Recall Example 10. in §7.1: Find $\int \sin\sqrt{x}\, dx$.

Or consider a similar example: $\int e^{\sqrt{x}}\, dx$.

Step 1. u-substitution: Let $u = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow du = \frac{1}{2}x^{-\frac{1}{2}}dx$

$dx = 2x^{\frac{1}{2}}du$

$$\int e^{\sqrt{x}}dx = \int e^{u}\cdot 2x^{\frac{1}{2}}du = 2\int e^{u}\cdot u\, du$$

Step 2. Integration by parts

Choose $f = u \quad g' = e^{u}$

$f' = 1 \quad g = e^{u}$

$$\int e^{u}u\, du = fg - \int f'g\, du = ue^{u} - \int e^{u}du = ue^{u} - e^{u} + C$$

Step 3. So $\int e^{\sqrt{x}}dx = 2ue^{u} - 2e^{u} + 2C$

$$= 2x^{\frac{1}{2}}e^{\sqrt{x}} + 2e^{\sqrt{x}} + C$$

In Step 1, we have used u-substitution by letting $u = \sqrt{x}$.
Another way to look at this substitution is letting $\boxed{x = u^2.}$ Then $dx = 2udu$, and $\int e^{\sqrt{x}}\, dx = \int e^{u}2udu$.
This is called **inverse substitution**.

In this section, we focus on using the following inverse substitutions, which are called **Trigonometric Substitutions.**

$1-\sin^2\theta=\cos^2\theta$ (1) $x = a\sin\theta$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$1+\tan^2\theta=\sec^2\theta$ (2) $x = a\tan\theta$, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\sec^2\theta-1=\tan\theta$ (3) $x = a\sec\theta$, for $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

We use Trigonometric Substitutions to calculate two known formulas.

Example 1. Find $\int \frac{1}{x^2+a^2}\,dx$, for $a > 0$

Let $x = a\tan\theta$, then $dx = a\sec^2\theta\,d\theta$ $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$$\int \frac{1}{x^2+a^2}\,dx = \int \frac{1}{a^2\tan^2\theta + a^2}\cdot a\sec^2\theta\,d\theta = \int \frac{1}{a^2(\tan^2\theta+1)}\cdot a\sec^2\theta\,d\theta$$

$$= \int \frac{1}{a}\,d\theta = \frac{1}{a}\cdot\theta + C = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

or $\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

Example 2. Find $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$, for $a > 0$

Let $x = a\sin\theta$ then $dx = a\cos\theta\,d\theta$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}\,a\cos\theta\,d\theta \qquad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$$

$$= \int \frac{1}{a\cos\theta}\,a\cos\theta\,d\theta$$

$$= \int d\theta = \theta + C = \sin^{-1}\left(\frac{x}{a}\right) + C$$

Example 3. Find $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

Step1 Let $x = 2\sin\theta$ then $dx = 2\cos\theta\, d\theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

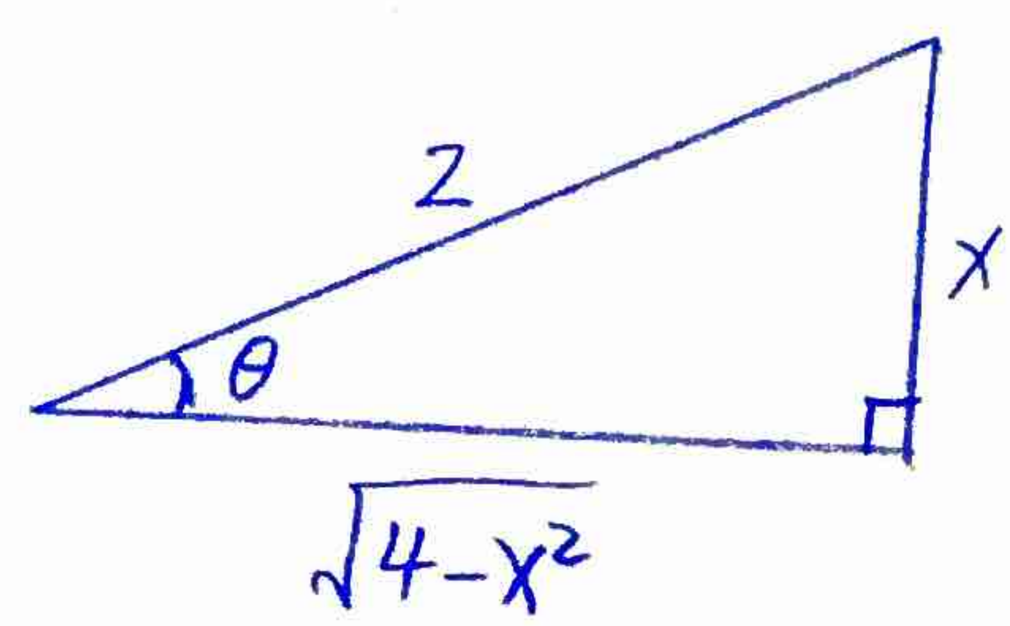
$$\int \frac{\sqrt{4-4\sin^2\theta}}{4\sin^2\theta}\, 2\cos\theta\, d\theta = \int \frac{2\cos\theta}{4\sin^2\theta}\cdot 2\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \cot^2\theta\, d\theta = \int \csc^2\theta - 1\, d\theta = -\cot\theta - \theta + C$$

