Type I: Infinite integrals

(1.) If $\int_a^t f(x)\,dx$ exists for all $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

(2.) If $\int_t^b f(x)\,dx$ exists for all $t \leq b$, then

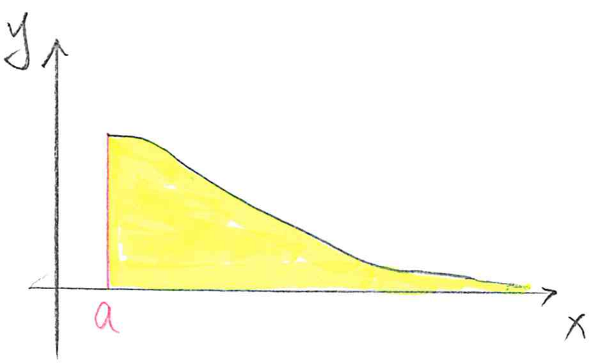
$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

The improper integral is called **convergent** if the limit exists (as a finite number) and **divergent** if the limit does not exist.

(3.) If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

(1)



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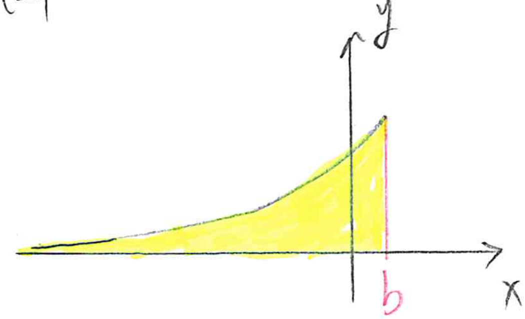

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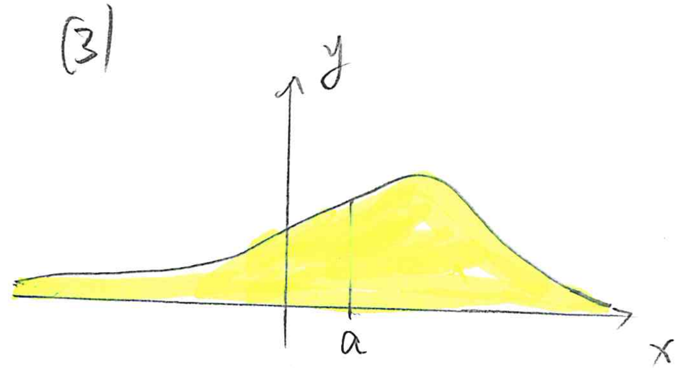
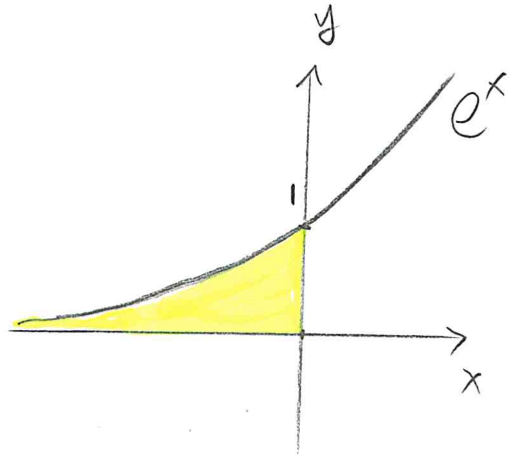


Example 1. Show that $\int_{-\infty}^{0} e^x \, dx = 1$.



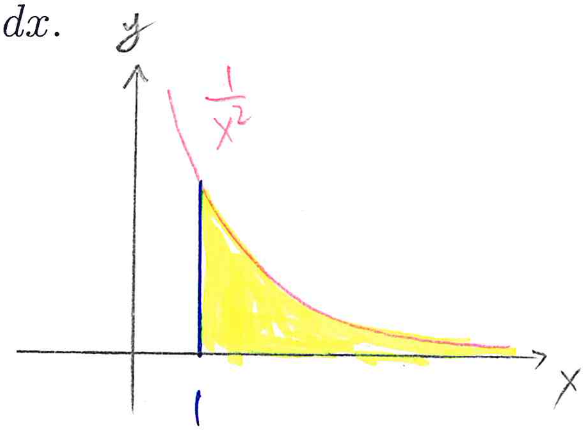
$$\int_{-\infty}^{0} e^x dx = \lim_{t \to -\infty} \int_t^0 e^x dt$$

$$= \lim_{t \to -\infty} e^x \Big|_t^0$$

$$= \lim_{t \to -\infty} (1 - e^t)$$

$$= 1$$

Example 2. Find the integral $\int_1^{\infty} \frac{1}{x^2} \, dx$.



