_n - v_m\| \to 0 \; when \; n, m \to \infty$

Claim 2.

By Taylor series, $v_m \to e^x$ when $m \to \infty$ uniformly on [0,1]

Claim 3. A sequence of functions can't converge to two different functions.

So v_m will not converge to a polynomial.

So v_m will not converge to a element in P.

So, P is not complete.

Remark: The set $P_2 = \{polynomials\ of\ degree \leq 2\ on\ [0,1]\}$ is a Hilbert space.

➢ Measurable functions and Lebesgue integrals

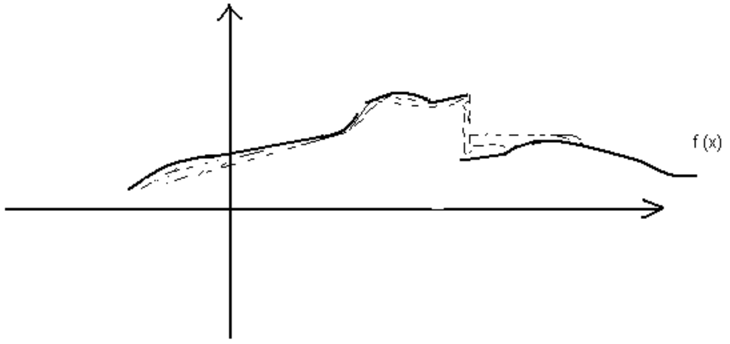
Let C be the set of continuous functions on $\mathbb{R}$. Let M be the set of measurable functions:

Definition: The set M of **measurable functions** on $\mathbb{R}$ (or an interval of $\mathbb{R}$) is the set of functions that are limits of continuous functions, i.e.

$$M = \left\{f(x) \middle| f(x) = \lim_{n\to\infty} f_n(x)\right\},$$

where $f_n(x)$ is continuous for all $x \in \mathbb{R}$

For example, the function $f(x)$ as a limit of continuous functions



Measurable functions

In fact, lots of functions (even discontinuous ones) can be viewed as limits of continuous functions.

For example

$$f(x) = \chi_{[0,1]}(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}.$$

is a discontinuous but measurable function

Note: ordinary notion of integral is difficult to use for functions as complicated as measurable functions.

To integrate measurable functions (Lebesgue integral):

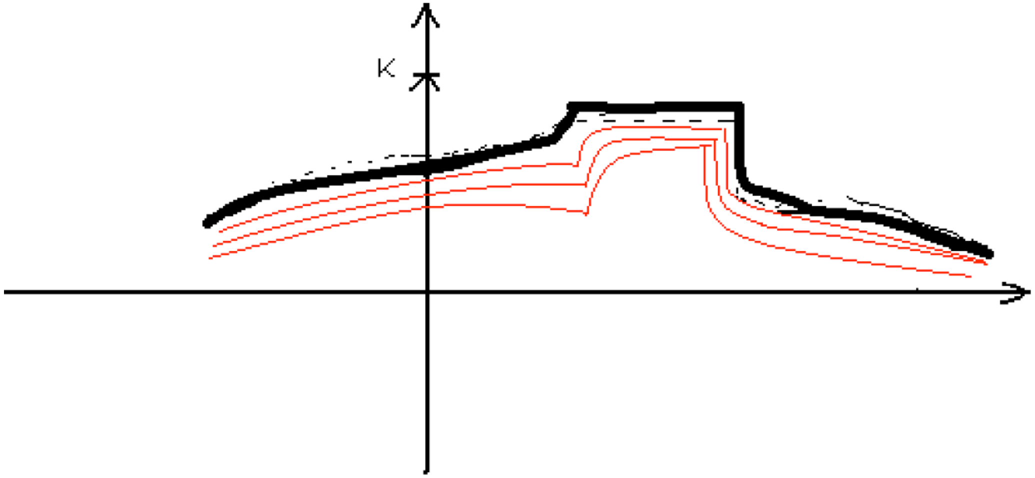
Lebesgue integral

Theorem: Given a non-negative measurable function $f\colon \mathbb{R}^p \to \mathbb{R}$, there is always an *increasing* sequence $\{f_n(\vec{x})\}_{n=1}^{\infty}$ of continuous functions (i.e. with the property that $f_n(\vec{x}) \geq f_{n-1}(\vec{x})$for all $\vec{x}$) which converges to $f(\vec{x})$.

Definition: If $f(\vec{x}) \geq 0$ is a positive measurable function, define

$$\int_{\mathbb{R}^p} f(x)\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^p} f_n(x)\, dx$$ Limit of ordinary Riemann integrals

where $f_n(\vec{x})$ is *any* increasing sequence of continuous functions which converges to $f(\vec{x})$.



