Notes on spectral sequence

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Long exact sequence coming from short exact sequence of (co)chain complex in (co)homology is a fundamental tool for computing (co)homology. Instead of considering short exact sequence coming from pair (X, A), one can consider filtered chain complexes coming from a increasing of subspaces $X^0 \subset X^1 \subset \cdots \subset X$. We can see it as many pairs (X^p, X^{p+1}). There is a natural generalization of a long exact sequence, called spectral sequence, which is more complicated and powerful algebraic tool in computation in the (co)homology of the chain complex. Nothing is original in this notes.

Contents

1 Homological Algebra

1.1 Definition of spectral sequence

Definition 1.1. A **differential bigraded module** over a ring R, is a collection of R-modules, $\{E^{p,q}\}$, where p, $q \in \mathbb{Z}$, together with a R-linear mapping,

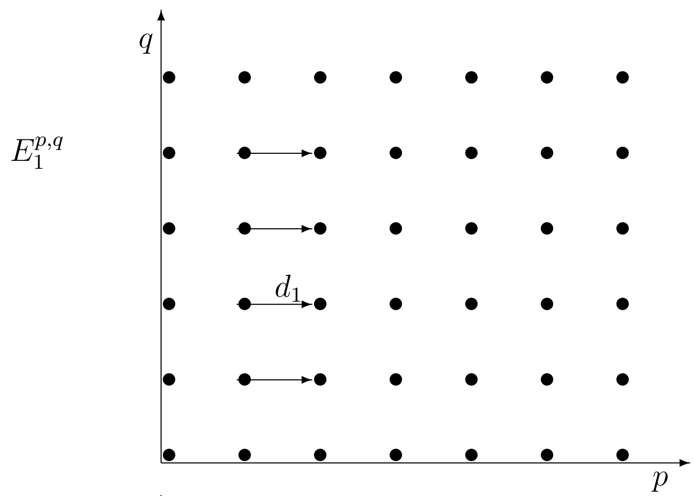
$d : E^{*,*} \to E^{*+s,*+t}$, satisfying $d \circ d = 0$. d is called the differential of bidegree (s, t).

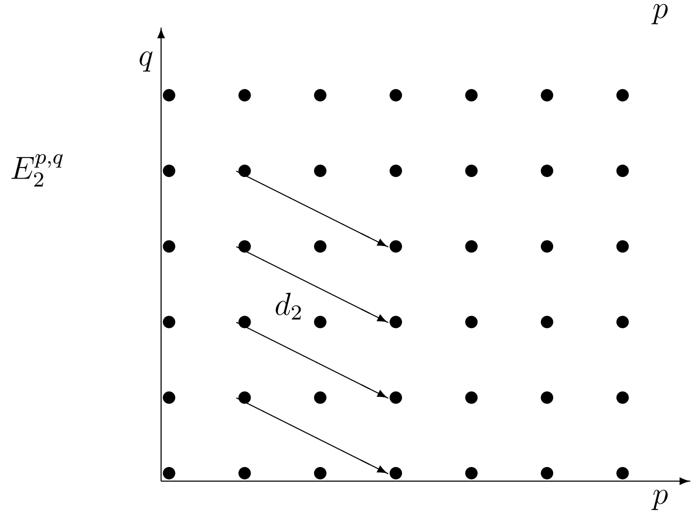
Definition 1.2. A **spectral sequence** is a collection of differential bigraded R-modules $\{E_r^{p,q}, d_r\}$, where $r = 1, 2, \cdots$ and

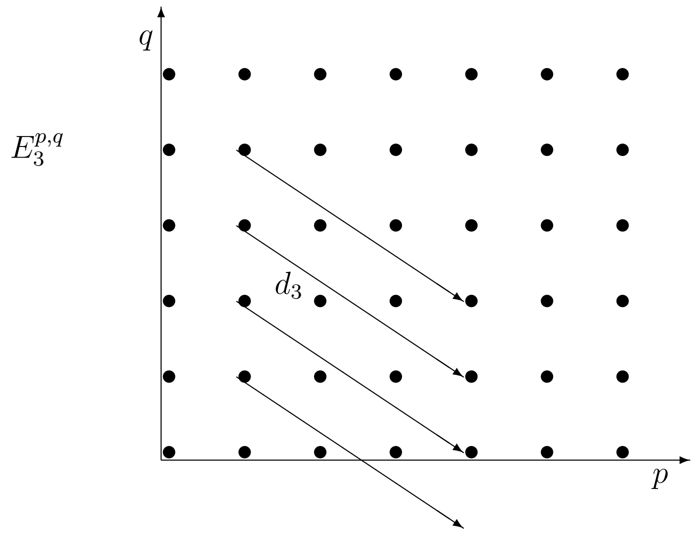
$$E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}) \cong ker(d_r : E_r^{p,q} \to E_r^{*,*})/im(d_r : E_r^{*,*} \to E_r^{p,q}).$$

In practice, we have the differential d_r of bidegree $(r, 1-r)$ (for a spectral sequence of cohomology type) or $(-r, r-1)$ (for a spectral sequence of homology type).

Let's look at what the (first quadrant) spectral sequence of cohomology type looks like.







An convention for spectral sequence of homology type is $\{E^r_{p,q}, d_r\}$ or $\{E^r_{p,q}, d^r\}$. In fact, some spectral sequences may have one grading(Bockstein spectral sequence), or three gradings.

Next consider the $E_\infty^{*,*}$-term. For the first quadrant cohomological spectral sequence, consider $E_r^{p,q}$, when $r > max(p, q+1)$, the differentials $d_r = 0$. Thus, $E_{r+1}^{p,q} = E_r^{p,q}$, also $E_{r+k}^{p,q} = E_r^{p,q}$, so, we can define $E_\infty^{p,q} = E_{r+1}^{p,q}$. In general case, we define $E_\infty^{*,*}$-term in the following way.(You can skip it at the first time)

Start from $E_2^{*,*}$-term. In order make the argument clear, let us suppress the bigrading. Denote

$$Z_2 = ker\ d_2 \text{ and } B_2 = im\ d_2.$$

Then, $B_2 \subset Z_2 \subset E_2$ and $E_3 \cong Z_2/B_2$ with short exact sequence

$$0 \to B_2 \to Z_2 \xrightarrow{j} E_3 \to 0.$$

Denote $\bar{Z}_3 = ker\ d_3 : E_3 \to E_3$ which is a submodule of E_3. So $\bar{Z}_3 \cong Z_3/B_2$, where $Z_3 = j^{-1}(\bar{Z}_3)$ is a submodule of Z_2. Similar $\bar{B}_3 = im\ d_3 \cong B_3/B_2$ with $B_3 = j^{-1}(\bar{B}_3)$. Then we have

$$E_4 \cong \bar{Z}_3/\bar{B}_3 \cong Z_3/B_3$$

and

$$B_2 \subset B_3 \subset Z_3 \subset Z_2 \subset E_2.$$

Iterating this process, we can present the spectral sequence as an infinite tower of submodules of E_2:

$$\boxed{B_2 \subset B_3 \subset \cdots \subset B_n \subset \cdots \subset Z_n \subset \cdots \subset Z_3 \subset Z_2 \subset E_2} \tag{1.1}$$

with property that

$$E_{n+1} \cong Z_n/B_n$$

and the differential d_{n+1} can be taken as a mapping $Z_n/B_n \to Z_n/B_n$, which has kernel Z_{n+1}/B_n and image B_{n+1}/B_n. This d_{n+1} induces a short exact sequence

$$0 \to Z_{n+1}/B_n \to Z_n/B_n \xrightarrow{d_{n+1}} B_{n+1}/B_n \to 0$$

which gives isomorphisms

$$Z_n/Z_{n+1} \cong B_{n+1}/B_n.$$

Conversely, a tower of submodules of E_2, together with a set of isomorphisms, determines a spectral sequence, by the following diagram.