$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t \to \infty} \int_1^t x^{-2} dx$$

$$= \lim_{t \to \infty} -x^{-1} \Big|_1^t$$

$$= \lim_{t \to \infty} \left(-\frac{1}{t} + 1\right)$$

$$= 1$$

Example 3. Determine whether the integral $\int_1^{\infty} \frac{1}{x} \, dx$ is convergent or divergent.

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t \to \infty} \int_1^t \frac{1}{x} dx$$

$$= \lim_{t \to \infty} \ln|x| \Big|_1^t$$

$$= \lim_{t \to \infty} (\ln t - \ln 1) = \lim_{t \to \infty} \ln t = \infty$$

Example 4. Determinant the values of p the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

- We already know that when $p=1$, it is divergent.
- Suppose $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{1}{1-p}\left(\frac{1}{t^{p-1}} - 1\right)$$

If $p-1>0$, then the limit is $\frac{1}{p-1}$. So $\int_1^\infty \frac{1}{x^p}\,dx$ convergent.

If $p-1<0$, then $\frac{1}{t^{p-1}} \to \infty$. So $\int_1^\infty \frac{1}{x^p}\,dx$ diverge.

Example 5. Show that $\int_{-1}^\infty 6x^2 e^{-x^3}\,dx = 2e$.

Step 1 Let $u(x) = -x^3$

$$du = -3x^2\,dx$$

$$dx = -\frac{1}{3x^2}\,du$$

Step 2 $\int 6x^2 e^{-x^3}\,dx = \int 6x^2 e^u \left(-\frac{1}{3x^2}\right) du$

$$= -2\int e^u\,du = -2e^u + C = -2e^{-x^3} + C$$

Step 3 $\int_{-1}^\infty 6x^2 e^{-x^3}\,dx = \lim_{t\to\infty} -2e^{-x^3}\Big|_{-1}^t$

$$= \lim_{t\to\infty}\left(-2e^{-t^3} + 2e\right)$$

$$= 2e$$

Example 6. Show that $\int_e^\infty \frac{1}{x\ln x}\,dx = \infty.$

Step1 Let $u(x)=\ln x$

$du=\frac{1}{x}dx$

Step2 $\int \frac{1}{x\ln x}dx$

$=\int \frac{1}{u}du$

$=\ln|u|+c$

$=\ln(\ln x)+c$

Step3 $\int_e^\infty \frac{1}{x\ln x}dx$

$=\lim_{t\to\infty}\int_e^t \frac{1}{x\ln x}dx$

$=\lim_{t\to\infty}\ln(\ln(x))\Big|_e^t$

$=\lim_{t\to\infty}\ln(\ln t)-0$

$=\infty$

Example 7. Show that $\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx = 2.$ $\ln 1=0$

Step1 Let $u(x)=\ln x$

$du=\frac{1}{x}dx$

Step2 $\int \frac{1}{x\sqrt{\ln x}}dx$

$=\int u^{-\frac{1}{2}}du$

$=2u^{\frac{1}{2}}+c$

$=2\sqrt{\ln x}+c$

Step3 $\int_1^e \frac{1}{x\sqrt{\ln x}}dx$

$=\lim_{t\to 1^+} 2\sqrt{\ln x}\Big|_t^e$

$=\lim_{t\to 1} 2\sqrt{\ln e}-2\sqrt{\ln t}$

$=2$

Type II: discontinuous integrals

(1.) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b-} \int_a^t f(x)\,dx$$

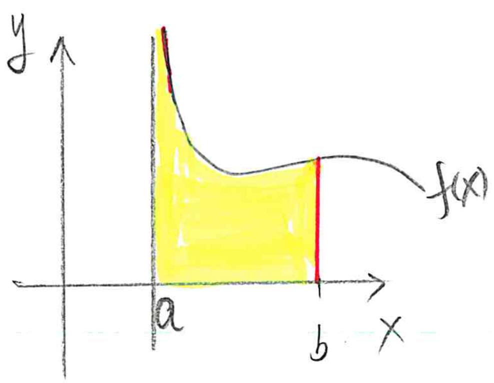
(2.) If f is continuous on $(a, b]$ and is discontinuous at a, then

