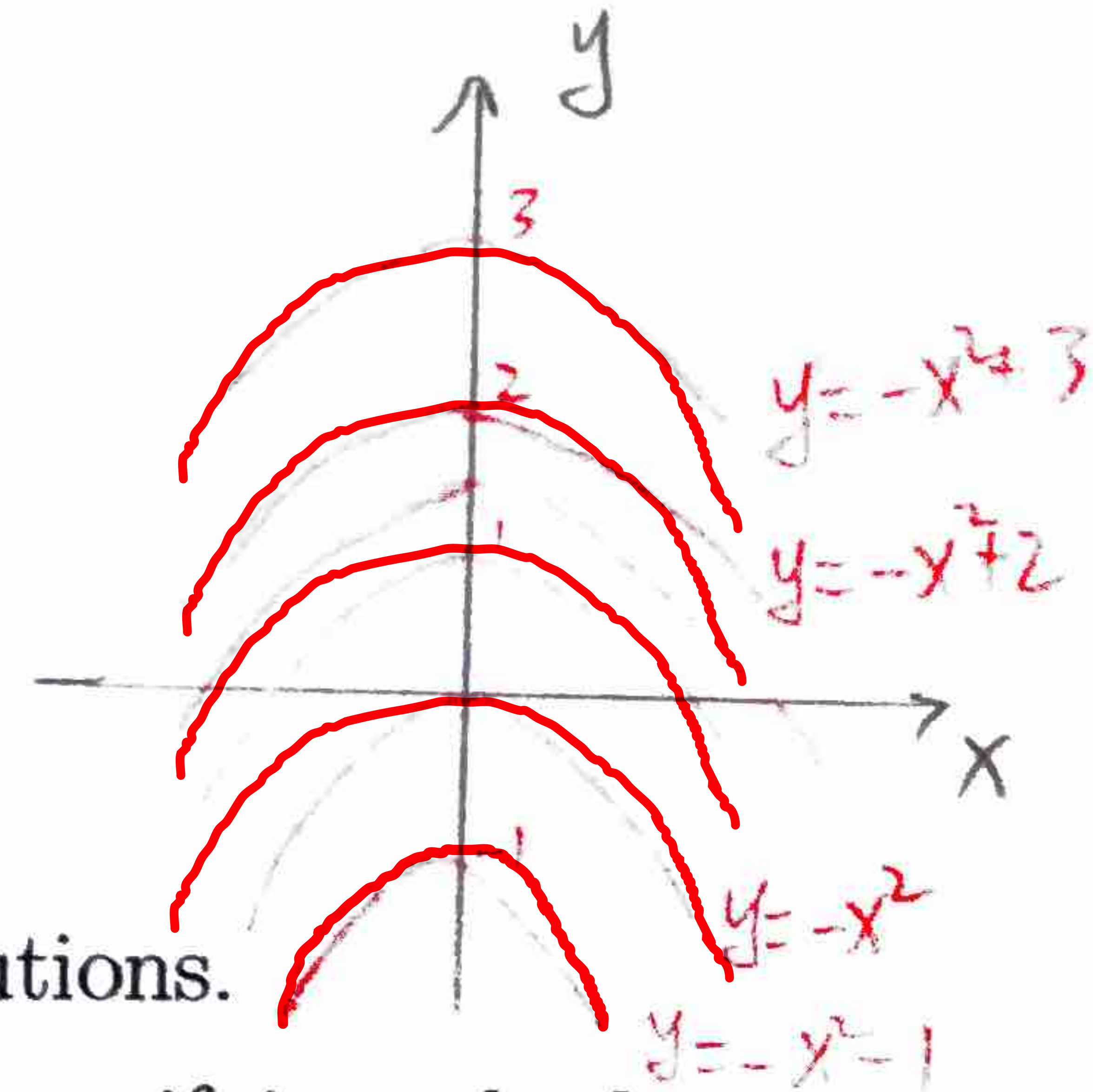
We have an introduction about differential equations in this chapter. There will be a class **Math 285**–Differential Equations after the class **Math 283**–Calculus 3.

To solve some real world problems, we need to set up an equation. Some times, we need differential equations.

A **differential equation** is an equation containing an *unknown function* $f(x)$ (or denoted by y) and some of its *derivatives* $y', y'', \ldots$.

Example 1. $y' = -2x$.

$$y = \int -2x\,dx = -x^2 + C$$



- A differential equation often has infinitely many solutions.
- Solving an equation is hard in general, but verifying whether or not a function $f(x)$ is a solution is easy.

