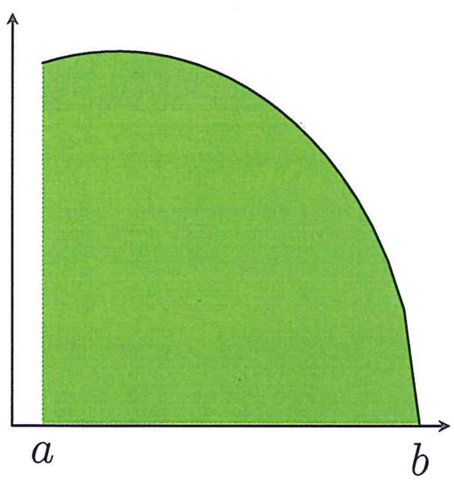
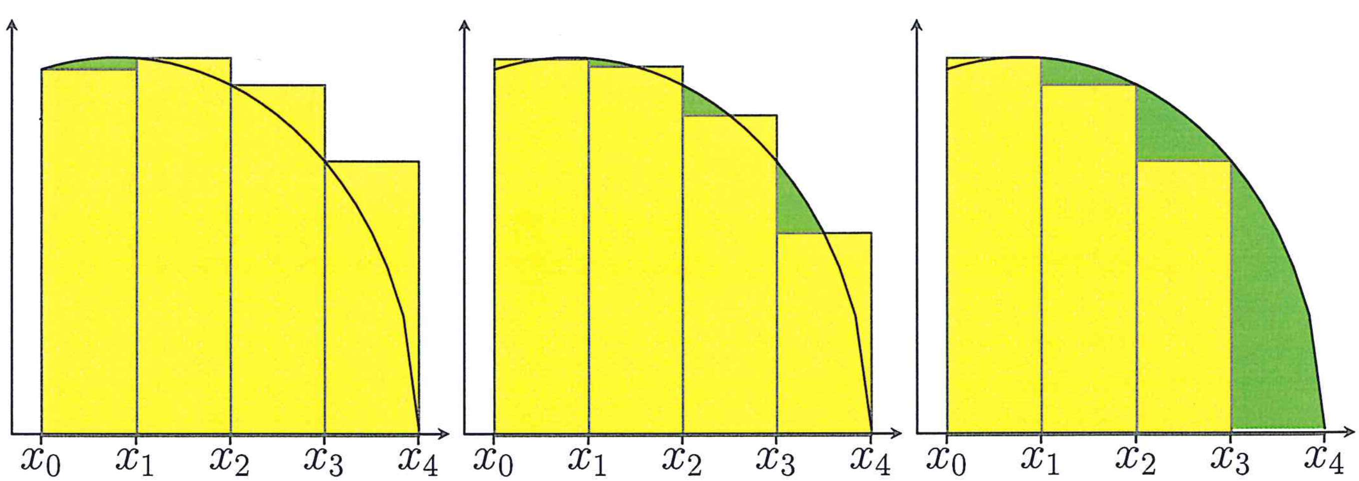
In many real world models $f(x)$, the definite integrals $\int_a^b f(x)dx$ can not be evaluated exactly.
For example, it is impossible to calculate the precise value of the following integrals:

$$\int_a^b \sqrt[3]{x^4+e^x-1}\,dx, \quad \int_a^b \sqrt{\sin 2x^3+1}\,dx, \quad \int_a^b e^{x^2}dx.$$

In these cases, an approximation for the value $\int_a^b f(x)dx$ will be useful.




## 1. Riemann Sum.




**1. Left endpoint approx.** **2. Midpoint approx.** **3. Right endpoint approx.**

$x_0 = a$ and $x_4 = b$

Let $\Delta x = \dfrac{b-a}{n}$. (In the above pictures, $n = 4$)

**(1). Left endpoint approximation.**

$$\int_a^b f(x) \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

**(2). Midpoint approximation**

$$\int_a^b f(x) \approx M_n = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x$$

Here, $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ is the midpoint of $[x_{i-1}, x_i]$.

