MATH 5110 – Applied Linear Algebra and Matrix Analysis

❖ Class Work-Basis

Instructor: He Wang
Department of Mathematics
Northeastern University

Subspace P of polynomials in $\mathbb{R}[X]$ of degree at most n.

1. The polynomials $1; X; X^2, \dots ; X^n$ form a canornical basis.

2. The Bernstein polynomials

$$\binom{n}{k}(1-X)^{n-k}X^k \text{ for } k=0,\dots,n,$$

also form a basis of that space P. These polynomials play a major role in the theory of spline curves.

Example: Bases for $\mathbb{R}^4$

1. **Standard basis U** $= \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$

2. We already know that any **four independent vectors** in $\mathbb{R}^4$ form a basis for $\mathbb{R}^4$.

3. In particular, the following set W of vectors forms a basis of $\mathbb{R}^4$ known as the **Haar basis.**

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{w}_c = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \vec{w}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

These vectors are pairwise orthogonal, so they are indeed linearly independent.

This basis and its generalization to dimension 2^n are crucial in **wavelet theory,** which play an important role in audio and video signal processing.

The **change of basis matrix** P from U to W is given by

$$P = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

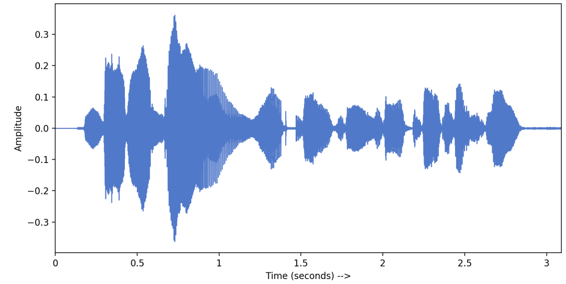
we easily find that the inverse of P by scale of P^T

$$P^{-1} = \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

So, for example, the vector $\vec{v} = \begin{bmatrix} 6 \\ 4 \\ 5 \\ 1 \end{bmatrix}$ over basis U, becomes $\vec{c} = P^{-1}\vec{v} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix}$

$$\begin{pmatrix} 4 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \\ 5 \\ 1 \end{pmatrix}$$

Given a **signal** $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$,



We first transform $\vec{v}$ into its coefficients $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$ over the Haar basis by computing

$$\vec{c} = P^{-1}\vec{v}.$$

Observe that

$$c_1 = \frac{v_1 + v_2 + v_3 + v_4}{4}$$

is the overall **average** value of the signal $\vec{v}$. The coefficient c_1 corresponds to the background of the image (or of the sound).

Then,
c_2 gives the coarse details of $\vec{v}$,
c_3 gives the details in the first part of $\vec{v}$,
c_4 gives the details in the second half of $\vec{v}$.

Reconstruction of the signal

$$\vec{v} = P\vec{c}$$

Compression

The trick for good compression is similar as SVD, FFT.

We throw away some of the coefficients of $\vec{c}$ (set them to zero), obtaining a **compressed signal** $\hat{\vec{c}}$, and still retain enough crucial information so that the reconstructed signal

$$\hat{\vec{v}} = P\hat{\vec{c}}$$

looks almost as good as the original signal $\vec{v}$.

Compression Process:

- Input signal $\vec{v}$
- Coefficients $\vec{c} = \mathrm{P}^{-1}\vec{v}$
- Compressed Coefficients $\hat{\vec{c}}$,
- Compressed signal $\hat{\vec{v}} = P\hat{\vec{c}}$

This kind of compression scheme makes modern video conferencing possible.

Similarly as Fast Fourier Transform (FFT), it turns out that there is a **faster** way to find $\vec{c} = \mathrm{P}^{-1}\vec{v}$, without actually using P^{-1} (**amazing**!).

This has to do with the multiscale nature of Haar wavelets.