- A **polynomial** is a function with variable x of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, \cdots, a_1, a_0$ are real numbers.

- If $a_n \neq 0$, we say the **degree** of $p(x)$ is n.
- A ratio of polynomials $f(x) = \dfrac{p(x)}{q(x)}$ is called a **rational function**.
- If the degree $\deg(p(x)) \geq \deg(q(x))$, we say that the rational function $f(x) = \dfrac{p(x)}{q(x)}$ is **improper**.
- On the other side, if the degree $\deg(p(x)) < \deg(q(x))$, we say that the rational function $f(x) = \dfrac{p(x)}{q(x)}$ is **proper**.

- We can simplify an improper rational function $f(x) = \dfrac{p(x)}{q(x)}$ as the sum of a polynomial and a proper rational function:

$$f(x) = s(x) + \frac{r(x)}{q(x)}$$

such that $\deg(r(x)) < \deg(q(x))$.

In this section, the title is §*7.4 Integration of Rational Functions (by Partial Fractions.)*

The integration of a polynomial function $s(x)$ is easy. So, we only focus on solving Integration of Proper Rational Functions $\dfrac{r(x)}{q(x)}$.

Let's see some easy examples first.

Example (1). Find $\int \frac{x^2+1}{x-2}\,dx$

$$\begin{array}{r} x+2 \\ x-2\,\overline{)\,x^2 \qquad +1} \\ \underline{x^2-2x} \\ 2x+1 \\ \underline{2x-4} \\ 5 \end{array}$$

$$= \int (x+2) + \frac{5}{x-2}\,dx$$

$$= \frac{x^2}{2} + 2x + 5\ln|x-2| + C$$

Example (2). Find $\int \frac{2x+3}{x^2+4x+4}\,dx$

$$= \int \frac{2(x+2)-1}{(x+2)^2}\,dx$$

$$= \int \frac{2}{(x+2)} - \frac{1}{(x+2)^2}\,dx$$

$$= 2\ln|x+2| - \frac{(x+2)^{-1}}{-1} + C$$

$$= 2\ln|x+2| + \frac{1}{x+2} + C$$

(H.W.)

Example (3). Find $\displaystyle\int \frac{x^3+x^2-3x}{x-1}\,dx$

$$\begin{array}{r} x^2+2x-1 \\ x-1\,\overline{)\,x^3+x^2-3x} \\ \underline{x^3-x^2} \\ 2x^2-3x \\ \underline{2x^2-2x} \\ -x \\ \underline{-x-1} \\ 1 \end{array}$$

$$= \int x^2+2x-1+\frac{1}{x-1}\,dx$$

$$= \frac{x^3}{3}+x^2-x+\ln|x-1|+C$$

Example (4). Find $\displaystyle\int \frac{3x-7}{x^2+4}\,dx$

$$= \int \frac{3x}{x^2+4}-\frac{7}{x^2+4}\,dx$$

$$= \frac{3}{2}\ln|x^2+4| - \frac{7}{2}\tan^{-1}\frac{x}{2} + C$$

(by u-substitution) (by Ex 1 in §7.3)

We can always express a *simple* rational function $f(x) = \dfrac{r(x)}{q(x)}$ as a sum of simpler fractions, called **partial fractions:**

$$\text{Type (I): } \frac{A}{(ax+b)^m} \quad \text{or} \quad \text{Type (II): } \frac{Ax+B}{(ax^2+bx+c)^n}$$

In Type (II), $b^2 - 4ac < 0$.

Case 1. The sum only involves the first type of partial fractions.

Example 1. Find $\displaystyle\int \frac{1}{x^2-a^2}\,dx$

$$\frac{1}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a} = \frac{A(x-a)+B(x+a)}{(x+a)(x-a)}$$

So,

$$1 = A(x-a)+B(x+a) = (A+B)x + (B-A)a$$

So $A+B=0$

$B-A = \frac{1}{a}$

So $A = -\frac{1}{2a}$

$B = \frac{1}{2a}$

$$= \int \frac{1}{(x+a)(x-a)}\,dx$$

$$= \frac{1}{2a}\int \frac{-1}{x+a} + \frac{1}{x-a}\,dx$$

$$= \frac{1}{2a}\left(-\ln|x+a| + \ln|x-a|\right) + C$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

