A differential equation of the form

$$\frac{dy}{dx} = f(x)g(y)$$

is called a **separable equation**.

- If $g(y_0) = 0$, then $y = y_0$ is an equilibrium solution.
- If $g(y) \neq 0$, we can write the equation as

$$\frac{dy}{g(y)} = f(x)dx.$$

Integrate both sides, we have

$$\int \frac{dy}{g(y)} = \int f(x)dx.$$

We can solve y from the last equation.

Example 1. Solve $\dfrac{dy}{dx} = ky$.

① $g(y) = y$ So $y = 0$ is an equilibrium solution.

② When $y \neq 0$,

$$\frac{dy}{y} = k\,dx$$

$$\int \frac{1}{y}\,dy = \int k\,dx$$

$$\ln|y| = kx + C$$

$$|y| = e^{kx+C} = e^C \cdot e^{kx}$$

$$\boxed{y = a \cdot e^{kx}}$$ denote $a = \pm e^C$

Example 2. Solve $\dfrac{dy}{dx} = xy$ with initial condition $y(1) = 1$ and compare the approximating result $y(1.5)$ with Example 3 in §9.2.

Since $y(1)=1$, so, $y \neq 0$.

$$\frac{dy}{y} = x\,dx$$

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$|y| = e^{\frac{1}{2}x^2+C} \Rightarrow y = a\cdot e^{\frac{1}{2}x^2} \text{ if denote } a = \pm e^C$$

By the initial condition $y(1)=1$

$$1 = a\cdot e^{\frac{1}{2}} \Rightarrow a = e^{-\frac{1}{2}}$$

So the solution is

$$y = e^{-\frac{1}{2}}\cdot e^{\frac{1}{2}x^2} = e^{\frac{1}{2}(x^2-1)}$$

Example 3. Solve $\dfrac{dy}{dx} = \dfrac{2x}{6y^2 - \sin y}$.

$$(6y^2 - \sin y)\,dy = 2x\,dx$$

$$\int (6y^2 - \sin y)\,dy = \int 2x\,dx$$

$$2y^3 + \cos y = x^2 + C$$

Example 4. Solve $\frac{dy}{dx} = 4x^3y$ with the initial condition $y(0) = 3$.

$$\frac{dy}{y} = 4x^3\,dx$$

$$\int \frac{1}{y}\,dy = \int 4x^3\,dx$$

$$\ln|y| = x^4 + C$$

$$|y| = e^{x^4+C} = e^C \cdot e^{x^4}$$

$$y = a \cdot e^{x^4}$$ if denote $a = \pm e^C$

By the initial condition

$$y(0) = 3$$

$$3 = a \cdot e^0 = a$$

So the solution is

$$y = 3e^{x^4}$$

Example 5. Solve $(\sec^2 y)x^{-1}y' = e^{2x^2}$.