$$\begin{array}{ccc} Z_n^{p,q}/B_n^{p,q} & \xrightarrow{epic} Z_n^{p,q}/Z_{n+1}^{p,q} \cong B_{n+1}^{p+r,q+1-r}/B_n^{p+r,q+1-r} \xrightarrow{monic} & Z_n^{p+r,q+1-r}/B_n^{p+r,q+1-r} \\ \| & & \| \\ E_{n+1}^{p,q} & \xrightarrow{\qquad\qquad d_{n+1} \qquad\qquad} & E_{n+1}^{p+r,q+1-r} \end{array}$$

Let $Z_\infty^{p,q} = \cap_n Z_n^{p,q}$ and $B_\infty^{p,q} = \cup_n B_n^{p,q}$. Define

$$\boxed{E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}.}$$

Definition 1.3. A **filtration** F^* on an R-module A is a family of submodules $\{F^pA\}$, such that

$$\cdots \subset F^{p+1}A \subset F^pA \subset F^{p-1}A \subset \cdots \subset A \text{ (decreasing filtration)}$$

or

$$A \supset \cdots \supset F^{p+1}A \supset F^pA \supset F^{p-1}A \supset \cdots \text{ (increasing filtration)}$$

A filtration is said to be **convergent** if $\cap_s F^sA = 0$ and $\cup_s F^sA = A$.

Definition 1.4. A spectral sequence $\{E_r^{*,*}, d_r\}$ is said to converge to a graded module H^* if there is a (decreasing) filtration F^* on H^* such that

$$E_\infty^{p,q} \cong F^p H^{p+q}/F^{p+1}H^{p+q}. \tag{1.2}$$

A filtration F^* on graded R-module H^* means that a filtration on each H^n. In fact, we can examine the filtration on each degree by letting

$$F^p H^n = F^p H^* \cap H^n.$$

How to get information from a spectral sequence converging to H^*? We can see that in the following example.

Example 1.5. Suppose the filtration of H^* is bounded above and below. That is

$$0 \subset F^n H^* \subset F^{n-1}H^* \subset \cdots \subset F^1 H^* \subset F^0 H^* \subset H^*$$

By formula (1.2), we have a series of short exact sequence

$$\begin{array}{ccccccccc}
 & & 0 & \to & F^n H^{p+q} & \to & E_\infty^{n,p+q-n} & \to & 0 \\
0 & \to & F^n H^{p+q} & \to & F^{n-1}H^{p+q} & \to & E_\infty^{n-1,p+q-n+1} & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & F^k H^{p+q} & \to & F^{k-1}H^{p+q} & \to & E_\infty^{k-1,p+q-k+1} & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & F^1 H^{p+q} & \to & F^0 H^{p+q} & \to & E_\infty^{0,p+q} & \to & 0 \\
0 & \to & F^0 H^{p+q} & \to & H^{p+q} & \to & E_\infty^{-1,p+q+1} & \to & 0
\end{array}$$

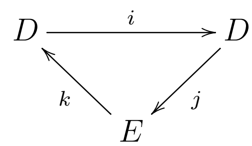
If H^* is vector space, then

$$\boxed{H^{p+q} \cong \bigoplus_{i+j=p+q} E_\infty^{i,j}}$$

In the above example, two assumptions are important, (H^* is vector space and the filtration of H^* is bounded above and below). If we don't have this tow assumption, the problem is hard. For general module H, we will meet the extension problem. For example, $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ is not splitting.

1.2 Construction of spectral sequence

Definition 1.6. An **exact couple** is an exact sequence of R-modules of the form

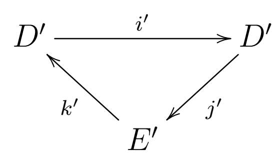


where i, j and k are R-module homomorphisms. Define $d : E \to E$ by $d = j \circ k$. Then

$$d^2 = j(kj)k = 0.$$

So, homology $H(E) = ker\ d/im\ d$ is defined.

From the above exact couple, we can construct a new exact couple, called derived couple,



by making the following definitions.

1. $D' = i(D)$.
2. $E' = H(E)$.
3. $i' = i|_{D'}$.
4. $j' : D' \to E'$ is defined to be $j'(a') = [j(a)]$, where $a' = i(a)$. This is well defined: (a.) ja is a cycle. $d(ja) = jkj(a) = 0$. (b.) $[ja]$ is independent of the choice of a. Suppose $a' = ia_1$. Then because $i(a - a_1) = 0$, we have $a - a_1 = kb$ for some $b \in E$. Thus,

   $$ja - ja_1 = j(a - a_1) = jk(b) = db.$$

   So, $[ja] = [ja_1]$.
5. $k' : E' \to D'$ is defined to be $k'[e] = k(e)$. This is well defined, since we can get $k(e) = i(a) \in i(A)$ from $jke = 0$.

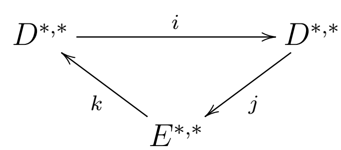
Proposition 1.7. The derived couple $\{D', E', i', j', k'\}$ is also an exact couple.

Proof: It is straightforward to check the exactness. (Exercise) □

Iterate this process, we can get the nth derived exact couple

$$\{D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)}\}.$$

Theorem 1.8. *Suppose $D^{*,*} = \{D^{p,q}\}$ and $E^{*,*} = \{E^{p,q}\}$ are graded modules over R equipped with homomorphisms i of bidegree $(-1, 1)$, j of bidegree $(0, 0)$ and k of bidegree $(1, 0)$.*



These data determine a spectral sequence $\{E_r^{*,*}, d_r\}$ of cohomology type, with $E_r^{*,*} = (E^{*,*})^{(r-1)}$ and $d_r = j^{(r)} \circ k^{(r)}$.