Example 2. Find $\displaystyle\int \frac{4x+3}{2x^2+3x-2}\,dx$

$$= \int \frac{2x-1}{(2x-1)(x+2)}\,dx$$

Suppose $\displaystyle\frac{4x+3}{(2x-1)(x+2)} = \frac{A}{2x-1} + \frac{B}{x+2}$

$$= \frac{A(x+2)+B(2x-1)}{(2x-1)(x+2)}$$

So $x(A+2B) + 2A - B = 4x+3$

So $\begin{cases} A+2B=4 \\ 2A-B=3 \end{cases} \Rightarrow \begin{cases} A+2B=4 \\ -5B=-5 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=1 \end{cases}$

So $\displaystyle\int \frac{4x+3}{2x^2+3x-2}\,dx = \int \frac{2}{2x-1} + \frac{1}{x+2}\,dx$

$$= \ln|2x-1| + \ln|x+2| + C$$

Partial fraction technique: Express a *simple* rational function $f(x) = \dfrac{r(x)}{q(x)}$ as a sum of **partial fractions.**

1. Any polynomial can be decomposed as the products of powers of linear functions $(ax+b)^m$ and powers of quadratic polynomials $(ax^2+bx+c)^n$.

2. Then we can suppose the partial fraction decomposition is the sum of

$$\text{Type (I):} \quad \frac{A}{(ax+b)^m} \quad \text{or} \quad \text{Type (II):} \quad \frac{Bx+C}{(ax^2+bx+c)^n}$$

Then, solve all A, B, C in the decomposition.

Example (a) Try to find the partial fraction decomposition for

$$\frac{2x^3 - x^2 + 4}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

$$= \frac{\cdots\cdots}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

, then compare the numerators and solve $A, B, \cdots, I, J$.

Least Common denominator

Many computer algebra systems can do this

- Mathematica
- Maple
- Sage math
- Matlab

Example (b) Find the partial fraction decomposition for

$$\frac{4-3x+x^2+2x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$= \frac{A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x}{x(x^2+1)^2}$$

$$\vdots$$

$$= \frac{(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A}{x(x^2+1)^2}$$

Compare the numerators

$(A+B)=0$ $C=2$ $2A+B+D=1$ $C+E=-3$ $A=4$

MATH330
Linear Algebra $\Rightarrow$ $A=4$ $B=-4$ $C=2$ $D=-3$ $E=-5$
Solve any linear equation.

So $$\frac{4-3x+x^2+2x^3}{x(x^2+1)^2} = \frac{4}{x} + \frac{-4x+2}{x^2+1} + \frac{-3x-5}{(x^2+1)^2}$$

Example 3. Find $\displaystyle\int \frac{4x}{x^3 - x^2 - x + 1}\,dx = \int \frac{4x}{(x-1)^2(x+1)}\,dx$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$= \frac{A(x-1)(x+1) + B(x-1) + C(x-1)^2}{(x-1)^2(x+1)}$$

$$= \frac{(A+C)x^2 + (B-2C)x + (-A+B+C)}{(x-1)^2(x+1)}$$

Compare the numerators

$$\begin{aligned} A+C&=0 \\ B-2C&=4 \\ -A+B+C&=0 \end{aligned} \quad\Rightarrow\quad \begin{aligned} A&=-C \\ B+2A&=4 \\ B-2A&=0 \end{aligned} \quad\Rightarrow\quad \begin{aligned} A&=1 \\ B&=2 \\ C&=-1 \end{aligned}$$

MATH330
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ -1 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\int \frac{4x}{(x-1)^2(x+1)}\,dx = \int \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\,dx$$

$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

$$= \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C$$

Case 2. The sum involves partial fractions of Type (II).

We need to use the result

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

or substitution. $x = a\tan\theta$

Example 10 in §7.3: Find $\displaystyle\int \frac{A}{x^2-2x+5}\,dx$

$$= \int \frac{A}{(x-1)^2+4}\,dx = \frac{A}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+C$$

Example 4. Find $\displaystyle\int \frac{Bx}{x^2-2x+5}\,dx$

$$= \int \frac{Bx}{(x-1)^2+4}\,dx$$

$u = x-1$

$du = dx$

$x = u+1$

$$= \int \frac{B(u+1)}{u^2+4}\,du$$

$$= \int \frac{Bu}{u^2+4} + \frac{B}{u^2+4}\,du$$

$$= \frac{B}{2}\ln|u^2+4| + \frac{B}{2}\tan^{-1}\left(\frac{u}{2}\right)+C$$

$$= \frac{B}{2}\ln|(x-1)^2+4| + \frac{B}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+C$$

