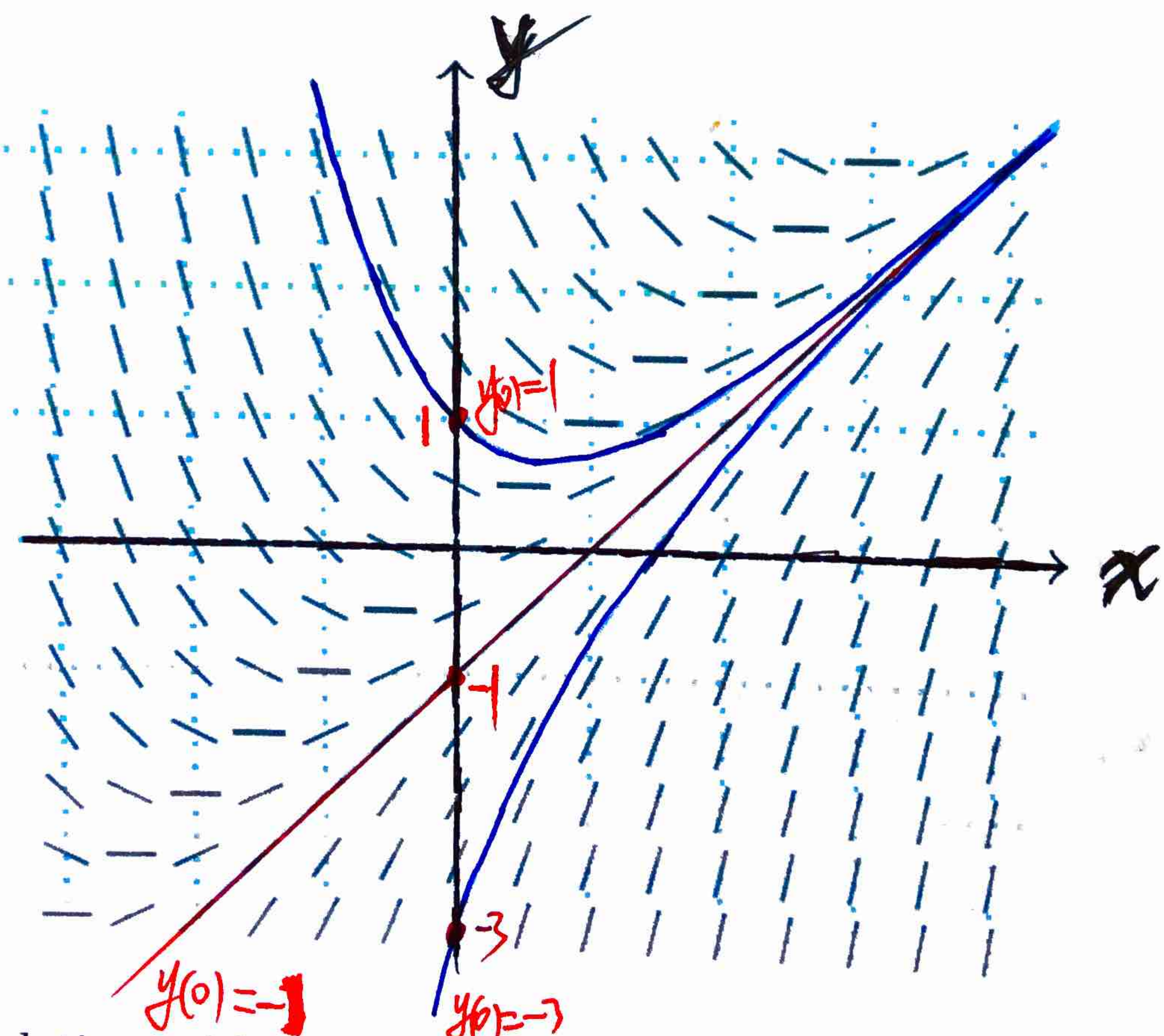
It is very hard or even impossible to find the *precise solutions* for most differential equations from our real world problem.
So, it is useful to get an estimation for the solutions using graphical approach or a numerical approach.