Proof: We only need to check that the differential have bidegree $(r, 1-r)$. We can prove this by induction and notice that

$$deg(i^{(n)}) = deg(i),$$

$$deg(k^{(n)}) = deg(k),$$

$$deg(j^{(n)}) = deg(j) - (n-1)[deg(i)].$$

$$deg(d_r) = deg(j^{(r)}) + deg(k^{(r)}).$$

So, $j^{(n)}$ has bidegree $(n-1, -n+1)$ and $deg(d_r) = (r, 1-r)$. □

Remark: For homomorphisms i of bidegree $(1, -1)$, j of bidegree $(0, 0)$ and k of bidegree $(-1, 0)$, we can get the spectral sequence with homology type, with d_r the bidegree $(-r, r-1)$. □

Denote

$$D_\infty = \bigcap_{n=1}^{\infty} D^{(n+1)} = \bigcap_{n=1}^{\infty} Im[i^{(n)}], D_0 = \bigcup_{m=1}^{\infty} Ker[i^{(n)}]$$

Then, in [5], we have

$$E_\infty \cong k^{-1} D_\infty / j D_0.$$

Exercise: Verify $Z_n = k^{-1}(Im[i^{(n)})]$ and $B_n = j(Ker[i^{(n)}])$ satisfy tower (1.1).

Example 1.9. An important example of exact couple comes from the long exact sequence in homology and a short exact sequence of coefficient. Let

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p \to 0,$$

(C_*, d) be a differential graded free abelian group. Then we have short exact sequence

$$0 \to C_* \xrightarrow{\times p} C_* \to C_* \otimes \mathbb{Z}/p \to 0,$$

and an exact couple

$$\begin{array}{ccc} H_*(C_*) & \xrightarrow{H(\times p)} & H_*(C_*) \\ & {\scriptstyle k}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & H_*(C_* \otimes \mathbb{Z}/p) & \end{array}$$

Then from this exact couple, we can have the **Bockstein spectral sequence** with $E_1^n = H_n(X, \mathbb{Z}/p)$, and $d_1 = \beta$ the Bockstein homomorphism, and converging to $(H_*(X)/torsion) \otimes \mathbb{Z}/p$. (see [2] Chapter 10)

One important method to get an exact couple is from filtered differential module.

Definition 1.10. A R-module is a **filtered differential graded module** (FDGM) if

1. (Graded) A is a direct sum of submodules, $A = \bigoplus_{n=0}^{\infty} A^n$.

2. (Differential) There is an R-linear mapping $d : A \to A$ of degree 1($d : A^n \to A^{n+1}$) (or of degree -1) satisfying $d \circ d = 0$.

3. (Filtration) A has a filtration F^* and the differential d respects the filtration, that is, $d : F^pA \to F^pA$.

Remark 1.11. Since the differential d preserve the filtration F^*, d induces a well-defined differential $d : F^pA^n/F^{p+1}A^n \to F^pA^{n+1}/F^{p+1}A^{n+1}$, and make $\{F^pA/F^{p+1}A, d\}$ into an cochain complex.

Example 1.12. For topological space X, the (co)chain complex $(C_*(X), d)$ $((C^*(X), d))$ can give a differential graded module C. If we can give filtration on each $C_n(X)$ $(C^n(X))$ such that the filtration preserve the differential, then we have a FDGM , $C = \bigoplus_{n=0}^{\infty} C_n(X)$, (or $C = \bigoplus_{n=0}^{\infty} C^n(X)$). More details will be discussed in the Section 2.

Theorem 1.13. *(Cohomology type) Each decreasing filtrated differential graded module (A, d, F^*) of degree 1 determines a spectral sequence, $\{E_r^{*,*}, d_r\}$, with d_r of bidegree $(r, 1-r)$ and*

$$E_1^{p,q} \cong H^{p+q}(F^pA/F^{p+1}A).$$

Moreover, if the filtration F^* is convergent, then the spectral sequence converges to $H^*(A, d)$, that is

$$E_\infty^{p,q} \cong F^pH^{p+q}(A, d)/F^{p+1}H^{p+q}(A, d)$$

The filtration on $H^{p+q}(A)$ is given by

$$F^p(H^{p+q}(A)) = ker[H^{p+q}(A) \to H^{p+q}(F^pA)].$$

Proof: For each filtration degree p, there is a short exact sequence of graded modules

$$0 \to F^{p+1}A \to F^pA \to F^pA/F^{p+1}A \to 0$$

When we apply the homology functor, we obtain, for each p, the long exact sequence

$$\cdots H^{p+q}(F^{p+1}A) \xrightarrow{i} H^{p+q}(F^pA) \xrightarrow{j} H^{p+q}(F^pA/F^{p+1}A) \xrightarrow{k}$$
$$H^{p+q+1}(F^{p+1}A) \xrightarrow{i} H^{p+q+1}(F^pA) \xrightarrow{j} \cdots$$

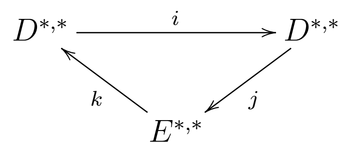
Define

$$E^{p,q} = H^{p+q}(F^pA/F^{p+1}A) \text{ and } D^{p,q} = H^{p+q}(F^pA).$$

Then we have the long exact sequence

$$\cdots D^{p+1,q-1} \xrightarrow{i} D^{p,q} \xrightarrow{j} E^{p,q} \xrightarrow{k} D^{p+1,q} \xrightarrow{i} D^{p,q+1} \xrightarrow{j} \cdots$$

with i of bidegree $(-1,1)$, j of bidegree $(0,0)$ and k of bidegree $(1,0)$. This gives an exact couple



By Theorem 1.8, this yields a spectral sequence with cohomology type.

The converging part is difficult. (See [2] Ch2 and 3, or [7] Ch9) □

The homology type of the above theorem is the following theorem.

Theorem 1.14. *(Homology type) Each increasing filtrated differential graded module (A, d, F^*) of degree -1 determines a spectral sequence, $\{E^r_{*,*}, d_r\}$, with d_r of bidegree $(-r, r-1)$ and*

$$E^1_{p,q} \cong H_{p+q}(F^pA/F^{p-1}A).$$

Moreover, if the filtration F^* is convergent, then the spectral sequence converges to $H_*(A, d)$, that is