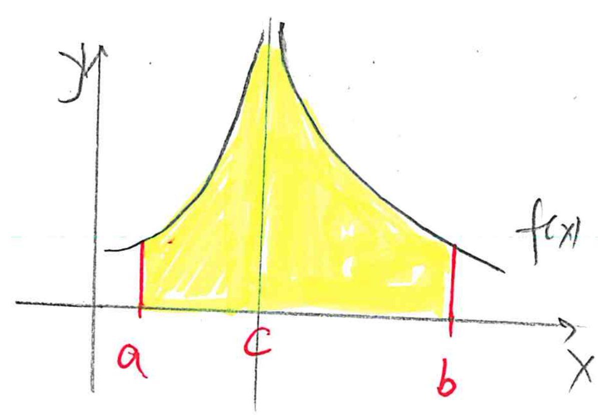
$$\int_a^b f(x)\,dx = \lim_{t \to a+} \int_t^b f(x)\,dx$$

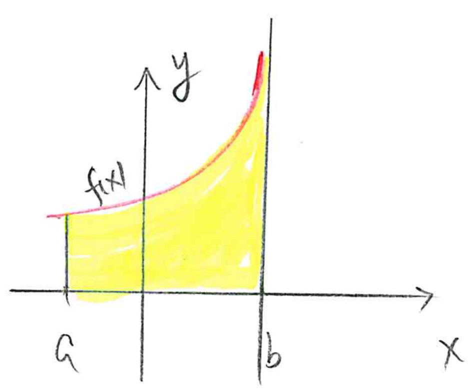
The improper integral is called **convergent** if the limit exists (as a finite number) and **divergent** if the limit does not exist.

(3.) If f is a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ convergent, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$







Example 8. Find $\int_{2}^{6} \frac{1}{\sqrt{x-2}}\, dx$.

$$\lim_{t\to 2+}\int_t^6 (x-2)^{-\frac{1}{2}}\,dx$$

$$= \lim_{t\to 2+} 2\sqrt{x-2}\,\Big|_t^6$$

$$= \lim_{t\to 2+} 2\sqrt{4} - 2\sqrt{t-2}$$

$$= 4$$

Let $u = x-2$

$du = dx$

$$\int \frac{1}{\sqrt{x-2}}\,dx$$

$$= \int u^{-\frac{1}{2}}\,du$$

$$= 2\sqrt{u} + C$$

$$= 2\sqrt{x-2} + C$$

Example 9. Determine whether $\int_0^4 \frac{1}{x-2}\, dx$ converges or diverges.

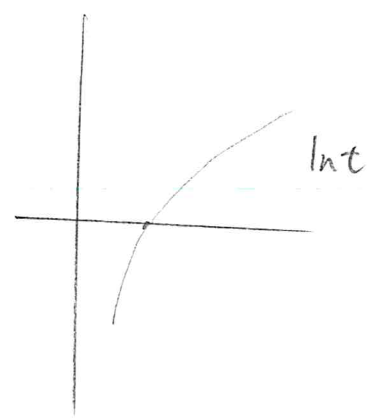
$x \neq 2$

$$\int_0^4 \frac{1}{x-2}\,dx$$

$$= \int_0^{2} \frac{1}{x-2}\,dx + \int_{2}^4 \frac{1}{x-2}\,dx$$

$$= \lim_{t\to 2-} \ln|x-2|\,\Big|_0^t + \lim_{t\to 2+} \ln|x-2|\,\Big|_t^4$$

$$= \lim_{t\to 2-} (\ln|t-2| - \ln 2) + \lim_{t\to 2+}(\ln 2 - \ln|t-2|).$$



$\because$ Each limit is divergent.

So $\int_0^4 \frac{1}{x-2}\,dx$ is divergent

Comparison Theorem: Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for x on $[a, \infty]$.

(1) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(2) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Example 10. Determine whether $\int_1^\infty e^{-x^2}\,dx$ converges or diverges.

$x^2 \geq x$ on $[1, \infty]$

So $-x^2 \leq -x$

So $e^{-x^2} \leq e^{-x}$

$$\int_1^\infty e^{-x}dx = \lim_{t\to\infty} \int_1^t e^{-x}dx = \lim_{t\to\infty} -e^{-x}\Big|_1^t$$

$$= \lim_{t\to\infty} -e^{-t} + e^{-1}$$

$$= e^{-1} \quad \text{convergent.}$$

So $\int_1^\infty e^{-x^2}dx$ is convergent.

Example 11. Determine whether $\int_0^\infty e^{-x^2}\,dx$ converges or diverges.

$$= \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$$

Both convergent.