Recall in §5.5,

> We have the u-**Substitution Rule** for indefinite integral:
> $$\int f(u(x))u'(x)\,dx = F(u(x)) + C$$
> where $F'(u) = f(u)$.

- **Trick:** find the right u function.
- Recall that we can look at u-Substitution Rule as the "backwards" of the chain rule, or the "anti-chain rule".
- Now, in this chapter, we will learn the backwards of the product rule, or consider it as "anti-product rule".

The **product rule** for derivative is

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$$

Or write it short by omitting the variable, $u = u(x)$, $v = v(x)$,

$$(uv)' = u'v + uv'$$

So, we can write it as

$$uv' = (uv)' - u'v$$

Integrating both sides, we have the formula for Integration by Parts.

> • Integration by Parts:
> $$\int uv'dx = uv - \int u'vdx$$
> or
> $$\int udv = uv - \int vdu$$

- **Trick:** Find the right u and v functions.

[1] This chapter is another difficult and technical chapter.

When you solve a problem, follow the steps!

Example 1. Find $\int x \sin x \, dx$

Step 1 Choose $u = x \quad v' = \sin x$

Step 2 Then $u' = 1 \quad v = -\cos x$ any anti-derivative of v'

Integration by Parts:

Step 3: $\int x \sin x \, dx = uv - \int u' v \, dx$

$$= x(-\cos x) - \int -\cos x \, dx$$

$$= -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x + C$$

Example 2. (practice) Find $\int x \cos 2x \, dx$

Step 1 Choose $u = x \quad v' = \cos 2x$

Step 2 Then $u' = 1 \quad v = \frac{1}{2}\sin 2x$

Integration by parts:

Step 3. $\int x \cos x \, dx = uv - \int u' v \, dx$

$$= \frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \, dx$$

$$= \frac{1}{2} x \sin 2x - \frac{1}{4}(-\cos 2x) \, dx$$

$$= \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x .$$

Example 3. Find $\int \ln x \, dx$

Step1 Choose $u = \ln x$ $v' = 1$

Step2 Then $u' = \frac{1}{x}$ $v = x$

Integration by parts:

Step3 $\int \ln x \, dx = uv - \int u'v \, dx$

$$= x \ln x - \int \frac{1}{x} \cdot x \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

Together with the Fundamental Theorem of Calculus, we have

$$\int_a^b uv' dx = uv\Big|_a^b - \int_a^b u'v dx$$

Example 4. (practice) Show that $\int_1^e \ln x \, dx = 1$

$$\int_1^e \ln x \, dx = x \ln x - x \Big|_1^e$$

$$= (e \ln e - e) - (1 \ln 1 - 1)$$

$$= (e - e) - (0 - 1)$$

$$= 1$$

Example 5. Find $\int x^2 e^x \, dx$

Choose $u = x^2$ $v' = e^x$

Then $u' = 2x$ $v = e^x$

Integration by parts

$$\int x^2 e^x dx = uv - \int u'v \, dx$$

$$= x^2 e^x - \boxed{\int 2x e^x dx}$$

we need integration by parts AGAIN

Choose $u = 2x$ $v' = e^x$

$u' = 2$ $v = e^x$

$$\boxed{\int 2x e^x dx} = uv - \int u'v \, dx$$

$$= 2x e^x - \int 2 e^x dx$$

$$= 2x e^x - 2e^x + C$$

So $\int x^2 e^x dx = x^2 e^x - (2x e^x - 2e^x + C)$

$$= x^2 e^x - 2x e^x + 2e^x \underbrace{- C}_{}$$

(or $+C$)

Example 5'. Find $\int_0^1 \frac{x}{e^x}\,dx = \int x e^{-x}\,dx$

Choose $u = x \qquad v' = e^{-x}$

Then $u' = 1 \qquad v = -e^{-x}$

Integration by Parts

$$\int_0^1 x e^{-x}\,dx = uv\Big|_0^1 - \int_0^1 u'v\,dx$$

$$= x(-e^{-x})\Big|_0^1 - \int_0^1 -e^{-x}\,dx$$

$$= -xe^{-x}\Big|_0^1 - e^{-x}\Big|_0^1$$

$$= (-e^{-1} - 0) - (e^{-1} - 1)$$

$$= -2e^{-1} + 1$$

Example 6. Find $\int e^x \sin x \, dx$

Try to use
$u = \sin x \quad v = e^x$

Choose $u = e^x \quad v' = \sin x$

Then $u' = e^x \quad v = -\cos x$

Integration by Parts:

$$\int e^x \sin x \, dx = uv - \int u'v \, dx$$
$$= -e^x \cos x + \boxed{\int e^x \cos x \, dx}$$

• Choose $u = e^x \quad v' = \cos x$

$u' = e^x \quad v = \sin x$

Integration by Parts:

$$\boxed{\int e^x \cos x \, dx} = uv - \int u'v \, dx$$
$$= e^x \sin x - \int e^x \sin x \, dx$$

plug in the red box to the first formula

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

So $2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$

So $\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$

Example 7. (practice) Find $\int e^{2x} \cos x \, dx$

Choose $u = e^{2x}$ $v' = \cos x$

Then $u' = 2e^{2x}$ $v = \sin x$

Integration by Parts:

$$\int e^{2x} \cos x \, dx = uv - \int u'v \, dx$$

$$= e^{2x} \sin x - 2 \boxed{\int e^{2x} \sin x \, dx}$$

Choose $u = e^{2x}$ $v' = \sin x$

Then $u' = 2e^{2x}$ $v = -\cos x$

$$\boxed{\int e^{2x} \sin x \, dx} = -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx$$