Integral of an arbitrary function

To find the integral of a negative measurable function $f(x)$, we just compute the integral of $-f(x)$ (which is positive), and put a minus sign in front of it.

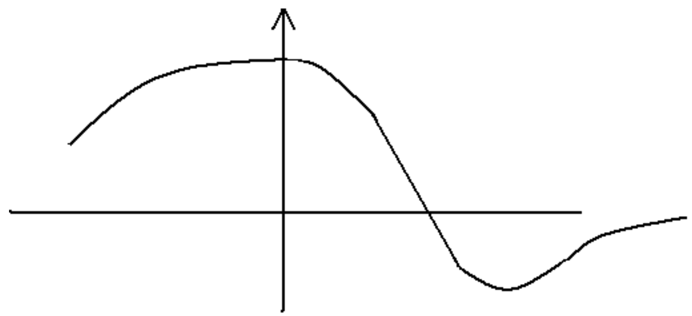
Since **every** function $f(x)$ is the sum of a positive plus a negative function

$$f(x) = f_1(x) + f_2(x)$$

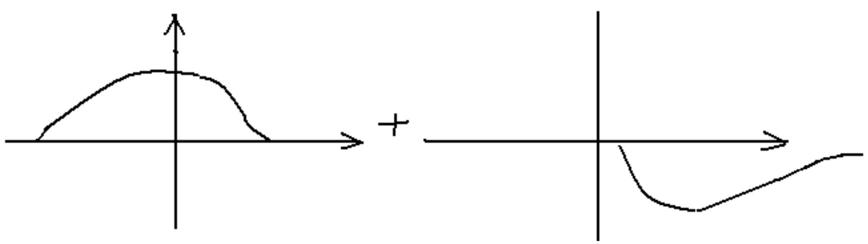
the integral of $f(x)$ is defined as

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f_1(x)dx + \int_{-\infty}^{\infty} f_2(x)dx$$

Example. $f(x)$ has positive and negative part



Then integral of $f(x)$ is integral of a positive plus a negative function:



All the properties of integrals we are used to also hold for this more general **Lebesgue** integral.

For example, we still have linear properties:

$$\int_{-\infty}^{\infty} f(x) + g(x) dx = \int_{-\infty}^{\infty} f(x) dx + \int_{-\infty}^{\infty} g(x) dx$$

➢ Hilbert spaces of functions

Consider the space

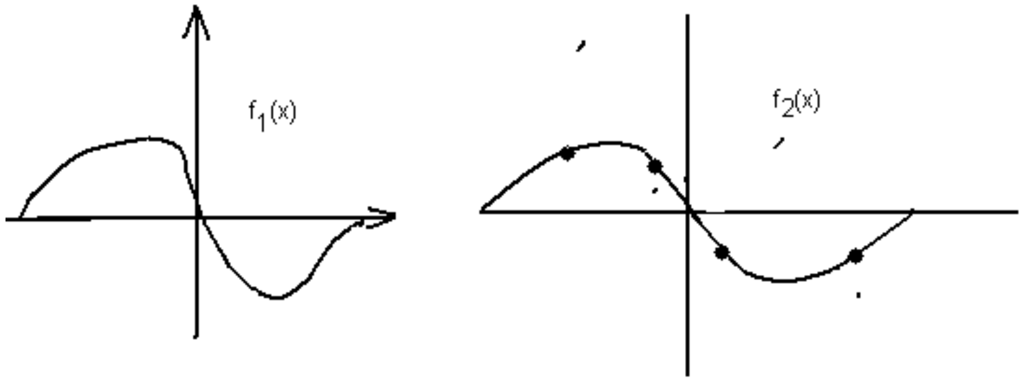
$$H = L^2[-\pi, \pi]$$

$$= \left\{ \text{measurable real functions } f(x) on[-\pi, \pi] \text{ with } \int_{-\pi}^{\pi} f^2(x)dx < \infty \right\}$$

Can show that H is complete (i.e., every Cauchy sequence converges to a function f in H).

Theorem: H is a Hilbert space

we always consider two measurable functions the same if they differ just at a finite number of points



same integral

$$\int |f_1 - f_2|\, dx = 0$$

Ref:

What is a tensor?
https://helper.ipam.ucla.edu/publications/tmtut/tmtut_17117.pdf
https://helper.ipam.ucla.edu/publications/tmtut/tmtut_17145.pdf

Tensor Methods and Emerging Applications to the Physical and Data Sciences Tutorials
https://www.ipam.ucla.edu/programs/workshops/tensor-methods-and-emerging-applications-to-the-physical-and-data-sciences-tutorials/?tab=schedule

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