$$E^\infty_{p,q} \cong F^pH_{p+q}(A, d)/F^{p-1}H_{p+q}(A, d)$$

The filtration on $H_{p+q}(A)$ is given by

$$F^p(H_{p+q}(A)) = im[H_{p+q}(F^pA) \to H_{p+q}(A)].$$

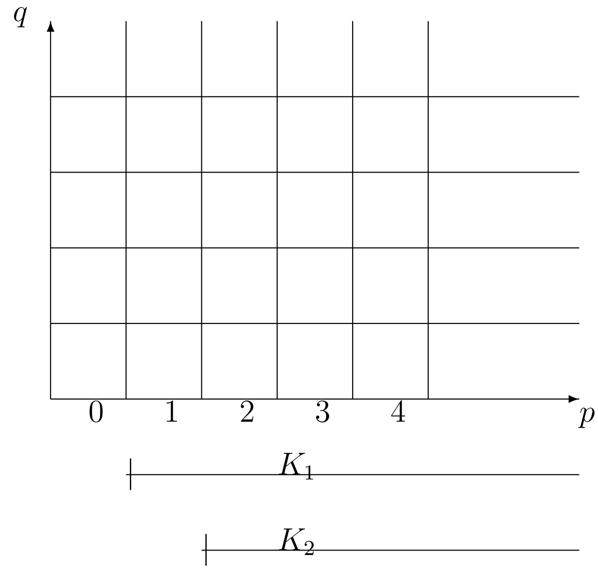
□

Example 1.15. [1]. If $K = \oplus_{p,q\geq 0} K^{p,q}$ is a **double complex** with horizontal δ and vertical operator d, we can form a single complex out of it in the usual way, by letting $K = \oplus C^k$, where $C^k = \oplus_{p+q=k} K^{p,q}$, and defining the differential operator $D : C^k \to C^{k+1}$ to be

$$D = \delta + (-1)^p d.$$

Then the sequence of subcomplexes indicated below is a decreasing filtration on K:

$$K_p = \bigoplus_{i\geq p} \bigoplus_{q\geq 0} K^{i,q}$$



Theorem 1.16. *[1]. Given a double complex $K = \oplus_{p,q\geq 0} K^{p,q}$ there is a spectral sequence $\{E_r, d_r\}$ converging to the total cohomology $H_D(K)$ such that d_r has bidegree $(r, 1-r)$, and*

$$E_1^{p,q} = H_d^{p,q}(K),$$

$$E_2^{p,q} = H_\delta^{p,q} H_d(K);$$

furthermore,

$$E_\infty^{p,q} = F^p H_D^{p+q} / F^{p+1} H_D^{p+q}$$

Proof: See [1]. □

Example 1.17. [4]. Let (C_*, d) and (C'_*, d') be two DGMs over R, the tensor product $(C_* \otimes C'_*, \partial)$ of these two DGMs over R is defined by

$$(C \otimes C')_k = \bigoplus_{i+j=k} C_i \otimes C'_j$$

and the differential is defined by

$$\partial(a \otimes b) = (da) \otimes b + (-1)^i a \otimes (d'b).$$

To compute the homology of $C \otimes C'$, define a filtration on it by

$$F^p(C \otimes C')_k = \bigoplus_{i \le p} C_i \otimes C_{k-i}.$$

Verify the following results

$$E^0_{p,q} = C_p \otimes C'_q, d_0 = (-1)^p \otimes \partial';$$

$$E^1_{p,q} = C_p \otimes H_q(C'_*), d_1 = d \otimes 1;$$

$$E^2_{p,q} = H_p(C_* \otimes H_q(C'_*)), d_2 = 0;$$

$$E^\infty_{p,q} = E^2_{p,q}.$$

2 Spectral sequence in Topology

2.1 General method

Let X be a topological space with a increasing filtration $\{X^q\}$: that is, the X^q are closed subspaces of X such that $X^q \subset X^{q+1}$, $X = \cup_q X^q$, $X^q = \emptyset$ for $q < 0$ and every compact subset of X is contained in some X^q.

(I) Homology. The filtration on X gives a increasing filtration on $C_*(X)$ by $F^pC_*(X) = C_*(X^p)$. The filtration F of C_* induce a filtration on $H_*(C_*)$ defined by

$$F^p(H(C_*)) = im[H(F^pC_*) \to H(C_*)].$$

Because the homology functor commutes with direct limits, if F^* is a convergent filtration of C_*, it follows that $\cup_s F^s H_*(C_*) = H_*(C_*)$. But in general $\cap_s F^s H(C_*) = 0$ may NOT true. However, if F is convergent and bounded below on C_* as in our case, that is $F^{t(n)}C^n_* = 0$ for any n, the same is true of the induced filtration on $H_*(C_*)$.

(II) Cohomology. Similar with homology case.(See [7] for details.) The filtration on X gives a decreasing filtration on $C^*(X)$ by

$$F^pC^*(X) = Ann(F^{p-1}C_*).$$

The filtration on $H^*(C^*)$ is defined by

$$F^s(H^*) = ker[H^*(C^*) \to H^*(F^{s-1}C^*)].$$

Go back to homology case.

For each filtration degree p, there is a short exact sequence of graded modules

$$0 \to C_*(X^{p-1}) \to C_*(X^p) \to C_*(X^p)/C_*(X^{p-1}) \to 0$$

When we apply the homology functor, we obtain, for each p, the long exact sequence

$$\cdots H_{p+q}(X^{p-1}) \xrightarrow{i} H_{p+q}(X^p) \xrightarrow{j} H_{p+q}(X^p, X^{p-1}) \xrightarrow{k}$$
$$H_{p+q-1}(X^{p-1}) \xrightarrow{i} H_{p+q-1}(X^p) \xrightarrow{j} \cdots$$

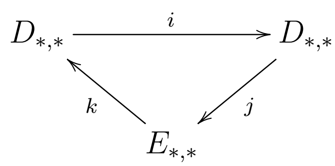
Define

$$E_{p,q} = H_{p+q}(X^p, X^{p-1}) \text{ and } D_{p,q} = H_{p+q}(X^p).$$

Then we have the long exact sequence

$$\cdots D_{p-1,q+1} \xrightarrow{i} D_{p,q} \xrightarrow{j} E_{p,q} \xrightarrow{k} D_{p-1,q} \xrightarrow{i} D_{p,q-1} \xrightarrow{j} \cdots$$

with i of bidegree $(1,-1)$, j of bidegree $(0,0)$ and k of bidegree $(-1,0)$. This gives an exact couple



Theorem 2.1. *Let X be a topological space with a increasing filtration $\{X^q\}$, then there is a spectral sequence, $\{E^r_{*,*}, d_r\}$, with d_r of bidegree $(-r, r-1)$ and*

$$E^1_{p,q} \cong H_{p+q}(X^p, X^{p-1}).$$

The spectral sequence converges to $H_*(X,d)$, that is

$$E^{p,q}_\infty \cong F^p H_{p+q}(X)/F^{p-1}H_{p+q}(X)$$

The filtration on $H_{p+q}(X)$ is given by $F^p(H_{p+q}(X)) = im[H_{p+q}(X^p) \to H_{p+q}(X)]$.

Proof: This follows from Theorem 1.14, since the filtration F^* on chain complex is convergent.

In fact, people can construct $Z^r_{s,t}$ and $B^r_{s,t}$ directly by

$$Z^r_{s,t} = k^{-1}(im[i^{r-1} : H_{s+t-1}(X^{s-r}) \to H_{s+t-1}(X^{s-1})])$$

$$B^r_{s,t} = j(ker[i^{r-1} : H_{s+t}(X^s) \to H_{s+t}(X_{s+r-1})]).$$

Another method to define $Z^r_{s,t}$ and $B^r_{s,t}$ is in [6] □

The cohomology version of this theorem is similar from 1.13.