Step2 Return to x, we need to find $\cot\theta$.

- we know $x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2}$

Draw a right triangle:



$$\cot\theta = \frac{\sqrt{4-x^2}}{x}$$

So $\int \frac{\sqrt{4-x^2}}{x^2}\,dx = -\cot\theta - \theta + C = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C$

Example 4. Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+9}}\,dx$

Step 1 Let $x = 3\tan\theta$ Then $dx = 3\sec^2\theta\, d\theta$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\int \frac{1}{x^2\sqrt{x^2+9}}\,dx = \int \frac{1}{9\tan^2\theta\sqrt{9\tan^2\theta+9}} \cdot 3\sec^2\theta\, d\theta$$

$$= \int \frac{3\sec^2\theta}{9\tan^2\theta\, 3\sec\theta}\,d\theta = \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{9}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

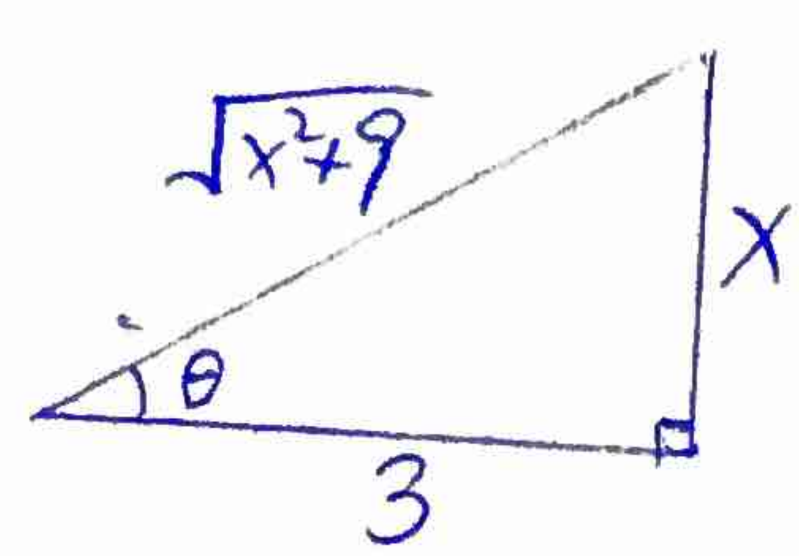
$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \frac{du}{u^2} = \frac{1}{9}\int u^{-2}\,du = \frac{-1}{9}\frac{1}{u} + C$$

u-substitution by $u = \sin\theta$

$$= -\frac{1}{9}\frac{1}{\sin\theta} + C$$

Step 2. we know $x = 3\tan\theta$. $\Rightarrow$ $\tan\theta = \frac{x}{3}$

Draw a right triangle



So $\sin\theta = \dfrac{x}{\sqrt{x^2+9}}$

So $\displaystyle\int \frac{1}{x^2\sqrt{x^2+9}}\,dx = -\frac{1}{9}\frac{\sqrt{x^2+9}}{x} + C$

Example 5. Find $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$ or Find $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx$

Step 1 Let $x = 3\sec\theta = \frac{3}{\cos\theta}$ $\quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

