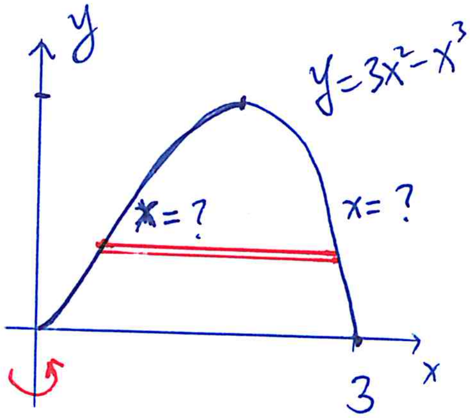
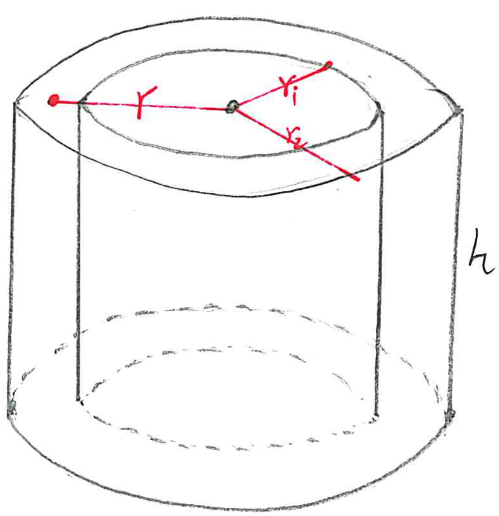
Example 1. The region R is in the first quadrant enclosed by the curves $y = 3x^2 - x^3$ and $y = 0$. Find the volume of the solid obtained by rotating R about the y-axis.



$A = \pi\,(\underline{\text{outer radius}})^2 - \pi\,(\underline{\text{inner radius}})^2$

we need to solve $x = (?)$, but we can't.

Method of cylindrical shells:



$$V = V_{out} - V_{in}$$
$$= \pi r_2^2 h - \pi r_1^2 h$$
$$= \pi (r_2^2 - r_1^2) h$$
$$= \pi (r_2 + r_1)(r_2 - r_1) h$$
$$= 2\pi \underbrace{\left(\frac{r_2 + r_1}{2}\right)}_{r} h \underbrace{(r_2 - r_1)}_{\Delta r}$$

The volume of a cylindrical shell is

$$\boxed{V = 2\pi r h \Delta r}$$

The volume of the solid is given by the sum of the volumes of the shells

$$V \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x.$$

The volume of the solid S obtained by rotating about the y-axis the region R under the curve $y = f(x)$ from a to b, is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x.$$

Hence, using the definition of definite integral

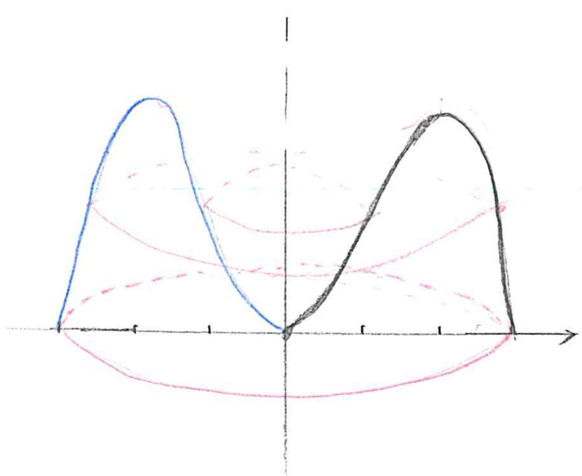
$$\boxed{V = \int_a^b 2\pi x f(x) dx}$$

Example 1.

$3x^2 - x^3 = 0$

$x^2(3-x) = 0$

$x = 0$ or 3



$$V = \int_0^3 2\pi x (3x^2 - x^3)\, dx$$

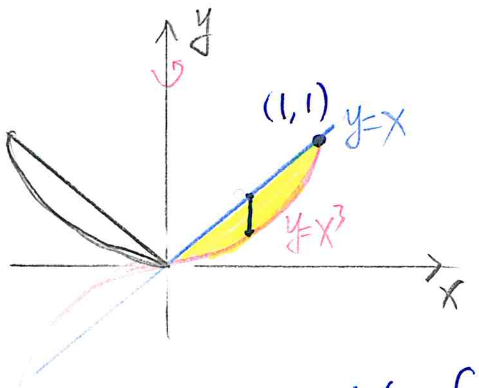
$$= \int_0^3 6\pi x^3 - 2\pi x^4\, dx$$

$$= \frac{6\pi x^4}{4} - \frac{2\pi x^5}{5} \Big|_0^3$$

$$= \frac{243\pi}{10}$$

Ex 6 in §6.2

Example 2. The region R is in the first quadrant enclosed by the curves $y = x$ and $y = x^3$. Use cylindrical shells, find the volume of the solid obtained by rotating R about the y-axis.



