MATH 5110 – Applied Linear Algebra and Matrix Analysis

❖ Fields-Class Work

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Example of groups:

1. The set $\mathbb{Z}$ of **integers** under **addition** is an abelian group with identity 0.

2. The set $\mathbb{Z}$ of **integers** under **multiplication** is not a group.

3. The set $\mathbb{Q}$ of rational numbers under **addition** is an abelian group.

4. The set $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ under **multiplication** is an abelian group with identity 1.

5. Given any nonempty finite set S, the set of **bijections** $f\colon S \rightarrow S$, called **permutations** of S, is a (non-abelian) group under function composition.

6. The set of $n \times n$ matrices with real (or complex) coefficients $\mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) is an abelian group under **addition** of matrices, with identity element the null-matrix.

7. The set of $n \times n$ **invertible** matrices with real (or complex) coefficients is a group under **matrix multiplication**, with identity element the identity matrix I. This group is called the **general linear group** and is usually denoted by $\boldsymbol{GL(n;\mathbb{R})}$ **or** $\boldsymbol{GL(n;\mathbb{C})}$.

8. The set of $n \times n$ **invertible** matrices with real (or complex) coefficients and **determinant 1** is a group under **matrix multiplication**, with identity element the identity matrix I. This group is called the **special linear group** and is usually denoted by $\boldsymbol{SL(n;\mathbb{R})}$ $(\boldsymbol{or\ SL(n;\mathbb{C})})$.

9. The set of $n \times n$ **orthogonal** matrices with **real** coefficients (i.e., $AA^T = A^T A = I$) is a group (under matrix multiplication) called the **general orthogonal group** and is usually denoted by **O(n).**

10. The set of $n \times n$ **orthogonal** matrices with **real** coefficients such that of **determinant +1** is a group (under matrix multiplication) called the **special orthogonal group** and is usually denoted by **SO(n)**

11. The set of $n \times n$ **unitary** matrices with **complex** coefficients (i.e., $AA^* = A^*A = I$) is a group (under matrix multiplication) called the **general unitary group** and is usually denoted by **U(n).**

12. The set of $n \times n$ **unitary** matrices such that of **determinant +1** is a group (under matrix multiplication) called the **special unitary group** and is usually denoted by **SU(n)**

https://math.mit.edu/~dav/classicalgroups.pdf

Those matrix groups are important examples in modern mathematics research, including group theory, differential geometry, algebraic topology, algebraic geometry, representation theory, Lie group, Lie algebra, symplectic geometry, etc. You can check some quick information from Wikipeida https://en.wikipedia.org/wiki/Classical_group

Real world application? Of course. Even SO(3) has lots of application in physics, computer vision, etc. (e.g., https://www.youtube.com/watch?v=zjMuIxRvygQ

Example (Heisenberg group):

Let H be the set of 3×3 upper triangular matrices given by

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

H with the binary operation of matrix multiplication is a group.

Questions:

- Find explicitly the inverse of every matrix in H.
- Is H abelian (commutative)?

https://en.wikipedia.org/wiki/Heisenberg_group