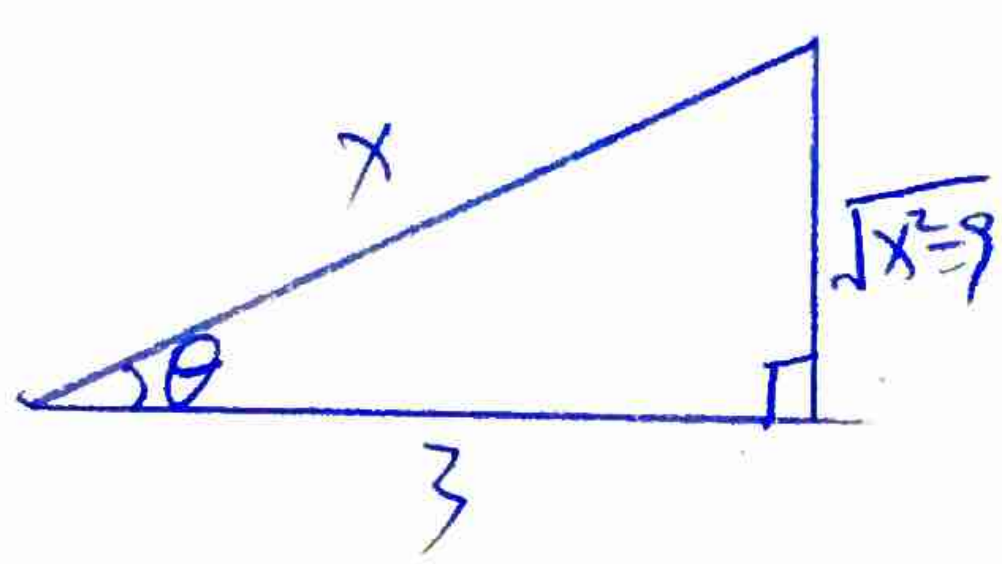
$dx = 3\sec\theta\tan\theta\,d\theta$

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \int \frac{1}{\sqrt{9\sec^2\theta-9}} \cdot 3\sec\theta\tan\theta = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

known result. (verify it yourse

Step2.



$x = 3\sec\theta$

$\sec\theta = \frac{x}{3}$

$\cos\theta = \frac{3}{x}$

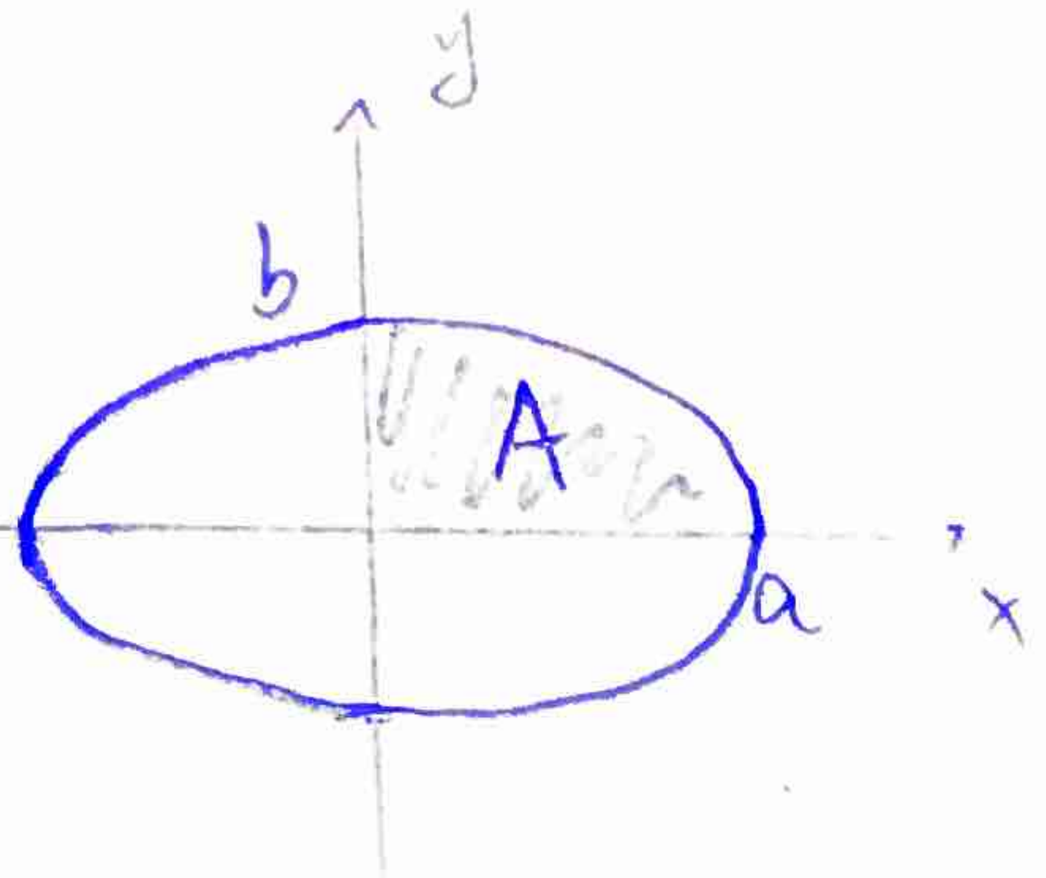
So $\sec\theta = \frac{x}{3}$ $\quad \tan\theta = \frac{\sqrt{x^2-9}}{3}$