1. Graphical approach (Slope fields, or direction fields)

If $y = f(x)$ is a solution for the differential equation $\frac{dy}{dx} = F(x, y)$, then the *slope of curve* $y = f(x)$ at (x, y) is $F(x, y)$.
At each point (x, y) draw a line segment with slope $F(x, y)$. The solution is the curve tangent at this point.

Example 1. $\frac{dy}{dx} = x - y$ (Use slope fields).



Three solutions with initial values $y(0) = -3$, $y(0) = -1$, $y(0) = 1$ are drawn in the graph.

2. Numerical approach (Euler's Method)

Find approximating solution for the differential equation $\dfrac{dy}{dx} = F(x, y)$ with initial value $y(x_0) = y_0$ using *Euler's Method with step size h:*

(1). Set step size h; (the smaller h, the better estimation.)

(2). Start with point (x_0, y_0);

(3). Define a sequence $\boxed{x_n = x_{n-1} + h;}$

(4). Then y_n is computed by the sequence

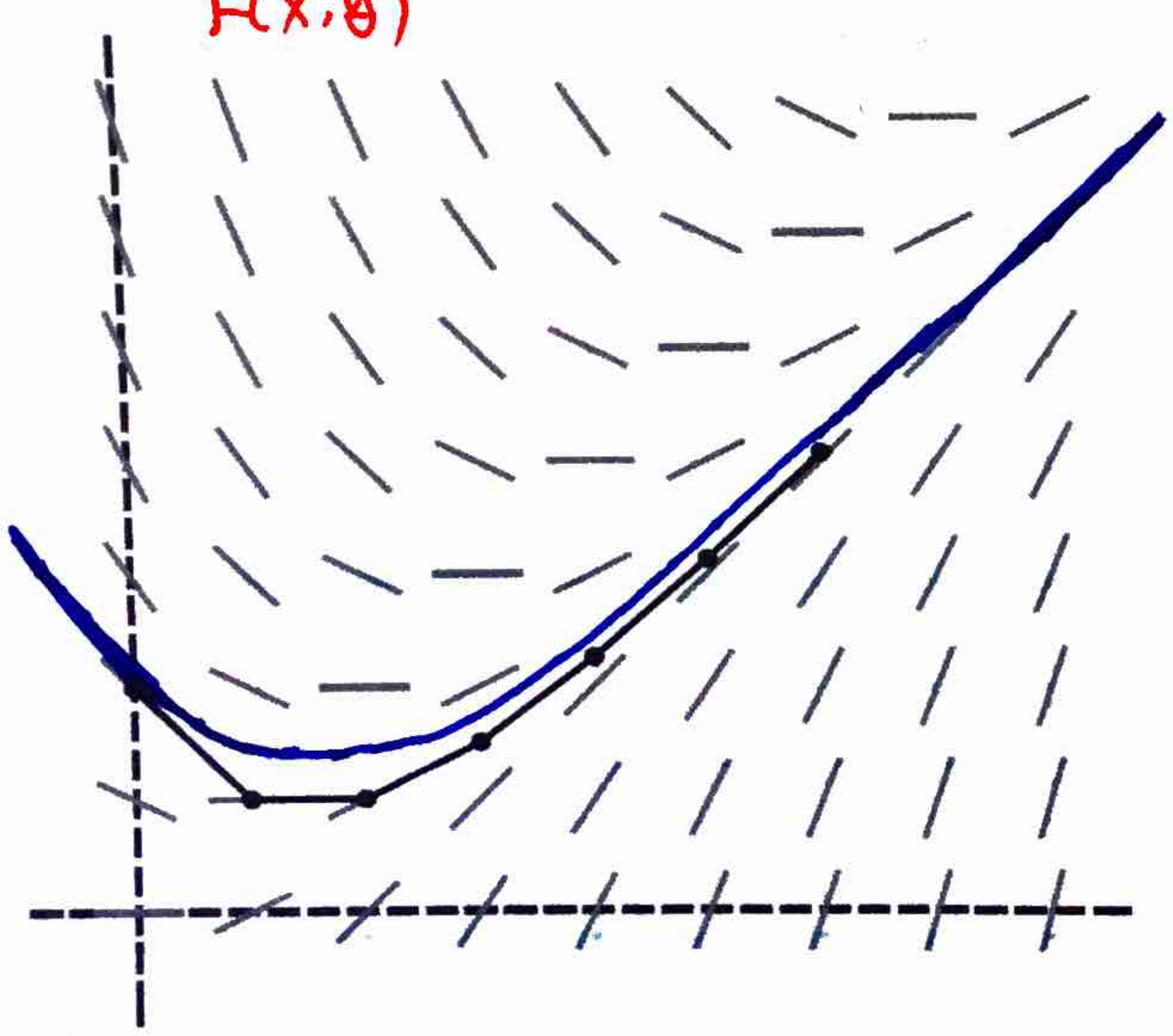
slope

$$\boxed{y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})}$$

$$\frac{y_n - y_{n-1}}{x_n - x_{n-1}} = F(x_{n-1}, y_{n-1}$$

Example 2. Use Euler's method with step size $h = 0.5$ to solve $\dfrac{dy}{dx} = \boxed{x - y}$ with initial value $y(0) = 1$.

$F(x, y)$



$x_0 = 0$

$x_1 = x_0 + h = 0.5$

$x_2 = x_1 + h = 1$

$x_3 = x_2 + h = 1.5$

