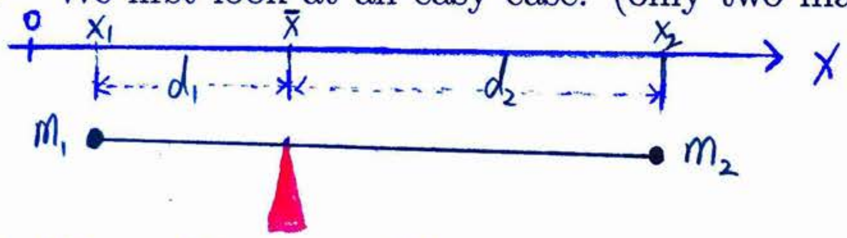
The ideal of all applications of definite integral is to divide the problem into n small parts; for each small part, we know how to find a formula for the problem; then we take the sum and limit to get a definite integral.

• Moments and Centers of Mass

Our Goal is to find the point (fulcrum) on which a *thin* plate of any given shape balances horizontally.

Case 1. We first look at an easy case: (only two mass points m_1 and m_2)



Archimedes' Law of the Lever tells us:

$$m_1 d_1 = m_2 d_2$$

Look at it in x-axis with m_1 at x_1 and m_2 at x_2. The center of the mass is at $\bar{x}$.

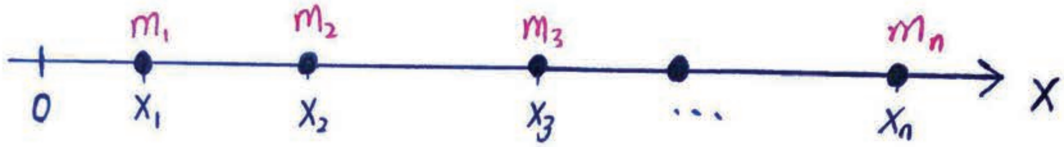
$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} + m_2\bar{x} = m_1 x_1 + m_2 x_2$$

The center of the mass is

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Case 2. Suppose we have n mass points $m_1, m_2, \ldots, m_n$ at points $x_1, x_2, \ldots, x_n$.

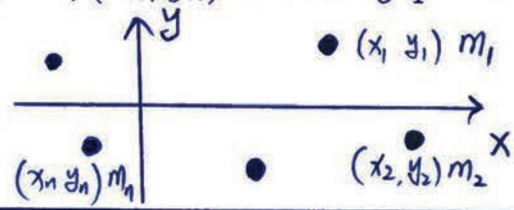


The center of the mass is

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^n m_ix_i}{\sum_{i=1}^n m_i}$$

Here, $\sum_{i=1}^n m_ix_i$ is the **moment of the system about the origin**. The total mass of the system is $m = \sum_{i=1}^n m_i$.

Case 3. More generally, suppose we have n mass points $m_1, m_2, \ldots, m_n$ at points $(x_1, y_2), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane.



The **moment of the system about the y-axis** is

$$M_y = \sum_{i=1}^n m_ix_i.$$

The **moment of the system about the x-axis** is

$$M_x = \sum_{i=1}^n m_iy_i.$$

The coordinates $(\bar{x}, \bar{y})$ of the center of the mass are given by

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

Example 1. Find the moments M_x and M_y, and center of mass $(\bar{x}, \bar{y})$ of the system. The objects have masses 2, 6, and 7 at the points $(-1, 2)$ $(1, 4)$ and $(2, 3)$.

The moment about y-axis:

$$M_y = \sum_{i=1}^{n} m_i x_i = 2(-1) + 6(1) + 7(2) = 18$$

The moment about x-axis:

$$M_x = \sum_{i=1}^{n} m_i y_i = 2(2) + 6(4) + 7(3) = 49$$

The Total mass

$$m = m_1 + m_2 + m_3 = 2 + 6 + 7 = 15$$

The coordinates of the center

$$\bar{x} = \frac{M_y}{m} = \frac{18}{15} = \frac{6}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{49}{15}$$

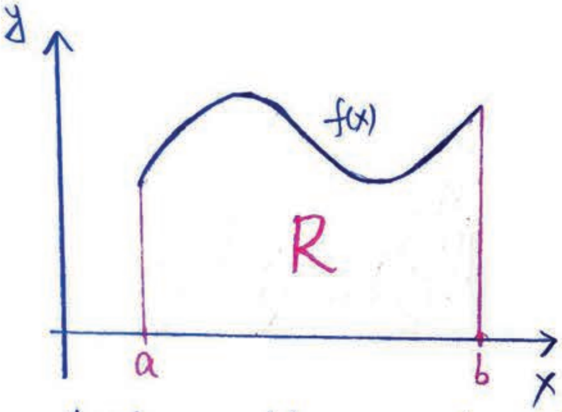
The center of mass is $\left(\frac{6}{5}, \frac{49}{15}\right)$