Example 5. Find $\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)x}{x(x^2+4)} = \frac{(A+B)x^2+Cx+4A}{x(x^2+4)}$$

compare the numerators $2x^2-x+4 = (A+B)x^2+Cx+4A$

So $A+B=2$, $C=-1$, $4A=4$ $\Rightarrow$ $A=1$, $B=1$, $C=-1$

So $\int \frac{2x^2-x+4}{x^3+4x}\, dx = \int \frac{1}{x} + \frac{x-1}{x^2+4}\, dx$

$$= \ln|x| + \int \frac{x}{x^2+4}\, dx - \int \frac{1}{x^2+4}\, dx$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Example 6. Find $\displaystyle\int \frac{x-1}{4x^2-4x+3}\,dx$

$$= \int \frac{x-1}{(2x-1)^2+2}\,dx$$

$u = 2x-1$

$du = 2\,dx$

$x = \frac{1}{2}(u+1)$

$$= \int \frac{\frac{1}{2}(u+1)-1}{u^2+2}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{4}\int \frac{u-1}{u^2+2}\,du$$

$$= \frac{1}{4}\int \frac{u}{u^2+2}\,du - \frac{1}{4}\int \frac{1}{u^2+2}\,du$$

$$= \frac{1}{8}\ln|u^2+2| - \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= \frac{1}{8}\ln|(2x-1)^2+2| - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$$

Example 7. Find $\displaystyle\int \frac{x}{(x^2+a^2)^n}\,dx$ $(n \geqslant 2)$

$u = x^2 + a^2$

$du = 2x\,dx$

$dx = \frac{1}{2x}\,du$

$$= \int \frac{x}{u^n}\frac{1}{2x}\,du$$

$$= \frac{1}{2}\int u^{-n}\,du$$

$$= \frac{1}{2}\,\frac{u^{-n+1}}{-n+1} + C = \frac{1}{2(1-n)}\cdot(x^2+a^2) + C$$

Example 8 Simply $\displaystyle\int \frac{1}{(x^2+a^2)^n}\,dx$ to a question in §7.2.

Let $x = a\tan\theta$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = a\sec^2\theta\,d\theta$

$$\int \frac{1}{(x^2+a^2)^n}\,dx = \int \frac{1}{a^{2n}\sec^{2n}\theta}\cdot a\sec^2\theta\,d\theta$$

$$= \frac{1}{a^{2n-1}}\int \frac{1}{\sec^{2n-2}\theta}\,d\theta$$

$$= \frac{1}{a^{2n-1}}\int \cos^{2(n-1)}\theta\,d\theta.$$

$$= \frac{1}{a^{2n-1}}\int \left[\frac{1}{2}(1+\cos 2\theta)\right]^{n-1}d\theta.$$

§7.2

Example 9 Find the indefinite integral of the rational function in Example (b).

$$f(x) = \frac{4-3x+x^2+2x^3}{x(x^2+1)^3} = \frac{4}{x} + \frac{-4x+2}{x^2+1} + \frac{-3x-5}{(x^2+1)^2}$$

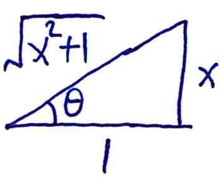
So $$\int f(x)\,dx = \int \frac{4}{x}\,dx + \int \frac{-4x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx + \int \frac{-3x}{(x^2+1)^2}\,dx + \int \frac{-5}{(x^2+1)^2}\,dx$$

$$= 4\ln|x| - 2\ln(x^2+1) + \tan^{-1}x + \frac{3}{2}\frac{1}{x^2+1} - 5\boxed{\int \frac{1}{(x^2+1)^2}\,dx}$$

For $$\int \frac{1}{(x^2+1)^2}\,dx \xlongequal{\text{Example 8 } (n=2)\ a=1 \quad x=\tan\theta} \int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \boxed{\frac{1}{2}\left(\tan^{-1}x + \frac{2x}{x^2+1}\right) + C}$$



$\sin\theta = \frac{x}{\sqrt{x^2+1}}$

$\cos\theta = \frac{1}{\sqrt{x^2+1}}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{2x}{x^2+1}$