So $$\int \frac{1}{\sqrt{x^2-9}}\,dx = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$$

- $$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

$$= \ln|x + \sqrt{x^2-a^2}|\ \boxed{-\ln|a| + C}$$ number C

Example 6. Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



we only calculate $\frac{1}{4}$ of the area in the first quadrant.

$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$ $\quad y=\frac{b}{a}\sqrt{a^2-x^2}$ $\quad 0\le x\le a$

$A=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$

Let $x=a\sin\theta$ $\quad dx=a\cos\theta$

$0\le\theta\le\frac{\pi}{2}$

$=\frac{b}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$

$=\frac{b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta\cdot a\cos\theta\,d\theta$

$=ab\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$

$=ab\int_0^{\frac{\pi}{2}}\frac{1}{2}(1+\cos 2\theta)\,d\theta$

$=\frac{1}{2}ab\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{2}}$

$=\frac{1}{2}ab\left(\frac{\pi}{2}\right)=\frac{1}{4}ab\pi$

So the area of the ellipse is $4A=ab\pi$.

Example 7. Find $\int \sqrt{a^2 - x^2}\, dx$

Let $x = a\sin\theta$ $dx = a\cos\theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - a^2\sin\theta}\; a\cos\theta$$

$$= \int a\cos\theta\; a\cos\theta\, d\theta$$

$$= a^2 \int \cos^2\theta\, d\theta$$

$$= a^2 \int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

Example 8. Find $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx$

Step 1 Let $x = a\tan\theta$ Then $dx = a\sec^2\theta\,d\theta$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \int \frac{1}{\sqrt{a^2\tan^2\theta + a^2}} \cdot a\sec^2\theta\,d\theta$$

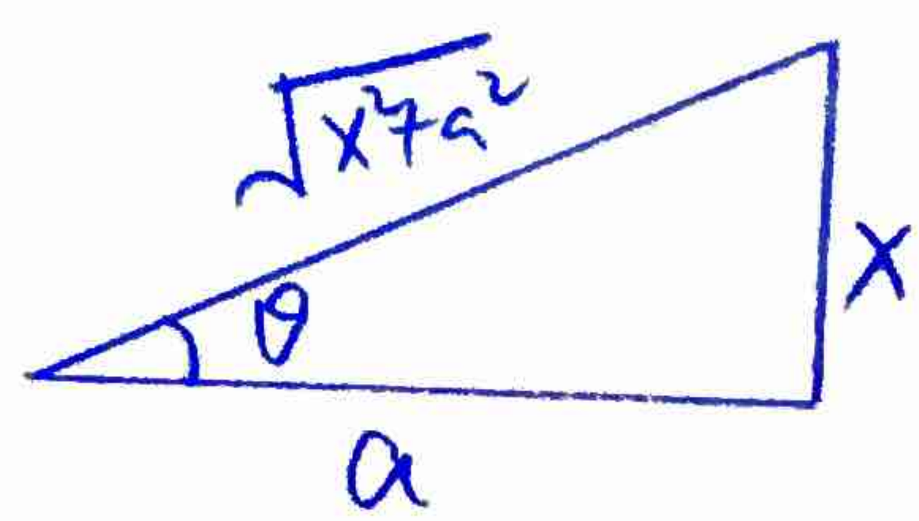
$$= \int \frac{a\sec^2\theta}{a\sec\theta}\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

Step 2 $x = a\tan\theta$

$\tan\theta = \dfrac{x}{a}$.