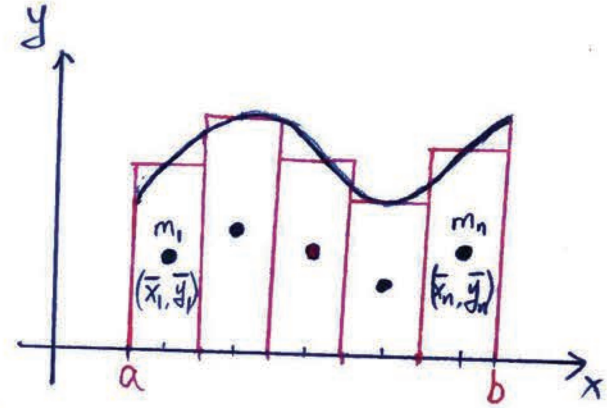
Up to now, we only considered **finite** number of mass points. We have not used the technique of Calculus yet.

Now, let us solve our original problem: Consider a flat thin plate (called a **lamina**) with a fixed density ρ that on a region R of the plane.

problem:

- Calculate the center of mass of the plate (called the centroid of R).
- Suppose the region R is below a function $y = f(x)$ on the interval $[a, b]$.





Again, consider our universal way of applying definite integral:
We divide the interval $[a, b]$ into n subintervals, hence, divide the region R into n rectangles.
For each rectangle R_i, the centroid of R_i is $(\bar{x}_i, \bar{y}_i)$ where

$$\bar{y}_i = \frac{1}{2} f(\bar{x}_i).$$

The **area of the rectangle** R_i is $A_i = f(\bar{x}_i)\Delta x$ and so the **mass of** R_i is

$$m_i = \rho A_i = \rho f(\bar{x}_i)\Delta x.$$

Now, we think our problem as in Case 3: n mass points $m_1, \ldots, m_n$ at $(\bar{x}_1, \bar{y}_1), \ldots, (\bar{x}_n, \bar{y}_n)$.

The **moment of R about the y-axis** is approximated by

$$M_y \approx \sum_{i=1}^{n} m_i x_i = \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i)\Delta x.$$

The moment of R about the y-axis: is

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i)\Delta x = \rho \int_a^b x f(x)dx$$

The **moment of R about the x-axis** is approximated by

$$M_x \approx \sum_{i=1}^{n} m_i y_i = \sum_{i=1}^{n} \rho \frac{1}{2}[f(\bar{x}_i)]^2 \Delta x.$$

The moment of R about the x-axis: is

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \frac{1}{2}[f(\bar{x}_i)]^2 \Delta x = \frac{1}{2}\rho \int_a^b [f(x)]^2 dx$$

The **mass** of the plate R is the product of its density ρ and its area $A = \int_a^b f(x)dx$:

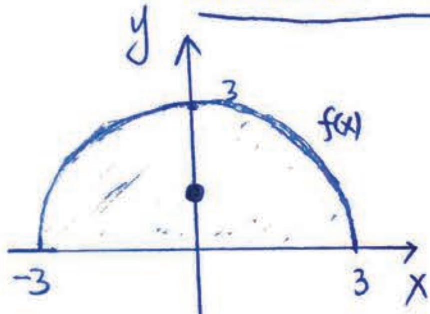
$$m = \rho A = \rho \int_a^b f(x)dx$$

The center of mass (**centroid** $(\bar{x}, \bar{y})$) of the plate R is

$$\bar{x} = \frac{M_y}{m} = \frac{1}{A}\int_a^b x f(x)dx \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{2A}\int_a^b [f(x)]^2 dx$$

Remark: The density ρ is cancelled in the formula of the centroid.

Example 2. Find the center of mass (centroid) of a semicircular plate of radius $r = 3$.



$x^2+y^2=r^2$

$f(x)=\sqrt{r^2-x^2}=\sqrt{9-x^2}$

$a=-3 \quad b=3$

① Area of R is $A=\frac{1}{2}\pi r^2=\frac{9}{2}\pi$

② $\bar{x}=\frac{1}{A}\int_a^b x f(x)\,dx=\frac{1}{A}\int_{-3}^{3} x\sqrt{9-x^2}\,dx=\frac{1}{A}\int_{-3}^{3}(9-x^2)^{\frac{1}{2}}\frac{dx^2}{2}$ u-substitution

$=\frac{1}{2A}\cdot\frac{2}{3}(9-x^2)^{\frac{3}{2}}\Big|_{-3}^{3}=0$

This make sense by symmetry.

