In Example 11 of §7.1, we have calculated $\int \sin^3 x \, dx$.

In this section, we use trigonometric identities to integrate more trigonometric functions:

Part 1. Evaluate $\int \sin^m x \cos^n \, dx$ for any m, n.

Some identities we need to use: $\sin x \cos x = \frac{1}{2}\sin 2x$

$$\sin^2 x + \cos^2 x = 1, \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x),$$

Example 1. $\int \cos^5 x \, dx$.

$$= \int \cos^4 x \cos x \, dx$$

$$= \int (1 - \sin^2 x)^2 \cos x \, dx$$

U-substitution: Let $u = \sin x$
$du = \cos x \, dx$

$$= \int (1 - u^2)^2 \, du$$

$$= \int 1 - 2u^2 + u^4 \, du$$

$$= u - \frac{2}{3}u^3 + \frac{u^5}{5} + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + C$$

Example 2. $\int \sin^m x \underline{\cos x\, dx}$. u-substitution

Let $u = \sin x$ $du = \cos x\, dx$

$$\int \sin^m x \cos x\, dx = \int u^m\, du = \frac{u^{m+1}}{m+1} + C$$

$$= \frac{\sin^{m+1} x}{m+1} + C$$

Example 3. $\int \sin^m x \cos^3 x\, dx$. u-substitution

$$= \int \sin^m x \cos^2 x \cos x\, dx$$

$$= \int \sin^m x (1 - \sin^2 x) \cos x\, dx$$

Let $u = \sin x$

$du = \cos x\, dx$

$$= \int u^m (1 - u^2)\, du$$

$$= \int u^m - u^{m+2}\, du$$

$$= \frac{u^{m+1}}{m+1} - \frac{u^{m+3}}{m+3} + C = \frac{\sin^{m+1} x}{m+1} - \frac{\sin^{m+3} x}{m+3} + C$$

Example 4. $\int \sin^m x \cos^5 x \, dx$.

$$= \int \sin^m x \cos^4 x \cos x \, dx$$

u-substitution

$$= \int \sin^m x (1-\sin^2 x)^2 \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$$= \int u^m (1-u^2)^2 \, du$$

$$= \int u^m (1-2u^2+u^4) \, du$$

$$= \int u^m - 2u^{m+2} + u^{m+4} \, du$$

$$= \frac{u^{m+1}}{m+1} - \frac{2u^{m+3}}{m+3} + \frac{u^{m+5}}{m+5} + C = \frac{\sin^{m+1} x}{m+1} - \frac{2\sin^{m+3} x}{m+3} + \frac{\sin^{m+5} x}{m+5} + C$$

1. **Strategy for Evaluating** $\int \sin^m x \cos^{2k+1} x \, dx$.

$$= \int \sin^m x \cos^{2k} x \cos x \, dx$$

$$= \int \sin^m x (1-\sin^2 x)^k \cos x \, dx$$

u-substitution

Let $u = \sin x$

$du = \cos x \, dx$

$$= \int u^m (1-u^2)^k \, du$$

$\vdots$ expand.

Example 5. (practice) $\int \cos^m x \sin^5 x \, dx$.

$$= \int \cos^m x \sin^4 x \sin x \, dx$$

U-substitution

Let $u = \cos x$

$du = -\sin x \, dx$

$$= \int \cos^m x \,(1-\cos^2 x)^2 \sin x \, dx$$

$$= \int u^m (1-u^2)^2 \, du$$

$$= -\int u^m - 2u^{m+2} + u^{m+4} \, du$$

$$= \frac{-u^{m+1}}{m+1} + \frac{2u^{m+3}}{m+3} - \frac{u^{m+5}}{m+5} + C$$

$$= \frac{-\cos^{m+1} x}{m+1} + \frac{2\cos^{m+3} x}{m+3} - \frac{\cos^{m+5} x}{m+5} + C$$

2, **Strategy for Evaluating** $\int \cos^m x \sin^{2k+1} x \, dx$.

$$= \int \cos^m x \sin^{2k} x \sin x \, dx$$

U-substitution

Let $u = \cos x$

$du = -\sin x \, dx$

$$= \int \cos^m x \,(1-\cos^2 x)^k \sin x \, dx$$

$$= -\int u^m (1-u^2)^k \, du$$

expand.

