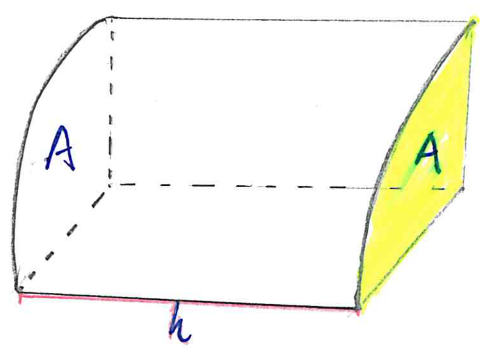
Find the volume of a solid S between $a \leq x \leq b$.

1. The volume of the cylinder is defined as

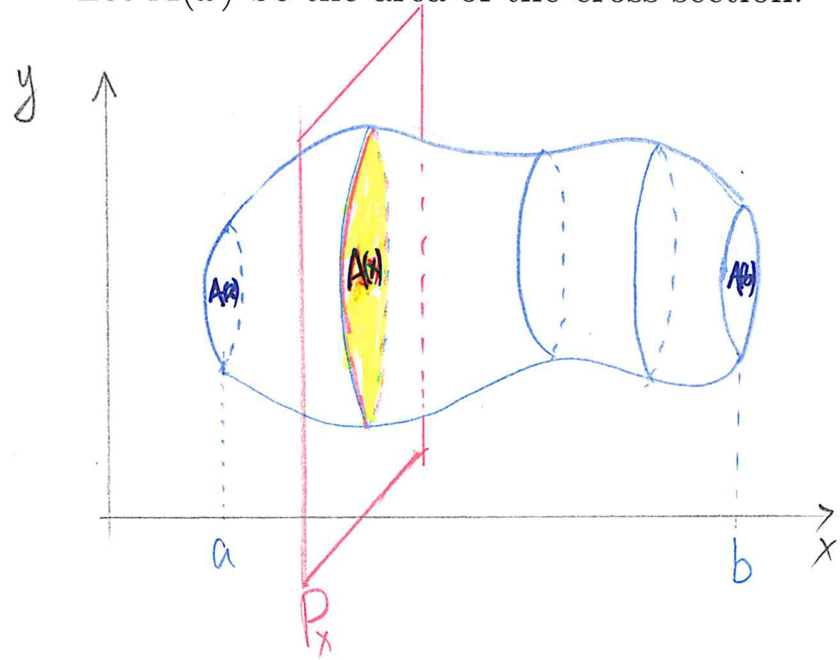
$$V = Ah$$

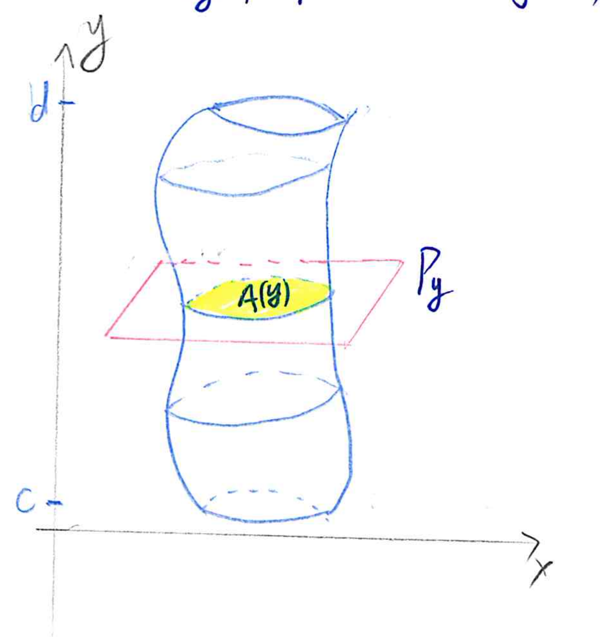
where, the area of the base is A and the height of the cylinder is h.



The **cross-section** of S is the intersection of a plane P_x perpendicular to x-axis with S.
Let $A(x)$ be the area of the cross-section.

(or P_y perpendicular to y-axis)





We divide $[a, b]$ by n parts. Then we also divide the solid S by n parts S_i. Each part S_i can be approximated by a cylinder, with volume

$$V(S_i) \approx A(x_i^*)\Delta x.$$

The volume of S can be approximated by the sum

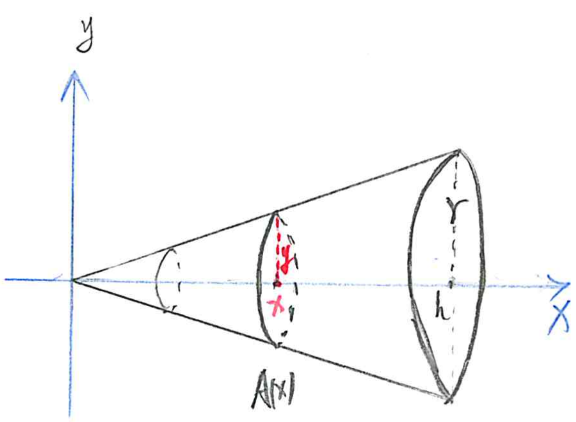
$$V(S_i) \approx \sum_{i=1}^{n} A(x_i^*)\Delta x.$$

If $A(x)$ is a continuous function, the **volume** of S is defined by the limit of Riemann sums

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx$$

Example 1. Let S be a cone of radius r and height h. Show that the volume of S is $V = \frac{1}{3}\pi r^2 h$.

