MATH 5110 – Applied Linear Algebra and Matrix Analysis

❖ Determinant - Class Work

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❑ Determinant of block matrices

Let M be a matrix written a as 2×2 block matrix

$$M = \begin{bmatrix} A & B \\ \mathbf{0} & D \end{bmatrix}$$

Here, A, D have the dimension $m \times m$, and $n \times n$.

By a direct proof or by **Schur complement**, we can show that

$$\det(M) = \det(A)\det(D)$$

More generally, let M be a matrix written a as 2×2 block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Here, A, D have the dimension $m \times m$, and $n \times n$.

By **Schur complement**, we can show that

$$\det(M) = \det(A) \det(D - CA^{-1}B) \text{ if } A \text{ is invertible}$$

and

$$\det(M) = \det(D) \det(A - BD^{-1}C) \text{ if } D \text{ is invertible}$$

Question: Do we have the formula in general?

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(AD - BC)$$

❑ Matrix determinant lemma

Suppose A is an $n \times n$ **invertible** matrix and $\vec{u}$, $\vec{v}$ are vectors in $\mathbb{R}^n$. Then

$$\det(A + \vec{u}\vec{v}^T) = (1 + \vec{v}^T A^{-1} \vec{u}) \det(A)$$

Proof:

$$\begin{pmatrix} I & 0 \\ \vec{v}^\mathsf{T} & 1 \end{pmatrix} \begin{pmatrix} I + \vec{u}\vec{v}^\mathsf{T} & \vec{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ -\vec{v}^\mathsf{T} & 1 \end{pmatrix} = \begin{pmatrix} I & \vec{u} \\ 0 & 1 + \vec{v}^\mathsf{T}\vec{u} \end{pmatrix}.$$

So,

$$\det(I + \vec{u}\vec{v}^T) = 1 + \vec{v}^T\vec{u}$$

The general case is

$$\det(A + \vec{u}\vec{v}^T) = \det(A(I + A^{-1}\vec{u}\vec{v}^T)) = \det(I + (A^{-1}\vec{u})\vec{v}^T)\det(A) = \cdots$$

If A is not invertible, it is possible to use adjugate matrix of A to compute

$$\det(A + \vec{u}\vec{v}^T) = \det(A) \; + \vec{v}^T(\mathrm{adj}(A))\vec{u}$$

More generally, for two $n{\times}m$ matrices U and V,

$$\det(A + UV^T) = \det(I + \vec{V}^T A^{-1} U)\det(A)$$

When $A = I$, we have the Weinstein–Aronszajn identity

$$\det(I + UV^T) = \det(I + \vec{V}^T U)$$

If $\lambda \neq 0$, then we have

$$\det(UV^T - \lambda I) = (-\lambda)^{\mathrm{m-n}}\det(\vec{V}^T U - \lambda I)$$

- This shows the that UV^T and $\vec{V}^T U$ have the same non-zero eigenvalues.
- It is useful in developing a Bayes estimator for multivariate Gaussian distributions.
- Applications in random matrix theory by relating determinants of large matrices to determinants of smaller ones.

https://terrytao.wordpress.com/2010/12/17/the-mesoscopic-structure-of-gue-eigenvalues/

An Introduction to Grids, Graphs, and Networks, Pozrikidis, C., Oxford University Press. (Appendix B:The Sherman-Morrison and Woodbury formulas.)

https://cran.r-project.org/web/packages/WoodburyMatrix/vignettes/WoodburyMatrix.html

https://en.wikipedia.org/wiki/Weinstein%E2%80%93Aronszajn_identity