**(3). Right endpoint approximation.**

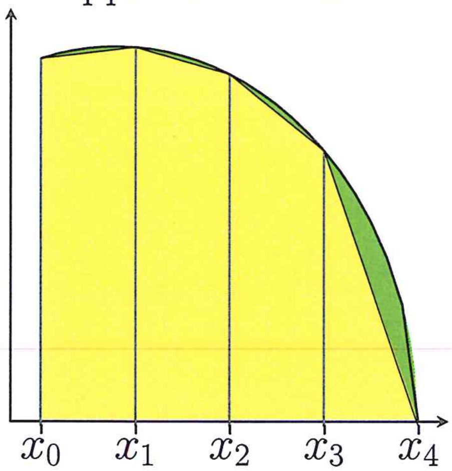
$$\int_a^b f(x) \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

In the previous picture example,

$$L_4 = [f(x_0) + f(x_1) + f(x_2) + f(x_3)]\Delta x.$$
$$R_4 = [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\Delta x.$$
$$M_4 = [f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4)]\Delta x.$$

## 2. Trapezoid Rule.

Instead of using rectangles as in Riemann Sum, we can use trapezoids to approximate the definite integral.




Trapezoid Rule:

$$\int_a^b f(x) \approx T_n = \sum_{i=1}^{n} [f(x_{i-1}) + f(x_i)]\frac{\Delta x}{2}$$

$$= [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]\frac{\Delta x}{2}$$

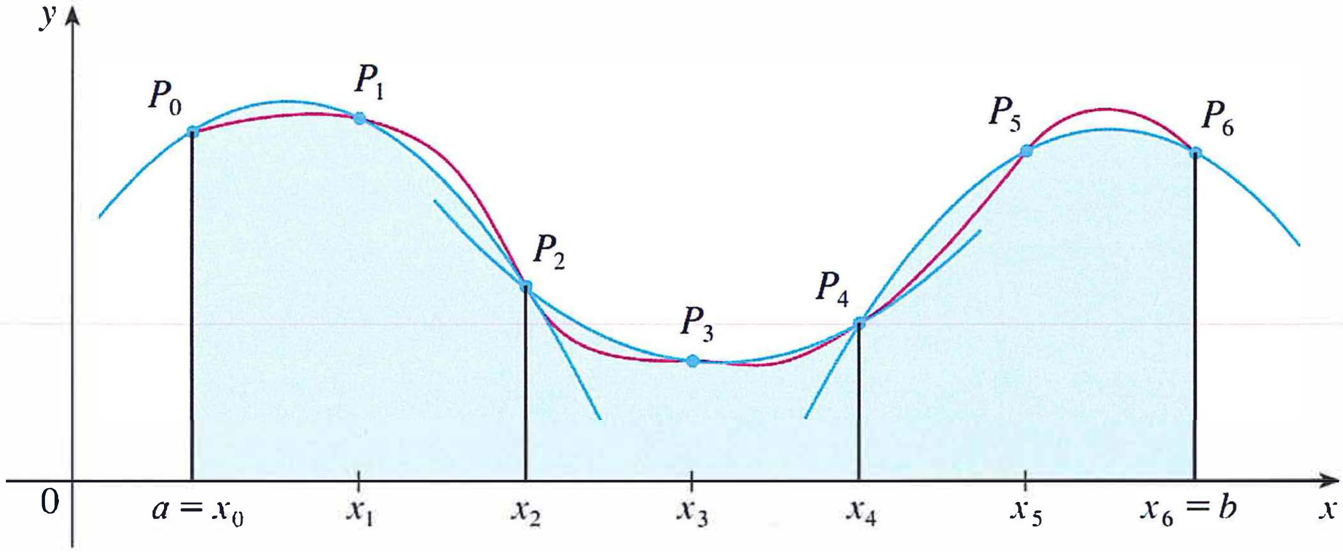
This is the average of the left endpoint approximation and the right endpoint approximation. That is, $T_n = \frac{1}{2}(L_n + R_n)$.

In the above picture example,

$$T_4 = [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]\frac{\Delta x}{2}$$

## 3. Simpson's Rule.

A modification of the Trapezoid Rule is the Simpson's Rule: connecting points by parabolas instead of by line segment.