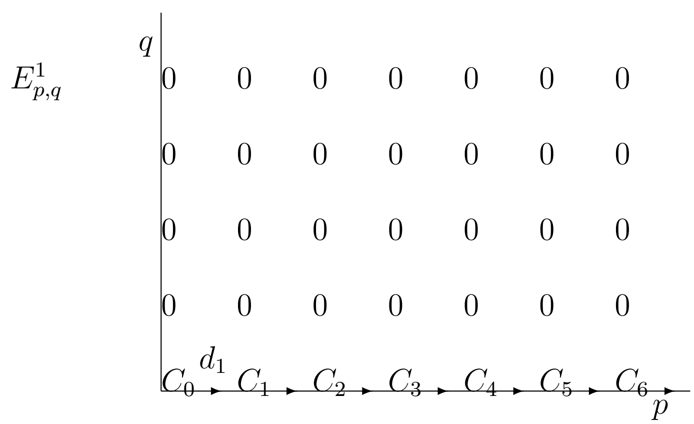
Example 2.2 (Cellular homology)**.** Let X be a CW-complex. Then X has a natural filtration by the p-skeleton X^p. By theorem 2.1, we have a spectral sequence, $\{E^r_{*,*}, d_r\}$ converging to $H_*(X, d)$, with

$$E^1_{p,q} \cong H_{p+q}(X^p, X^{p-1}).$$

Recall that

$$H_{p+q}(X^p, X^{p-1}) \cong \begin{cases} C^{cell}_p(X), & q = 0, \\ 0, & q \neq 0. \end{cases}$$

So, the $E^1_{*,*}$ is



And the first differential $d_1 : E_{p,0} \to E^1_{p-1,0}$ is the same as the cellular differential

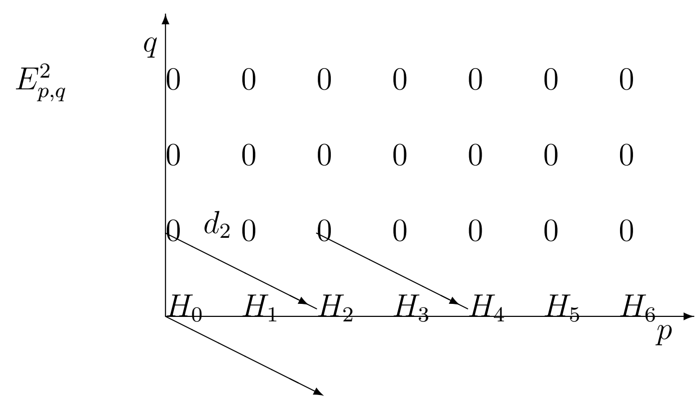
$$d : C^{cell}_p(X) = H_p(X^p, X^{p-1}) \to C^{cell}_{p-1}(X) = H_{p-1}(X^{p-1}, X^{p-2})$$

This cellular can be seen as two equivalent ways: (1) the boundary map of the long exact sequence of the triple (X^p, X^{p-1}, X^{p-2}). (2) The composition of $H_p(X^p, X^{p-1}) \xrightarrow{k} H_{p-1}(X^{p-1}) \xrightarrow{j} H_{p-1}(X^{p-1}, X^{p-2})$. The equivalence follows from the commutative diagram.

$$\begin{array}{ccccccc}
\longrightarrow & H_p(X^p, X^{p-1}) & \xrightarrow{k} & H_{p-1}(X^{p-1}) & \xrightarrow{i} & \\
& \| & & \downarrow j & & \\
\longrightarrow & H_p(X^p, X^{p-1}) & \xrightarrow{d} & H_{p-1}(X^{p-1}, X^{p-2}) & \xrightarrow{i} &
\end{array}$$

Therefore, E^2 is given in terms of the cellular homology by

$$E^2_{p,q} \cong \begin{cases} H^{cell}_p(X), & q = 0, \\ 0, & q \neq 0. \end{cases}$$



So, $d_r = o$ for $r \geq 2$. Then $E^r_{p,q} = E^2_{p,q}$ for $r \geq 2$. So, $H_p(X) = H^{cell}_p(X)$.

2.2 Leray-Serre spectral sequence

Suppose $F \hookrightarrow X \xrightarrow{\pi} B$ is a fibration, where B is a path-connected CW-complex. We can filter X by the subspaces $X^p = \pi^{-1}(B^p)$, B^p being the p-skeleton of B.

All fibers $F_b = \pi^{-1}(b)$ are homotopy equivalent to a fixed fiber since each path γ in B lifts to a homotopy equivalence $L_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$ between the fibers over the endpoints of γ. In particular, a loop γ at a basepoint of B gives homotopy equivalence $L_\gamma : F \to F$ for F the fiber over the base point. The association $\gamma \to L_{\gamma *}$ defines an action of $\pi_1(B)$ on $H_*(F)$. We may assume that this action is trivial in the following theorem, meaning that $L_{\gamma *}$ is the identity for all loops γ. Then the fibration is called **orientable** (This is a generalization of the concept of orientability of a sphere bundle).

Theorem 2.3. *[the homology Leray-Serre spectral sequence]*

Suppose $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibration, where B is path-connected. If $\pi_1(B)$ acts trivially on $H_*(F)$, then there is a spectral sequence $\{E^r_{p,q}, d_r\}$ converging to H_*, with

$$E^2_{p,q} \cong H_p(B; H_q(F)).$$

Furthermore, this spectral is natural with respect to fibre-preserving maps of fibrations.

Sketch of proof: When B is a CW-complex, $E^1_{p,q} = H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$. Calculate this relative homology as a direct sum over the p-cells $\sigma : D^p \to B$ of $H_{p+q}(\sigma^* E, (\sigma|_{S^{p-1}})^* E)$ to find that

$$E_{p,q} = C^{cell}_p(B; H_q(E_x)).$$

Similar with the cellular homology, try to find the differential to be the cellular differential.

When B is not a CW-complex, use CW approximation to B. ([3] for detail.) □

Remark 2.4. (1) When $X = B \times F$, we can compute $H_*(X)$ by K*ü*nneth formula. In general, $H_*(B \times F)$ provides an upper bound on the size of $H_*(X)$.

(2) Since (B^p, B^{p-1}) is $(p-1)$-connected, the homotopy lifting property implies that (X^p, X^{p-1}) is also $(p-1)$-connected. Hence, $E^1_{p,q} = H_{p+q}(X^p, X^{p-1})$ are nonzero only when $p \geq 0$ and $q \geq 0$(first quadrant spectral sequence).

(3) If the action is not trivial, the theorem is also true by regarding the homology with local coefficients.