$x_4 = x_3 + h = 2$

$x_5 = x_4 + h = 2.5$

$y_0 = 1$

$y_1 = y_0 + h(x_0 - y_0) = 1 + 0.5(0 - 1) = 0.5$

$y_2 = y_1 + h(x_1 - y_1) = 0.5 + 0.5(0.5 - 0.5) = 0.5$

$y_3 = y_2 + h(x_2 - y_2) = 0.5 + 0.5(1 - 0.5) = 0.75$

$y_4 = y_3 + h(x_3 - y_3) = 0.75 + 0.5(1.5 - 0.75) = 1.125$

$y_5 = y_4 + h(x_4 - y_4) = 1.125 + 0.5(2 - 1.125) = 1.5625$

So $y(2.5) \approx y_5 = 1.5625$

Example 3. Use Euler's method with step size $\underline{h = 0.1}$ to solve $\frac{dy}{dx} = xy$ with initial value $y(1) = \underset{y_0}{1}$. Find $y(1.4)$ and $y(1.5)$

$x_0 = 1$ $\qquad y_0 = 1$

$x_1 = x_0 + h = 1.1$ $\qquad y_1 = y_0 + h(x_0 y_0) = 1 + 0.1(1) = 1.1$

$x_2 = x_1 + h = 1.2$ $\qquad y_2 = y_1 + h(x_1 y_1) = 1.1 + 0.1(1.21) = 1.221$

$x_3 = x_2 + h = 1.3$ $\qquad y_3 = y_2 + h(x_2 y_2) = 1.221 + 0.1(1.2)(1.221) = 1.36752$

$x_4 = x_3 + h = 1.4$ $\qquad y_4 = y_3 + h(x_3 y_3) = 1.36752 + 0.1(1.3)(1.36752) = 1.5452976$

$x_5 = x_4 + h = 1.5$ $\qquad y_5 = y_4 + h(x_4 y_4) = 1.5452976 + 0.1(1.4)(1.5452976)$

$= 1.761639264$

So, $y(1.4) \approx y_4 = 1.5452976$

$y(1.5) \approx y_5 = 1.761639264$

- In fact, from Example 2 in §9.3, we know the solutions are

$$y = e^{-\frac{1}{2}} e^{\frac{x^2}{2}} = e^{\frac{1}{2}(x^2-1)}$$

So $y(1.5) = e^{0.625} \approx 1.868$