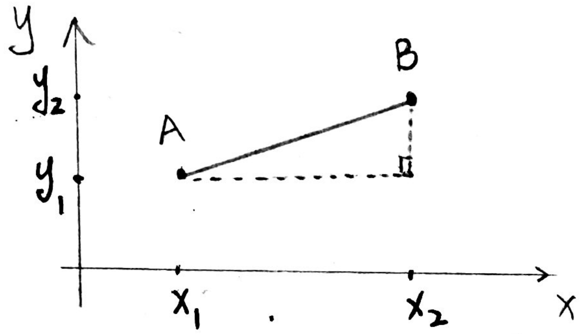
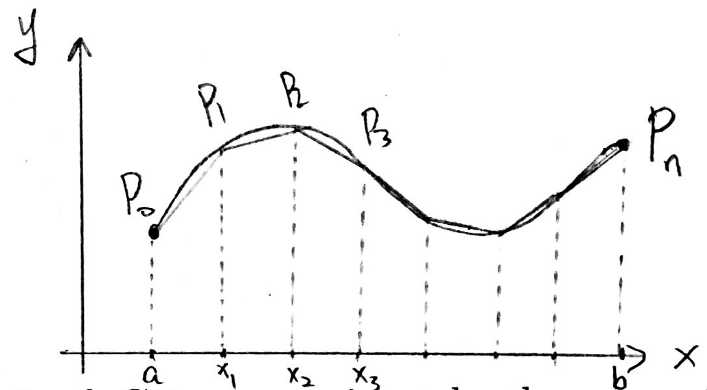
We know how to calculate the length of a line segment between A (x_1, y_1) and B (x_2, y_2) by Pythagorean Theorem:

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Question: How to calculate the length of a curve C defined by equation $y = f(x)$, with $a \le x \le b$?



The length L of C is approximately the sum of the lengths of line segments

$$L \approx \sum_{i=1}^{n} |P_{i-1}P_i|.$$

Here,

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i \\ &\approx \sqrt{1 + [f'(x_i)]^2}\, \Delta x_i \end{aligned}$$

The more line segments, the better approximating. To make it precise, we take the limit:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i)]^2}\Delta x_i$$
$$= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

The Arc Length Formula

If $f'(x)$ is continuous on $[a, b]$, then the length of the curve $y = f(x)$ on $[a, b]$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example 1. Use the arc length formula to find the length of the curve $y = 3x - 5$ when $1 \leq x \leq 3$. Check your answer by Pythagorean Theorem.

1. ① $y' = 3$

② $L = \int_1^3 \sqrt{1+3^2}\, dx = \int_1^3 \sqrt{10}\, dx = \sqrt{10}(3-1) = 2\sqrt{10}$

Arc length formula.

2. Pythagorean Theorem:

$x_1 = 1 \quad y_1 = -2$

$x_2 = 3 \quad y_2 = 4$

$L = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(3-1)^2 + (4+2)^2} = \sqrt{40} = 2\sqrt{10}$

Example 2. Use the arc length formula to find the length of the curve $y = 2x^{3/2} - 3$, for $1 \leq x \leq 4$.

① $y' = 2\left(\frac{3}{2}\right)x^{\frac{1}{2}} = 3x^{\frac{1}{2}}$

② By arc length formula

$$L = \int_1^4 \sqrt{1+(3x^{\frac{1}{2}})^2}\,dx = \int_1^4 \sqrt{1+9x}\,dx$$

$$\boxed{\begin{array}{ll} \text{Let } u = 1+9x & u(1)=10 \\ du = 9\,dx & u(4)=37 \\ dx = \frac{1}{9}du & \end{array}}$$

$$= \int_{10}^{37} u^{\frac{1}{2}} \cdot \frac{1}{9}\,du$$

$$= \frac{1}{9} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \Big|_{10}^{37} = \frac{2}{27}\left(37^{\frac{3}{2}} - 10^{\frac{3}{2}}\right)$$

Example 3. Use the arc length formula to find the length of the curve $y = 2(x-3)^{3/2}$, for $3 \leq x \leq 4$.

① $y' = 3(x-3)^{\frac{1}{2}}$

② By arc length formula

$$L = \int_3^4 \sqrt{1+(y')^2}\,dx = \int_3^4 \sqrt{1+9(x-3)}\,dx$$

$$= \int_3^4 \sqrt{9x-26}\,dx$$

Let $u = 9x-26$

$du = 9\,dx$

$dx = \frac{1}{9}du$

$u(3) = 27-26 = 1$

$u(4) = 36-26 = 10$

$$= \int_1^{10} u^{\frac{1}{2}} \cdot \frac{1}{9}\,du$$

$$= \frac{1}{9} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \Big|_1^{10}$$

$$= \frac{2}{27}\left(10^{\frac{3}{2}} - 1\right)$$

Similarly,

$\boxed{\begin{array}{l}\text{If } x = g'(y) \text{ is continuous on } c \le y \le d, \text{ then the length of the curve } x = g(y) \text{ on } c \le y \le d \text{ is} \\ \displaystyle L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy\end{array}}$

Example 4. Use the arc length formula to find the length of the curve $y^3 = x^2$, for $1 \le y \le 4$.

① $x = y^{\frac{3}{2}}$

② $x' = \frac{dx}{dy} = \frac{3}{2} y^{\frac{1}{2}}$

③ By arc length formula

$$L = \int_1^4 \sqrt{1+(x')^2}\, dy = \int_1^4 \sqrt{1+\frac{9}{4}y}\, dy$$

Let $u = 1+\frac{9}{4}y$

$du = \frac{9}{4} dy$

$dy = \frac{4}{9} du$

$u(1) = \frac{13}{4}$ $u(4) = 10$

$$= \int_{\frac{13}{4}}^{10} u^{\frac{1}{2}} \cdot \frac{4}{9}\, du$$

$$= \frac{4}{9} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} \Big|_{13/4}^{10}$$

$$= \frac{8}{27}\left(10^{\frac{3}{2}} - \left(\frac{13}{4}\right)^{\frac{3}{2}}\right)$$

Remark: Use $y = x^{2/3}$ will be hard to calculate.

The Arc Length Function:

It is useful to define a function $s(x)$ measuring the arc length of a curve from starting point $t = a$ to any other point $t = x$ on the curve C.

The Arc Length Function

If $y = f'(t)$ is continuous on $[a, b]$, then the the Arc Length **Function** is defined as

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

Here, x is the variable for the arc length function.
The Fundamental Theorem of Calculus gives

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Take squares both sides, It can also be written as

$$(ds)^2 = (dx)^2 + (dy)^2$$

Example 5. Find the arc length function for the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ taking $(1, 1/4)$ as the starting point.

$$y' = \frac{1}{2}x - \frac{1}{2x}$$

$$1+(y')^2 = 1 + \left(\frac{x}{2}\right)^2 - \frac{1}{2} + \left(\frac{1}{2x}\right)^2$$

$$= \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

$$S(x) = \int_1^x \sqrt{1+(y')^2}\, dt$$

$$= \int_1^x \frac{t}{2} + \frac{1}{2t}\, dt$$

$$= \frac{t^2}{4} + \frac{1}{2}\ln t \Big|_1^x$$

$$= \frac{x^2}{4} + \frac{1}{2}\ln x - \frac{1}{4}$$