(4) The theorem is also true for fiber bundles.

Example 2.5. Consider orientable fibration $K(\mathbb{Z}, 1) \hookrightarrow PK(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$, where $PK(\mathbb{Z}, 2)$ is the space of paths of $K(\mathbb{Z}, 2)$ starting at the base point, hence contractible. This is a spectral case of Gysin sequence in the following.

Also consider the general case $K(\mathbb{Z}, n) \hookrightarrow PK(\mathbb{Z}, 2) \to K(\mathbb{Z}, n+1)$.

Next consider the Leray-Serre spectral sequence for cohomology.

Theorem 2.6. *[the cohomology Leray-Serre spectral sequence]*

Consider the cohomology with coefficients in a ring R. Suppose $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibration, where B is path-connected. If $\pi_1(B)$ acts trivially on $H^*(F;R)$, then there is a spectral sequence $\{E_r^{p,q}, d_r\}$ converging to H^*, with

$$E_2^{p,q} \cong H^p(B; H^q(F;R)).$$

This spectral is natural with respect to fibre-preserving maps of fibrations.

Remark 2.7. It becomes much more powerful when cup products are brought into the picture. Consider the cohomology with coefficients in a ring R rather than just an abelian group. We think the cup product as the composition

$$H^*(X;R) \times H^*(X;R) \xrightarrow{\times} H^*(X \times X;R) \xrightarrow{\triangle^*} H^*(X;R)$$

of cross product with the map induced by the diagonal map $\triangle\colon X \to X \times X$. If $\alpha, \beta \in H^*(X;R)$, represented by two homomorphisms $f : C_* \to R$ and $g : C_* \to R$, then $\alpha \times \beta$ is represented by homomorphism

$$C_*(X \times X) \xrightarrow{\delta} C_*(X) \otimes C_*(X) \xrightarrow{f \otimes g} R \otimes R \to R.$$

Proposition 2.8. The Leray-Serre spectral sequence for cohomology can be provided with bilinear products $E_r^{p,q} \times E_r^{s,t} \xrightarrow{\star_r} E_r^{p+s,q+t}$ satisfying

1. $d_r(x \star_r y) = (d_r x) \star_r y + (-1)^{p+q} x \star_r (d_r y)$ for $x \in E_r^{p,q}$ and $y \in E_r^{s,t}$.

2. $\star_{r+1}$ is the product on the homology of (E_r, d_r) induced by $\star_r$.

3. $x \star_2 y = (-1)^{qs} x \cup y$, for $x \in E_2^{p,q}$ and $y \in E_2^{s,t}$, where the coefficients are multiplied via the cup product $H^q(F;R) \times H^t(F;R) \to H^{q+t}(F;R)$.

4. The cup product in $H^*(X;R)$ restricts to maps $F^pH^m \times F^sH^n \to F^{p+s}H^{m+n}$. These induce quotient maps

$$F^pH^m/F^{p+1}H^m \times F^sH^n/F^{s+1}H^n \to F^{p+s}H^{m+n}/F^{p+s+1}H^{m+n}.$$

   that coincide with the products $E_\infty^{p,m-p} \times E_\infty^{s,n-s} \to E_\infty^{p+s,m+n-p-s}$.

2.3 Application of Leray-Serre spectral sequence

Example 2.9. An orientable fibration $F \hookrightarrow E \xrightarrow{\pi} B$ with fiber an n-sphere, $n > 0$, is called spherical fibration. Then

$$H^q(F;R) = \begin{cases} R, & \text{if } q = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, in Leray-Serre spectral sequence,

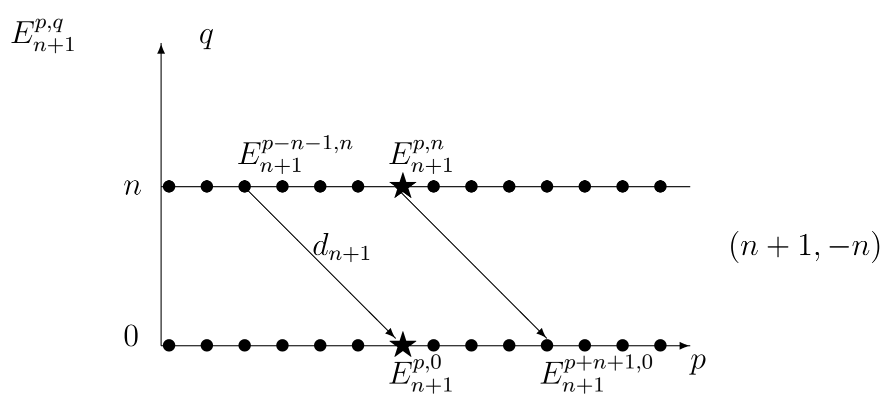
$$E_2^{p,q} = H^p(B; H^q(F;R)) \begin{cases} H^p(B;R), & \text{if } q = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

The bidegree of d_r is $(r, 1-r)$, so the only non-trivial differential is d_{n+1}. Hence,

$$E_2^{p,q} = E_3^{p,q} = \cdots = E_n^{p,q} = E_{n+1}^{p,q}$$

and

$$E_{n+2}^{p,q} = E_{n+3}^{p,q} = \cdots = E_\infty^{p,q}$$



Clearly,

$$E_{n+2}^{p,n} = ker[d_{n+1} : E_{n+1}^{p,n} \to E_{n+1}^{p+n+1,0}]$$
$$E_{n+2}^{p,0} = coker[d_{n+1} : E_{n+1}^{p-n-1,n} \to E_{n+1}^{p,0}].$$

So, we have exact sequence

$$0 \to E_{n+2}^{p,n} \to E_{n+1}^{p,n} \xrightarrow{d_{n+1}} E_{n+1}^{p+n+1,0} \to E_{n+2}^{p+n+1,0} \to 0.$$

That is

$$0 \to E_\infty^{p,n} \to E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} \to E_\infty^{p+n+1,0} \to 0. \tag{2.1}$$

The only nontrivial E_∞-term are $E_\infty^{p,0}$ and $E_\infty^{p,n}$ for $p \geq 0$. And,

$$E_\infty^{p,q} \cong F^p H^{p+q}/F^{p+1}H^{p+q}.$$

Consider $H^m(E;R)$, we have the only two nontrivial E_∞-term

$$\begin{cases} E_\infty^{m,0} = F^m H^m/F^{m+1}H^m \\ E_\infty^{m-n,n} = F^{m-n}H^m/F^{m-n+1}H^m \end{cases}$$

Then the filtration on $H^m(E;R)$ have the form

$$H^m = \cdots = F^{m-n}H^m \supset F^{m-n+1}H^m = \cdots = F^m H^m \supset F^{m+1}H^m = \cdots = \{0\}.$$

Then we have

$$\begin{cases} E_\infty^{m,0} = F^m H^m \\ E_\infty^{m-n,n} = H^m/F^{m-n+1}H^m \end{cases}$$

This yields a short exact sequence

$$0 \to E_\infty^{m,0} \to H^m(E;R) \to E_\infty^{m-n,n} \to 0. \tag{2.2}$$

Gluing exact sequence (2.1) and (2.2), and recalling that

$$E_2^{p,0} = E_2^{p,n} = H^p(B;R),$$

We get a long exact sequence

$$\boxed{\cdots \to H^m(E;R) \xrightarrow{\phi} H^{m-n}(B;R) \xrightarrow{d_{n+1}} H^{m+1}(B;R) \xrightarrow{\pi^*} H^{m+1}(E;R) \to \cdots,} \tag{2.3}$$

which is called **Gysin sequence** of a spherical fibration $\pi : E \to B$. (In fact, for $n = 0$, Gysin sequence is also true, but it need another proof.)