Example 6. $\int \sin^2 x \, dx.$

use $\sin^2 x = \frac{1-\cos 2x}{2}$

$$= \int \frac{1-\cos 2x}{2} dx$$

$$= \frac{1}{2} \int 1-\cos 2x \, dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Example 7. $\int \sin^4 x \, dx.$

$$= \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 dx$$

use $\cos^2 x = \frac{1+\cos 2x}{2}$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x \, dx$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x) \, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}\left(x + \frac{\sin 4x}{4}\right) + C$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Example 8. $\int \cos^2 x \sin^4 x \, dx$.

$$= \int (\cos x \sin x)^2 \sin^2 x \, dx$$

$$= \int \frac{1}{4} \sin^2 2x \; \frac{1}{2}(1-\cos 2x)\, dx$$

$$= \frac{1}{8} \int \sin^2 2x \, dx - \frac{1}{8}\int \sin^2 2x \cos 2x \, dx$$

$$= \frac{1}{8} \int \frac{1}{2}(1-\cos 4x)\, dx - \frac{1}{8} \int \sin^2 2x \left(\frac{1}{2} d \sin 2x\right)$$

$$= \frac{1}{16}\left(1 - \frac{\sin 4x}{4}\right) - \frac{1}{16}\,\frac{\sin^3 2x}{3} + C$$

3. Strategy for Evaluating $\int \cos^{2m} x \sin^{2n} x \, dx$.

use $\sin x \cos x = \frac{1}{2} \sin 2x$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Part 2. Evaluate $\int \tan^m x \sec^n x \, dx$ for any m, n.

$\sec x = \frac{1}{\cos x}$

Some identities we need to use:

$$\sec^2 x = 1 + \tan^2 x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

Example 9. $\int \tan^4 x \sec^4 x \, dx$.

$$= \int \tan^4 x \sec^2 x \sec^2 x \, dx$$

U-substitution

Let $u = \tan x$

$du = \sec^2 x \, dx$

$$= \int \tan^4 x \, (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int u^4 (1 + u^2) \, du$$

$$= \int u^4 + u^6 \, du$$

$$= \frac{u^5}{5} + \frac{u^7}{7} + C$$

$$= \frac{1}{5} \tan^5 x + \frac{1}{7} \tan^7 x + C$$

Example 10. (practice) $\int \tan^6 x \sec^4 x \, dx$.

$$= \int \tan^6 x \sec^2 x \sec^2 x \, dx$$

u-substitution

$$= \int \tan^6 x (1+\tan^2 x) \sec^2 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

$$= \int u^6 (1+u^2) \, du$$

$$= \int u^6 + u^8 \, du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

4. Strategy for Evaluating $\int \tan^m x \sec^{2k} x \, dx$.

$$= \int \tan^m x \sec^{2k-2} x \sec^2 x \, dx$$

$$= \int \tan^m x (1+\tan^2 x)^{k-1} \sec^2 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

$$= \int u^m (1+u^2)^{k-1} \, du$$

$\vdots$

Example 11. $\int \tan^3 x \sec^5 x \, dx.$

$$= \int \tan^2 x \sec^4 x \tan x \sec x \, dx$$

Let $u = \sec x$

$du = \tan x \sec x \, dx$

$$= \int (\sec^2 x - 1) \sec^4 x \tan x \sec x \, dx$$

$$= \int (u^2 - 1) u^4 \, du$$

$$= \int u^6 - u^4 \, du$$

$$= \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$$

5. **Strategy for Evaluating** $\int \tan^{2k+1} x \sec^m x \, dx.$

$$= \int \tan^{2k} x \sec^{m-1} x \tan x \sec x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{m-1} x \tan x \sec x \, dx$$

Let $u = \sec x$

$du = \tan x \sec x \, dx$

$$= \int (u^2 - 1)^k u^{m-1} \, du$$

Example 12. (practice) $\int \tan^5 x \sec^8 x \, dx$.

$$= \int \tan^4 x \sec^7 x \tan x \sec x \, dx$$

$$= \int (\sec^2 x - 1)^2 \sec^7 x \tan x \sec x \, dx$$

Let $u = \sec x$

$du = \tan x \sec x \, dx$

$$= \int (u^2 - 1)^2 u^7 \, du$$

$$= \int (u^4 - 2u^2 + 1) u^7 \, du$$

$$= \int u^{11} - 2u^9 + u^7 \, du$$

$$= \frac{u^{12}}{12} - \frac{2u^{10}}{10} + \frac{u^8}{8} + C$$

$$= \frac{\sec^{12} x}{12} - \frac{\sec^{10} x}{5} + \frac{\sec^8 x}{8} + C$$

Some more formulas we may need:

$$\int \tan x \, dx = \ln|\sec x| + C$$

Practice: Find the above formula.

Example 13. $\int \tan^3 x \, dx$.

$$= \int \tan x \tan^2 x \, dx$$

$$= \int \tan x (\sec^2 x - 1) \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int u \, du - \ln|\sec x| + C$$

$$= \frac{u^2}{2} - \ln|\sec x| + C$$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C.$$