$$(\sec^2 y)\,dy = x e^{2x^2}\,dx$$

$$\int \sec^2 y\,dy = \int x e^{2x^2}\,dx$$

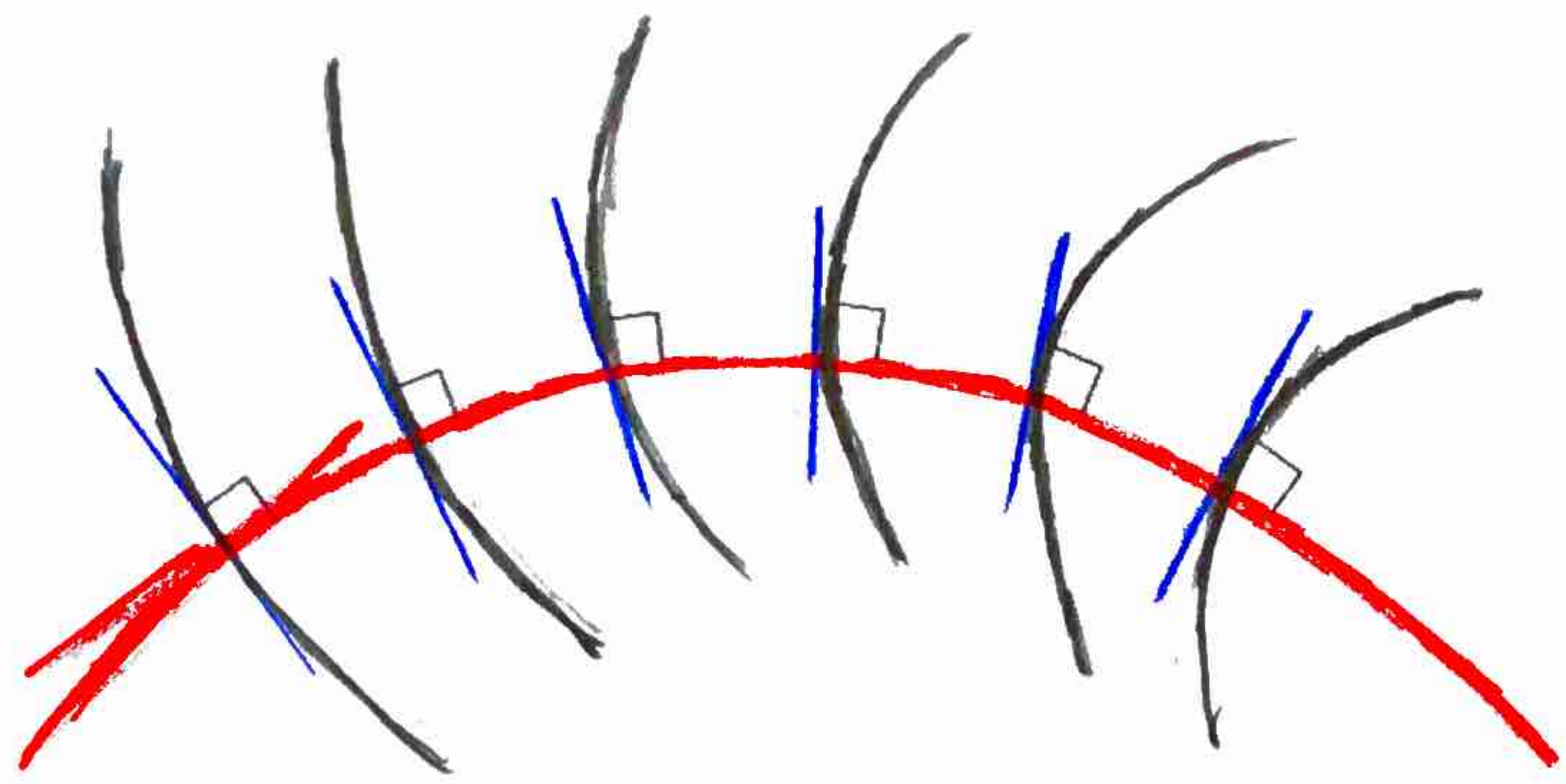
$$\tan y = \frac{1}{4} e^{2x^2} + C$$

left side integral

by u-substitution

An $\boxed{\textbf{orthogonal trajectory}}^{f}$ of a $\boxed{\text{family of curves}}^{g}$ is a curve that intersects each curve of the family orthogonally (or perpendicularly, or at right angles).

$$f' = \frac{1}{g'}$$



Example 6. Find the orthogonal trajectories of the family of curves $y = kx$ for $k \in \mathbb{R}$.

$\frac{dy}{dx} = k$ for the family of curves.

So, the derivatives of the orthogonal trajectories are $-\frac{1}{k}$

So, ~~the derivatives of~~ the orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{1}{k} = -\frac{x}{y}$$