Simpson's Rule:

$$\int_a^b f(x) \approx S_n$$

$$= [f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]\frac{\Delta x}{3}$$

Here, $n$ is an even number and the pattern of the coefficients is $1, 4, 2, 4, 2, 4, 2, \cdots, 4, 2, 4, 1$

$S_n$ is the area of the shape under the blue curve (parabolas).

In fact, $T_{2m} = \frac{1}{3}T_m + \frac{2}{3}M_m$

In the above picture example,

$$S_6 = [f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+f(x_6)]\frac{\Delta x}{3}$$

**Error Bounds:**

Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If $E_T$ and $E_M$ are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

and

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

Suppose $|f^{(4)}(x)| \leq C$ for $a \leq x \leq b$. If $E_S$ is the error in the Simpson's Rule, then

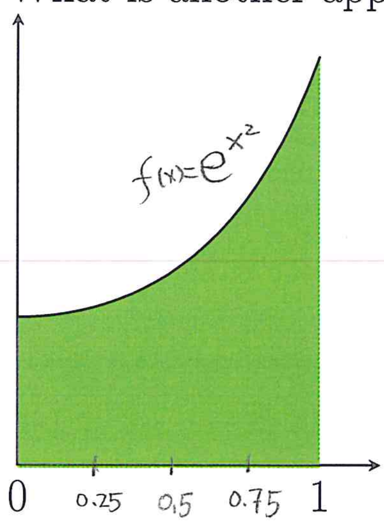
$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

1. In all of the methods we can get better approximations if we increase the value of $n$.
2. The Trapezoidal and Midpoint Rules are much more accurate than the left and right endpoint approximations.
3. The error in the Midpoint Rule is about half the size of the error in the Trapezoidal Rule.
4. The error bound is the worst case it can happen.

**Example:** Use Left endpoint, Right endpoint, Midpoint Rule, Trapezoidal Rule, Simpson's Rule with $n = 4$ to approximate the integral $\int_0^1 e^{x^2}\,dx$. $\approx 1.46265$

$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

- What is another approximation we can use?




$x_0 = 0$ $\quad \bar{x}_1 = 0.125$

$x_1 = 0.25$ $\quad \bar{x}_2 = 0.375$

$x_2 = 0.5$ $\quad \bar{x}_3 = 0.625$

$x_3 = 0.75$ $\quad \bar{x}_4 = 0.875$

$x_4 = 1$

**Left** endpoint approximation:

$$L_4 = [f(x_0) + f(x_1) + f(x_2) + f(x_3)]\Delta x.$$

$$= \left(e^0 + e^{0.25^2} + e^{0.5^2} + e^{0.75^2}\right) 0.25$$

$$\approx 1.27589$$

Taylor Series: $e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120} + \cdots$

$$\int_0^1 e^{x^2}\,dx \approx x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \frac{x^9}{24\times 9} + \frac{x^{11}}{120\times 11}\Big|_0^1$$

$$\approx 1.46253$$

**Right** endpoint approximation:

$$R_4 = [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\Delta x.$$

$$= \left[ e^{0.25^2} + e^{0.5^2} + e^{0.75^2} + e \right] 0.25$$

$$\approx 1.70546$$

**Midpoint Rule:** $\bar{x}_1 = 0.125 \quad \bar{x}_2 = 0.375 \quad \bar{x}_3 = 0.625 \quad \bar{x}_4 = 0.875$

$$M_4 = [f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4)]\Delta x.$$

$$= \left[ e^{0.125^2} + e^{0.375^2} + e^{0.625^2} + e^{0.875^2} \right] 0.25$$

$$\approx 1.44875$$

## Trapezoidal Rule:

$$T_4 = [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]\frac{\Delta x}{2}$$

$$= \left[ e^0 + 2e^{0.25^2} + 2e^{0.5^2} + 2e^{0.75^2} + e^1 \right] \frac{0.25}{2}$$

$$\approx 1.49068$$

## Simpson's Rule:

$$S_4 = [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4))]\frac{\Delta x}{3}$$

$$= \left[ e^0 + 4e^{0.25^2} + 2e^{0.5^2} + 4e^{0.75^2} + e^1 \right] \frac{0.25}{3}$$

$$\approx 1.46371$$