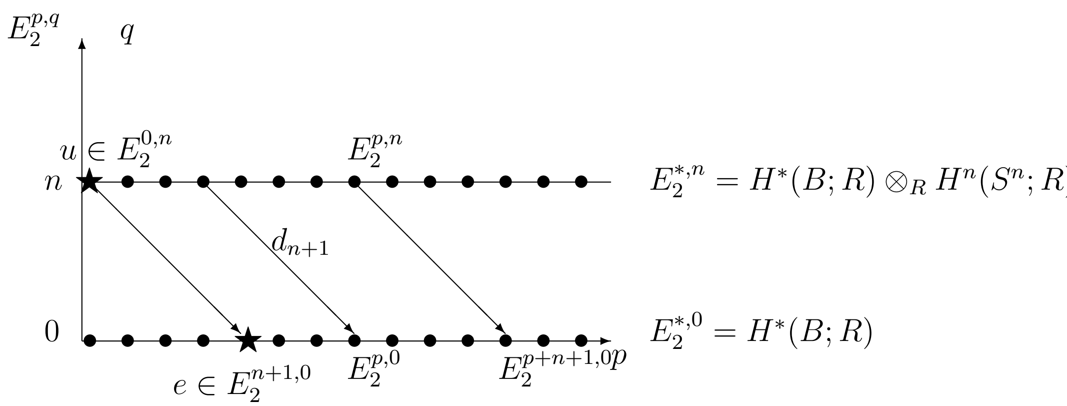
The following formula explain how the gluing works

$$\begin{array}{llll} 0 \to E_\infty^{m,0} \to & H^m(E;R) \to & E_\infty^{m-n,n} \to 0 & \\ & 0 \to & E_\infty^{m-n,n} \to E_2^{m-n,n} \xrightarrow{d_{n+1}} E_2^{m+1,0} \to E_\infty^{m+1,0} \to 0. & \end{array}$$

The map π^* need some argument. (See [6].) We are more interested in map d_{n+1}, which can be described in another useful way.

$$E_2^{p,n} = H^p(B; H^0(S^n; R)) \cong H^p(B; R) \otimes_R H^n(S^n; R)$$

$$E_2^{p,0} = H^p(B; H^0(S^n; R)) \cong H^p(B; R)$$



Let $u \in H^n(S^n; R)$ be a generator, we can also regard u as lying in $E_2^{0,n} = H^0(B; H^n(S^n; R)) \cong H^n(S^n; R)$. Let

$$e = d_{n+1}u \in E_2^{n+1,0} = H^{n+1}(B; H^0(S^n; R)) \cong H^{n+1}(B; R).$$

e is called the **Euler class** of the spherical fibration. We may regard elements of $E_2^{p,n}$ as being of the form $u \star x$, where $x \in H^p(B; R) \cong H^p(B; H^0(S^n; R)) = E_2^{p,0}$. Thus, by Proposition 2.8,

$$d_{n+1}(u \star x) = (d_{n+1}u) \star x + (-1)^n u \star d_{n+1}(x) = e \star x = e \cup x.$$

Thus,

$$d_{n+1} : H^{m-n}(B; R) \to H^{m+1}(B; R)$$

is simply cup-product with the class $x \to e \cup x$.

Example 2.10. The cohomology ring of $\mathbb{C}P^n$ and $\mathbb{C}P^\infty$ by Gysin sequence. Consider Hopf fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$ for every $n \geq 1$. The fiberation is orientable because $\pi_1(\mathbb{C}P^n, *) = 0$. The Euler class $e \in H^2(\mathbb{C}P^n)$ and for $0 < m < 2n$, Gysin sequence 2.3 becomes

$$0 \to H^{m-1}(\mathbb{C}P^n; R) \xrightarrow{\cup e} H^{m+1}(\mathbb{C}P^n; R) \to 0.$$

We know that $H^1(\mathbb{C}P^n; R) = 0$, and hence

$$H^{2n}(\mathbb{C}P^n; R) \cong H^{2n-2}(\mathbb{C}P^n; R) \cong \cdots \cong H^2(\mathbb{C}P^n; R) \cong H^0(\mathbb{C}P^n; R) \cong R;$$

$$H^{2n-1}(\mathbb{C}P^n; R) \cong H^{2n-3}(\mathbb{C}P^n; R) \cong \cdots \cong H^3(\mathbb{C}P^n; R) \cong H^1(\mathbb{C}P^n; R) \cong 0.$$

and $H^{2r}(\mathbb{C}P^n; R)$ is generated by e^r. Thus

$$H^*(\mathbb{C}P^n; R) \cong R[e]/(e^{n+1}),$$

a truncated polynomial algebra, where $e \in H^2(\mathbb{C}P^n; R)$.

Similarly $H^*(\mathbb{C}P^\infty; R) \cong R[e]$.

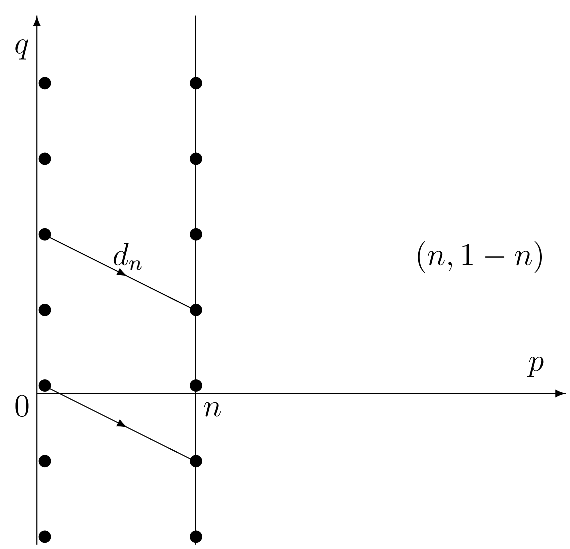
Example 2.11. If the base B is a sphere S^n for $n > 0$ in an orientable fibration $F \hookrightarrow E \xrightarrow{\pi} B$, Then

$$H^p(B; R) = \begin{cases} R, & \text{if } p = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, in Leray-Serre spectral sequence,

$$E_2^{p,q} = H^p(B; H^q(F; R)) \begin{cases} H^q(F; R), & \text{if } p = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

$E_2^{p,q}$



By the similar method of getting the Gysin sequence, we can get **Wang Sequence**.

$$\cdots \to H^m(E) \xrightarrow{i^*} H^m(F) \xrightarrow{d_n} H^{m-n+1}(F) \to H^{m+1}(E) \to \cdots .$$

where $i : F \to E$ is the inclusion.