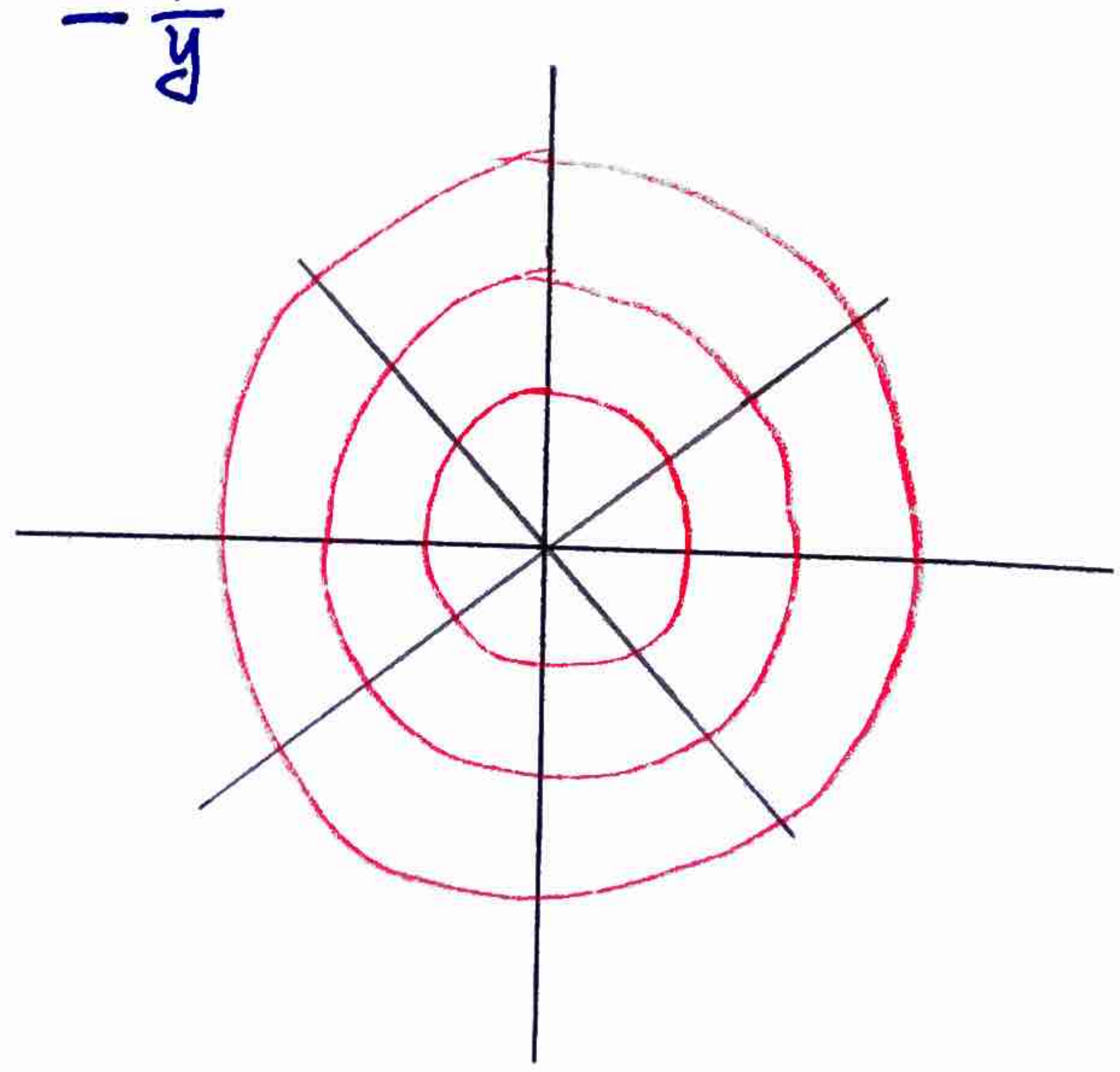
$y\,dy = -x\,dx$

$\int y\,dy = \int -x\,dx$

$\frac{y^2}{2} = \frac{-x^2}{2} + C$

$x^2 + y^2 = 2C$

circles with radius $r = \sqrt{2C}$



Example 7. Find the orthogonal trajectories of the family of curves $y = kx^2$ for $k \in \mathbb{R}$.

$\frac{dy}{dx} = 2kx$ are the slope of the family of curves.

So, the derivative of the orthogonal trajectories are $-\frac{1}{2kx}$

the orthogonal trajectories satisfy:

So $\frac{dy}{dx} = -\frac{1}{2kx} = -\frac{1}{2\left(\frac{y}{x^2}\right)x} = -\frac{x}{2y}$

$2y\,dy = -x\,dx$

$\int 2y\,dy = -\int x\,dx$

$y^2 = \frac{-x^2}{2} + C$

$\frac{x^2}{2} + y^2 = C$

ellipses