Example 2. Verify that $y = ce^{2x}$ (c is any real number) are solutions for the differential equation $y' = 2y$.

Plugin $y=ce^{2x}$:

Left side of the equation $y' = 2ce^{2x}$

Right side of the equation $2y = 2ce^{2x}$

Left side = Right side

So, $y=ce^{2x}$ are solutions for $y'=2y$

Example 3. One model for the growth of a population

$$\frac{dP}{dt} = kP \qquad P' = kP$$

where k is a constant number, t is the time (independent variable) and P is the number of individuals in the population (dependent variable).

$y = Ce^{kx}$ are solutions for $\frac{dP}{dt} = kP$ for any $C \in \mathbb{R}$

Example 4. Verify that $y = a\sin 3x + b\cos 3x$ (where, a, b are any real numbers) are solutions for the differential equation $y'' = -9y$.

$y = a\sin 3x + b\cos 3x$

$y' = 3a\cos 3x - 3b\sin 3x$

$y'' = -9a\sin 3x - 9b\cos 3x$

Left hand side of the differential equation

$= y'' = -9a\sin 3x - 9b\cos 3x$

$= -9(a\sin 3x + b\cos 3x)$

$= -9y$

= right hand side of the equation.

Example 5. (1) For what values of r does the function $y = e^{rx}$ a solution for $y'' - y' - 2y = 0$?

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

plug into the equation

$r^2 e^{rx} - re^{rx} - 2e^{rx} = 0$

$e^{rx}(r^2 - r - 2) = 0$

Since $e^{rx} \neq 0$, we have $r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r = 2 \quad r = -1$

So, $y = e^{2x}$ and $y = e^{-x}$ are solutions for $y'' - y' - 2y = 0$

(2) If r_1 and r_2 are the values of r that you found in part (1), show that every member of the family of functions $y = ae^{r_1 x} + be^{r_2 x}$ is a solution for $y'' - y' - 2y = 0$.

$y = ae^{2x} + be^{-x}$

$y' = 2ae^{2x} - be^{-x}$

$y'' = 4ae^{2x} + be^{-x}$

Left hand side of the equation $= y'' - y' - 2y$

$= 4ae^{2x} + be^{-x} - (2ae^{2x} - be^{-x}) - 2(ae^{2x} + be^{-x})$

$= 4ae^{2x} + be^{-x} - 2ae^{2x} + be^{-x} - 2ae^{2x} - 2be^{-x}$

$= 0$

$=$ Right hand side of the equation.

When we apply differential equations to real world problems, we are interested in finding a particular solution satisfying a condition of the form $y(t_0) = y_0$, which is called an **initial condition**.
This kind differential equation with initial condition is called **initial-value problem**.

Example 6. (1) For what values of r does the function $y = e^{x^r}$ a solution for $y' = 3x^2y$?

Left hand side $= y' = r \cdot x^{r-1} e^{x^r}$

right hand side $= 3x^2 y = 3x^2 e^{x^r}$

If we want left = Right for any x, r must be 3.

(2) Show that every member of the family of functions $y = ke^{x^3}$ is a solution for $y' = 3x^2y$. Here, k is any real number.

left hand side $= y' = k \cdot 3x^2 e^{x^3}$

right hand side $= 3x^2 y = 3x^2 k e^{x^3}$

Left side = right side

So $y = ke^{x^3}$ are solutions for $y' = 3x^2y$.

(3) Find a solution of differential equation $y' = 3x^2y$ with initial condition $y(0) = 2$.

plugin $y(0)=2$ to $y = ke^{x^3}$ we have $2 = ke^0$

So $k = 2$

So $y = 2e^{x^3}$ is the solution of $y' = 3x^2y$ with initial condition $y(0)=2$.

Many populations start by increasing in an exponential model, however, the populations decrease when they approach its **carrying capacity** M.

Example 7. [Pierre-Francois Verhulst, 1840s] The world population growth is modeled by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where M is the carrying capacity.

Let us see what can we obtain from this model.

1. The constant functions $P(t) = 0$ and $P(t) = M$ are solutions for the differential equation, which are called *equilibrium solutions.*
2. If the initial population $P(0) < M$, then the right side of the equation is positive, hence $\frac{dP}{dt} > 0$ and the population increases.
3. If the initial population $P(0) > M$, then the right side of the equation is negative, hence $\frac{dP}{dt} < 0$ and the population decreases.