③ $\bar{y}=\frac{1}{2A}\int_a^b f(x)^2\,dx=\frac{1}{2A}\int_{-3}^{3}9-x^2\,dx=\frac{1}{9\pi}\left(9x-\frac{x^3}{3}\right)\Big|_{-3}^{3}$

$=\frac{1}{9\pi}\left((27-9)-(-27+9)\right)=\frac{4}{\pi}$

④ The centroid is $(0,\frac{4}{\pi})$

Example 3. Calculate the moments M_x and M_y of the region Example 2 when the density $\rho = 2$.

① The moment of R about the y-axis is

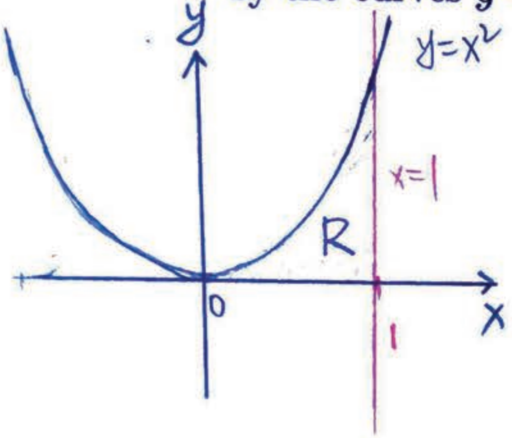
$$M_y=\rho\int_a^b x f(x)\,dx=2\int_{-3}^{3}x\sqrt{9-x^2}\,dx=0$$

② The moment of R about the x-axis is

$$M_x=\frac{1}{2}\rho\int_a^b f(x)^2\,dx=\frac{1}{2}(2)\int_{-3}^{3}(9-x^2)\,dx=\left(9x-\frac{x^2}{3}\right)\Big|_{-3}^{3}$$

$$=36$$

Example 4. Calculate the moments M_x and M_y and the center of mass (centroid) of a lamina with density $\rho = 3$ and shape R bounded by the curves $y = x^2$ and lines $x = 1$, $y = 0$.



$a=0 \quad b=1$

① The moment of R about the y-axis is

$$M_y = \rho \int_a^b x f(x)\,dx = 3\int_0^1 x^3\,dx = 3 \cdot \frac{x^4}{4}\Big|_0^1 = \frac{3}{4}$$

② The moment of R about the x-axis is

$$M_x = \frac{1}{2}\rho \int_a^b f(x)^2\,dx = \frac{3}{2}\int_0^1 x^4\,dx = \frac{3}{2}\cdot\frac{x^5}{5}\Big|_0^1 = \frac{3}{10}$$

③ The mass of R is

$$m = \rho A = \rho\int_0^1 x^2\,dx = 3\cdot\frac{x^3}{3}\Big|_0^1 = 1$$

④ $$\bar{x} = \frac{M_y}{m} = \frac{\frac{3}{4}}{1} = \frac{3}{4}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{3}{10}}{1} = \frac{3}{10}$$

So, The center of mass (centroid) is $\left(\frac{3}{4}, \frac{3}{10}\right)$

If the region R lies between two curves $y = f(x)$ and $y = g(x)$ for $f(x) \geq g(x)$, then the centroid $(\bar{x}, \bar{y})$ of R is given by

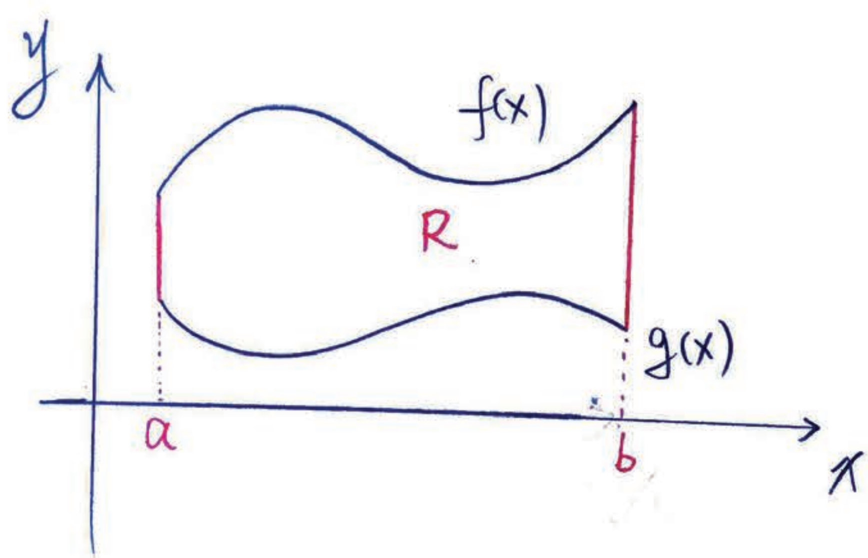
The center of mass (**centroid** $(\bar{x}, \bar{y})$) of the plate R is