So $\sec\theta = \dfrac{\sqrt{x^2+a^2}}{a}$

So $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|\frac{\sqrt{x^2+a}}{a} + \frac{x}{a}\right| + C_1$

$$= \ln|\sqrt{x^2+a} + x| - \ln|a| + C_1$$

$$= \ln|\sqrt{x^2+a} + x| + C$$

Example 6. Find $\displaystyle\int \frac{1}{x^2 - a^2}\,dx$ (H.W.) similar to Ex 5.

Step 1 Let $x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$$\int \frac{1}{x^2-a^2}\,dx = \int \frac{1}{a^2\sec^2\theta - a^2}\cdot a\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{1}{a^2\tan^2\theta}\cdot a\sec\theta\tan\theta\,d\theta$$

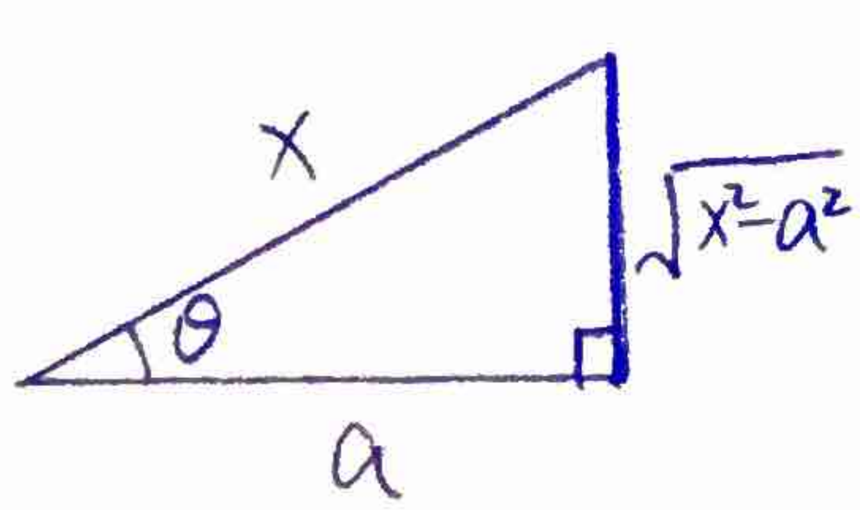
$$= \frac{1}{a}\int \frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= \frac{1}{a}\int \frac{1}{\cos\theta\cdot\frac{\sin\theta}{\cos\theta}}\,d\theta$$

$$= \frac{1}{a}\int \csc\theta\,d\theta$$

$$= \frac{1}{a}\ln|\csc\theta - \cot\theta| + C$$

Step 2.



So, $\csc\theta = \frac{x}{\sqrt{x^2-a^2}}$

$\cot\theta = \frac{a}{\sqrt{x^2-a^2}}$

So $$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{a}\ln\left|\frac{x}{\sqrt{x^2-a^2}} - \frac{a}{\sqrt{x^2-a^2}}\right| + C$$

$$= \frac{1}{a}\ln\left|\frac{x-a}{\sqrt{x^2-a^2}}\right| + C$$

$$= \frac{1}{a}\ln\left|\frac{x-a}{\sqrt{(x+a)(x-a)}}\right| + C$$

$$= \frac{1}{a}\ln\left|\left(\frac{x-a}{x+a}\right)^{\frac{1}{2}}\right| + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

Example 10. Find $\int \frac{1}{x^2 - 2x + 5}\, dx$

$$= \int \frac{1}{(x-1)^2 + 4}\, dx$$ complete the square

$$= \int \frac{1}{u^2+4}\, du$$ u-substitution $u = x-1$

$$= \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) + C$$ Example 1

$$= \frac{1}{2} \tan^{-1}\left(\frac{x-1}{2}\right) + C$$

Example 11. Find $\int \frac{1}{\sqrt{x^2 - 4x - 5}}\, dx$

$$= \int \frac{1}{\sqrt{(x-2)^2 - 1}}\, dx$$ complete the square

$$= \int \frac{1}{\sqrt{u^2-1}}\, du$$ u-substitution $u = x-2$

$$= \ln\left| u + \sqrt{u^2-1} \right| + C$$ Example 5

$$= \ln\left| x-2 + \sqrt{(x-2)^2-1} \right| + C$$