$x = x^3$

$x(1-x^2) = 0$

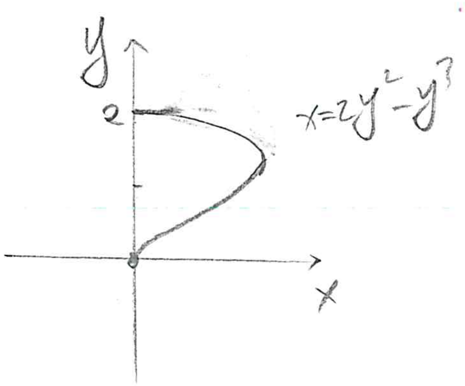
$x=0$ or $x=1$

$$V = \int_0^1 2\pi x(x - x^3)\,dx$$

$$= \int_0^1 2\pi x^2 - 2\pi x^4\,dx$$

$$= 2\pi\frac{x^3}{3} - 2\pi\frac{x^5}{5}\Big|_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{5}\right) = 2\pi\left(\frac{2}{15}\right)$$

Example 3. The region R is enclosed by the curves $x = 2y^2 - y^3$ and $x = 0$. Use cylindrical shells, find the volume of the solid obtained by rotating R about the x-axis.



$2y^2 - y^3 = 0$

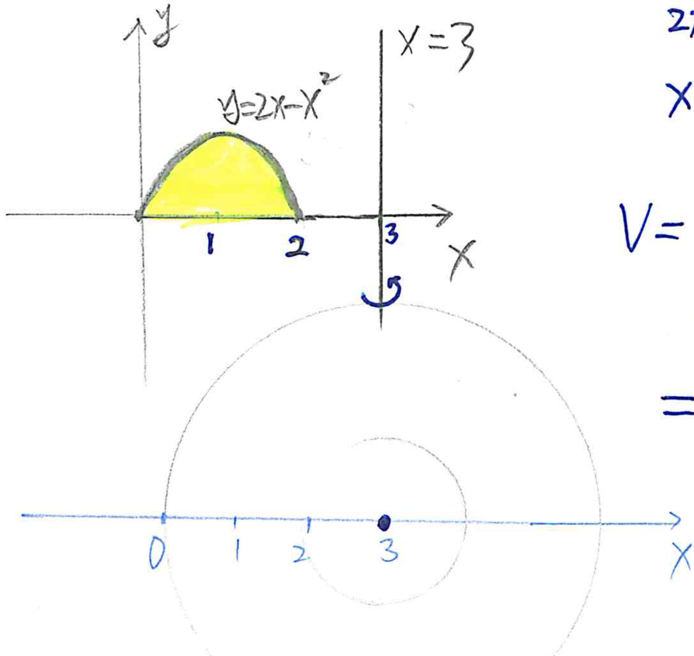
$y^2(2-y) = 0$ $y=0$ or $y=2$

$$V = \int_0^2 2\pi y(2y^2 - y^3)\,dy$$

$$= \int_0^2 4\pi y^3 - 2\pi y^4\,dy$$

$$= \pi y^4 - \frac{2\pi y^5}{5}\Big|_0^2 = \frac{16\pi}{5}$$

Example 4. Find the volume of the solid obtained by rotating the region bounded by $y = 2x - x^2$ and $y = 0$ about the line $x = 3$.



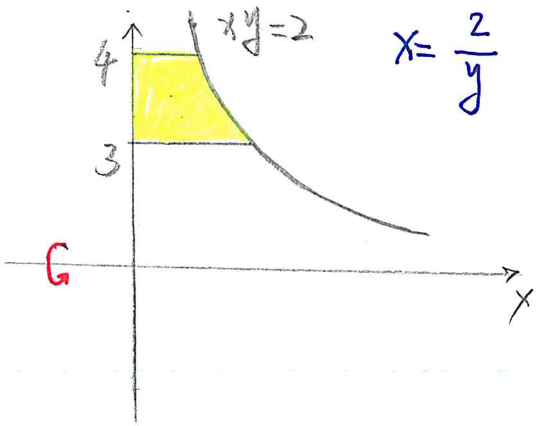
$2x - x^2 = 0$

$x(2-x) = 0 \quad x = 0 \text{ or } x = 2$

$$V = \int_0^2 2\pi(3-x)(2x-x^2)\,dx$$

$$= 2\pi \int_0^2 6x - 5x^2 + x^3\,dx$$

Example 5. The region R is in the first quadrant enclosed by the curves $xy = 2$, $x = 0$, $y = 3$, $y = 4$. Use cylindrical shells method to find the volume of the solid obtained by rotating R about the x-axis.



$x = \frac{2}{y}$

$$V = \int_3^4 2\pi y \left(\frac{2}{y}\right) dy$$

$$= \int_3^4 4\pi\,dy$$

$$= 4\pi y \Big|_3^4$$

$$= 4\pi$$