Plug in, then

$$\int e^{2x} \cos x \, dx = e^{2x} \sin x + 2 e^{2x} \cos x - 4 \int e^{2x} \cos x \, dx$$

So $5 \int e^{2x} \cos x \, dx = e^{2x} (\sin x + 2\cos x)$

So $\int e^{2x} \cos x \, dx = \frac{1}{5} e^{2x} (\sin x + 2\cos x)$

Notation:
$\tan^{-1}x = \arctan x$

Example 8. Find $\displaystyle\int_0^1 \tan^{-1} x\, dx$

Choose $u = \tan^{-1} x$ $\quad v' = 1$

Then $u' = \dfrac{1}{1+x^2}$ $\quad v = x$

Review

Integration by parts:

$$\int_0^1 \tan^{-1} x\, dx = x\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \tan^{-1} 1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - \boxed{\int_0^1 \frac{x}{1+x^2}\,dx}$$

u-substitution:

Let ① $u = 1+x^2$, $du = 2x\,dx$, $dx = \dfrac{1}{2x}\,du$.

② $u(0) = 1+0^2 = 1$, $u(1) = 1+1^2 = 2$

③ $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx = \int_1^2 \frac{x}{u}\,\frac{1}{2x}\,du = \frac{1}{2}\ln u\Big|_1^2 = \frac{1}{2}\ln 2$

So $\displaystyle\int_0^1 \tan^{-1} x\, dx = \frac{\pi}{4} - \frac{1}{2}\ln 2$

Example 9. (practice) Find $\int \sin^{-1} x \, dx$

Choose $u = \sin^{-1} x \quad v' = 1$

Then $u' = \frac{1}{\sqrt{1-x^2}} \quad v = x$

Integration by Parts

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \boxed{\int \frac{x}{\sqrt{1-x^2}} \, dx}$$

u-substitution

Let $u = 1 - x^2$

$du = -2x \, dx$

$dx = -\frac{1}{2x} \, du$

$$\int \frac{x}{\sqrt{1-x^2}} \, dx = \int \frac{x}{\sqrt{u}} \cdot \left(-\frac{1}{2x}\right) du$$

$$= -\frac{1}{2} \int u^{-\frac{1}{2}} \, du$$

$$= \left(-\frac{1}{2}\right) \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= -\sqrt{1-x^2} + C$$

So, $\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} \pm C$

Example 10. Find $\int \sin \sqrt{x}\, dx$

Step 1. u-substitution: Let $u = \sqrt{x} = x^{\frac{1}{2}}$

$$du = \frac{1}{2} x^{-\frac{1}{2}} dx$$

$$dx = 2x^{\frac{1}{2}} du$$

$$\int \sin\sqrt{x}\, dx = \int (\sin u)\, 2x^{\frac{1}{2}} du = 2 \boxed{\int u \cdot \sin u \, du}$$

Example 1

u is the variable!

Step 2. Choose $f = u \quad g' = \sin u$

$$f' = 1 \quad g' = -\cos u$$

Integration by parts

$$\int u \sin u \, du = -u\cos u - \int -\cos u \, du$$

$$= -u\cos u + \sin u + C$$

Step 3

So $$\int \sin\sqrt{x}\, dx = -2u\cos u + 2\sin u + \underbrace{2C}_{\text{ok}}$$

$$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$$

Example 11. Find $\int$ [redacted] $\sin^3 x \, dx = \int \sin^2 x \sin x \, dx$

Method 1: Choose $u = \sin^2 x$ $v' = \sin x$

Then $u' = 2 \sin x \cos x$ $v = -\cos x$

Integration by parts:

$$\int \sin^3 x \, dx = uv - \int u'v \, dx = -\sin^2 x \cos x + 2\int \sin x \cos^2 x \, dx$$

$$= -\sin^2 x \cos x + 2 \int \sin x (1 - \sin^2 x) \, dx$$

$$= -\sin^2 x \cos x + 2 \int \sin x \, dx - 2 \int \sin^3 x \, dx$$

So $3 \int \sin^3 x \, dx = -\sin^2 x \cos x - 2 \cos x + C$

So $\int \sin^3 x \, dx = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + \underset{\text{ok } (\frac{1}{3}C)}{\underline{\underline{C}}}$

Remark: This method can be generalized to compute $\int \sin^n x \, dx$

Method 2: $\int \sin^3 x \, dx = \int (1 - \cos^2 x) \sin x \, dx = \int \sin x \, dx - \int \cos^2 x \sin x \, dx$

u-subsitution $u = \cos x$

$$= -\cos x + \frac{\cos^3 x}{3} + C$$

Example 12. (practice) Find $\int x^2 \ln x \, dx$

Choose $u = \ln x$ $\quad v' = x^2$

Then $u' = \frac{1}{x}$ $\quad v = \frac{x^3}{3}$

Integration by parts

$$\int x^2 \ln x \, dx = uv - \int u'v \, dx$$
$$= \frac{x^3}{3}\ln x - \int \frac{1}{x}\cdot\frac{x^3}{3}\, dx$$
$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx$$
$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$