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]dx$$

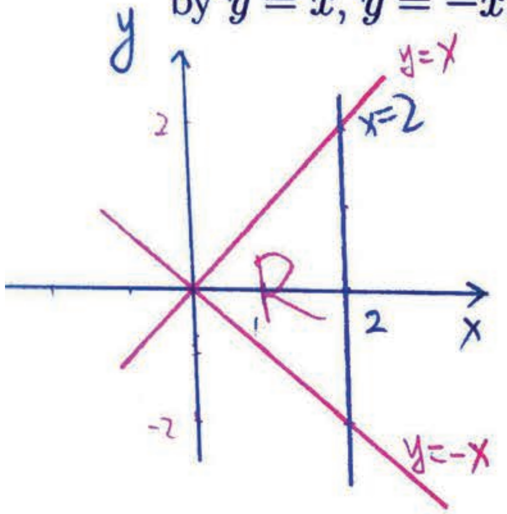
$$\bar{y} = \frac{1}{2A}\int_a^b [f(x)]^2 - [g(x)]^2 dx$$

Here A is the area of R calculated by

$$A = \int_a^b [f(x) - g(x)]dx.$$



Example 5. Calculate the moments M_x and M_y and the center of mass (centroid) of a lamina with density $\rho = 3$ and shape R bounded by $y = x$, $y = -x$, $x = 2$.



$a=0 \quad b=2$

① The moment of R about the y-axis is

$$M_y = \rho\int_a^b x(f(x)-g(x))\,dx = 3\int_0^2 x(x-(-x))\,dx$$

$$= 3\int_0^2 2x^2\,dx = 3\left(2\frac{x^3}{3}\right)\Big|_0^2 = 16$$

② The moment of R about the x-axis is

$$M_x = \frac{1}{2}\rho\int_a^b f(x)^2 - g(x)^2\,dx = \frac{1}{2}\rho\int_a^b x^2 - x^2\,dx = 0$$

③ The Mass of R is

$$m = \rho A = \rho\int_a^b f(x)-g(x)\,dx = 3\int_0^2 x-(-x)\,dx = 3\int_0^2 2x\,dx$$

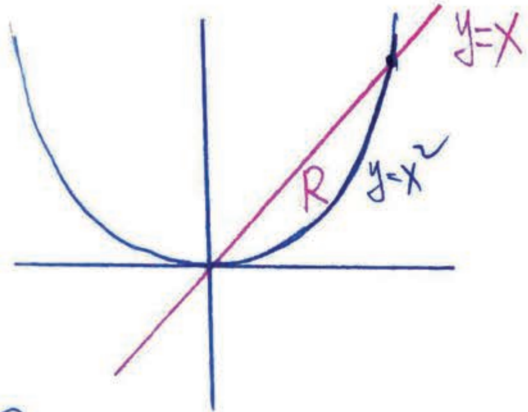
$$= 3\,(2)\frac{x^2}{2}\Big|_0^2 = 12$$

④ $\bar{x} = \dfrac{M_y}{m} = \dfrac{16}{12} = \dfrac{4}{3}$

$\bar{y} = \dfrac{M_x}{m} = \dfrac{0}{12} = 0$

The centroid is $\left(\frac{4}{3}, 0\right)$

Example 6. Calculate ~~the moments M_x and M_y and~~ the center of mass (centroid) of a lamina with density $\rho = 5$ and shape R bounded by $y = x$ and the parabola $y = x^2$.



① The area of the region R is

$$A = \int_0^1 x - x^2 \, dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{6}$$

⓪ The intersection

$x = x^2 \quad x(x-1) = 0$

$x=0$ and $x=1$

② $$\bar{x} = \frac{1}{A}\int_0^1 x(f(x) - g(x))\, dx$$

$$= \frac{1}{1/6}\int_0^1 x(x-x^2)\, dx$$

$$= 6\int_0^1 (x^2-x^3)\, dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{2}$$

③ $$\bar{y} = \frac{1}{2A}\int_0^1 f(x)^2 - g(x)^2 \, dx$$

$$= \frac{1}{2(1/6)}\int_0^1 x^2 - x^4 \, dx$$

$$= 3\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1$$

$$= \frac{2}{5}$$

So, The centroid of the lamina is $\left(\frac{1}{2}, \frac{2}{5}\right)$