Example 2.12. Using Gysin sequence and Wang sequence to compute the cohomology of Heisenberg manifold, and compare this two methods. (Exercise.)

Example 2.13. The Serre exact sequence. (Exercise.) [2] [6].

Theorem 2.14. *(Leray-Hirsch) Let $F \hookrightarrow E \xrightarrow{p} B$ be a fiber bundle such that, for some commutative coefficient ring R:*

1. *$H^n(F; R)$ is a finitely generated free R-module for each n.*
2. *There exist classes $c_j \in H^{k_j}(E; R)$ whose restrictions $i^*(c_j)$ form a basis for $H^*(F; R)$ as module in each fiber F, where $i : F \to E$ is the inclusion.*

Then, the map

$$\Phi : H^*(B; R) \otimes_R H^*(F; R) \to H^*(E; R),$$

given by

$$\Sigma_{ij} b_i \otimes i^*(c_j) \to \Sigma_{ij} p^*(b_i) \cup c_i$$

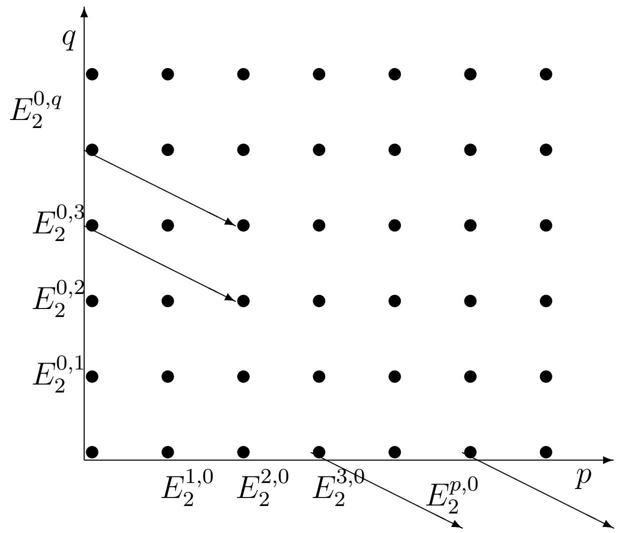
is an isomorphism.

Sketch of proof: (I didn't find a suitable proof for this theorem by Leray-Serre spectral sequence. In books[2] P148 and [6]P365, there are stronger and weaker proof. I change a little McCleary's proof in the following.)

We need to show that the Leray-Serre spectral sequence for this fiber bundle collapses at E_2-term.
Claim: (See [2] P147) The composite

$$H^q(E; R) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \subset E_q^{0,q} \subset \cdots \subset E_2^{0,q} = H^q(F; R)$$

is the homomorphism $i^* : H^q(E; R) \to H^q(F; R)$ induced by inclusion $i : F \hookrightarrow E$.



By assumption (2), $i^* : H^q(E; R) \to H^q(F; R)$ is surjective. So, all the inclusion must be equalities, and hence $d_r = 0$ when restricted to the q-axis. At E_2, by assumption (1), $H^n(F; R)$ is a finitely generated free R-module for each n, so,

$$E_2^{p,q} = H^p(B; H^q(F; R)) \cong H^p(B; R) \otimes_R H^q(F; R) = E_2^{p,0} \otimes E_2^{0,q}.$$

Since, d_2 is zero on $E_{p,0}$ already because of the degree reason and the spectral sequence is the first quadrant spectral sequence. So, $d_2 = 0$ and $E_2 = E_3$. The same argument can be applied on E_3 and so on. We have shown that the spectral sequence collapses at E_2.

□

In other words, $H^*(E; R)$ is a free $H^*(B; R)$-module with basis $\{c\}$, where we view $H^*(E; R)$ as a module over the ring $H^*(B; R)$ by defining scalar multiplication by $bc = p^*(b) \cup c$ for $b \in H^*(B; R)$ and $c \in H^*(E; R)$.

If $p : E \to B$ Fiber bundle and E' is a subspace of E, then (E, E') is a **Fiber bundle pair** if there is a subspace F' of the fiber F such that E' is an F'-bundle over B, and the local trivializations for E' are given by restrictions from E.

Remark 2.15. There are relative versions of Leray-Serre spectral sequence. In particular, suppose that $\dot{E} \subset E$ is a subset such that $(p|_{\dot{E}}) : \dot{E} \to B$ is also a fibration. Let $\dot{F} = F \cap \dot{E}$. Then there is a spectral sequence with

$$E_2^{p,q} \cong H^p(B; H^q(F, \dot{F}; R))$$

and converging to $H^*(E, \dot{E})$.

Theorem 2.16. *(Leray-Hirsch theorem for fiber bundle pair) Suppose that* $(F, F') \to (E, E') \xrightarrow{p} B$ *is a fiber bundle pair such that* $H^*(F, F'; R)$ *is a free* R*-module, finitely generated in each dimension. If there exist classes* $c_j \in H^*(E, E'; R)$ *whose restrictions form a basis for* $H^*(F, F'; R)$ *in each fiber* (F, F')*, then* $H^*(E, E'; R)$*, as a module over* $H^*(B; R)$*, is free with basis* $\{c_j\}$. □

For disk bundle $(D^n, S^{n-1}) \hookrightarrow (E, E') \xrightarrow{p} B$, an element $t \in H^n(E, E'; R)$ whose restriction to each fiber (D^n, S^{n-1}) is a generator of $H^n(D^n, S^{n-1}; R) \cong R$ is called a **Thom class** for the bundle.

Theorem 2.17. *(Thom Isomorphism Theorem) If the disk bundle* $(D^n, S^{n-1}) \hookrightarrow (E, E') \xrightarrow{p} B$ *has a Thom class* $t \in H^n(E, E'; R)$*, then the map* $\Phi : H^i(B; R) \to H^{i+n}(E, E'; R)$*,* $\Phi(b) = p^*(b) \cup t$*, is an isomorphism for all* $i \geq 0$*, and* $H^i(E, E'; R) = 0$ *for* $i < n$*.*

Proof: This is special case of Leray-Hirsch theorem for bundle pairs for $j = 1$. □

Remark 2.18. We can also derive the Gysin sequence for a sphere bundle $S^{n-1} \hookrightarrow E \xrightarrow{p} B$ by considering the mapping cylinder M_p, which is a disk bundle $D^n \hookrightarrow M_p \xrightarrow{p} B$ with E as its boundary sphere bundle.

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