$0 \leq x \leq h$



$\frac{y}{r} = \frac{x}{h}$ $\quad y = \frac{rx}{h}$

$A(x) = \pi y^2 = \frac{\pi r^2 x^2}{h^2}$

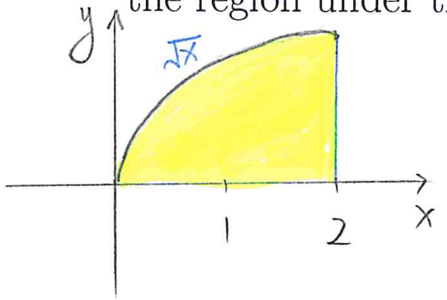
$\text{Volume} = \int_0^h \frac{\pi r^2 x^2}{h^2}\, dx$

$= \frac{\pi r^2}{h^2} \cdot \frac{x^3}{3}\Big|_0^h$

$= \frac{1}{3}\pi r^2 h.$

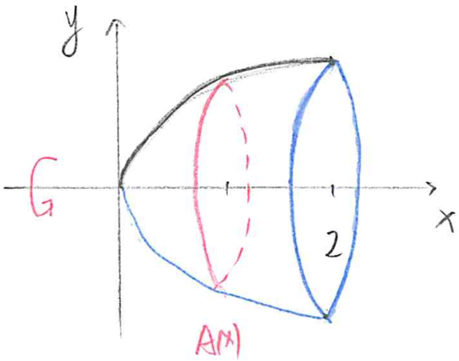
$A = \pi\,(\text{radius})^2$

Example 2. Let S be a solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 2.



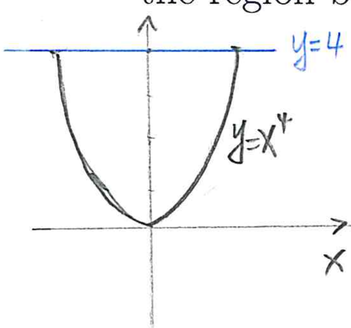
$A(x) = \pi y^2 = \pi(\sqrt{x})^2 = \pi x$

$$\text{Volume} = \int_0^2 A(x)\,dx = \int_0^2 \pi x\,dx = \pi \frac{x^2}{2}\Big|_0^2 = 2\pi$$



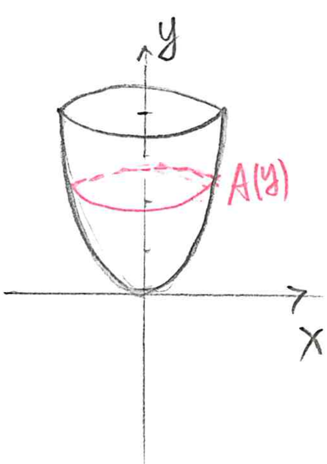
Example 3. Let S be a solid obtained by rotating about the y-axis the region bounded by $y = x^4$, $y = 4$.

$0 \le y \le 4$



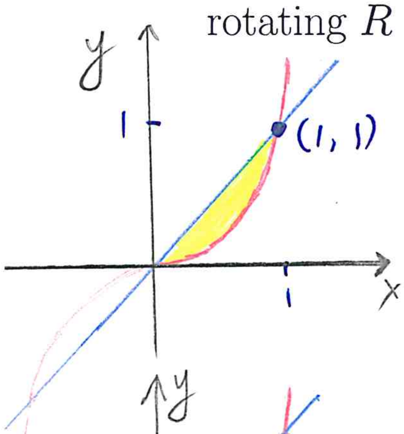
$A(y) = \pi x^2 = \pi(y^{\frac{1}{4}})^2 = \pi y^{\frac{1}{2}}$

$$\text{Volume} = \int_0^4 A(y)\,dy = \int_0^4 \pi y^{\frac{1}{2}}\,dy = \frac{2}{3}\pi y^{\frac{3}{2}}\Big|_0^4 = \frac{16}{3}\pi$$



$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$

Example 4. The region R is in the first quadrant enclosed by the curves $y = x$ and $y = x^3$. Find the volume of the solid obtained by rotating R about the x-axis.



intersection

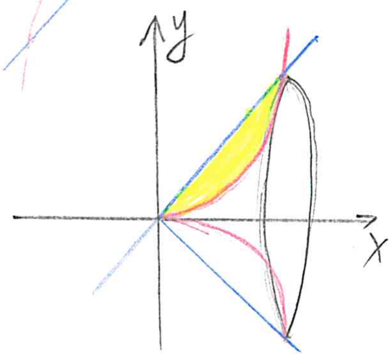
$x = x^3$

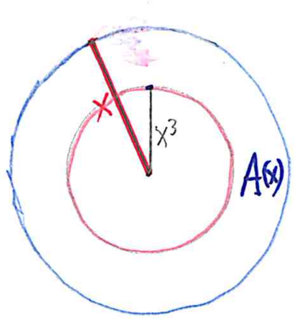
$x(x^2-1) = 0$

$x = -1,\ 0,\ 1.$

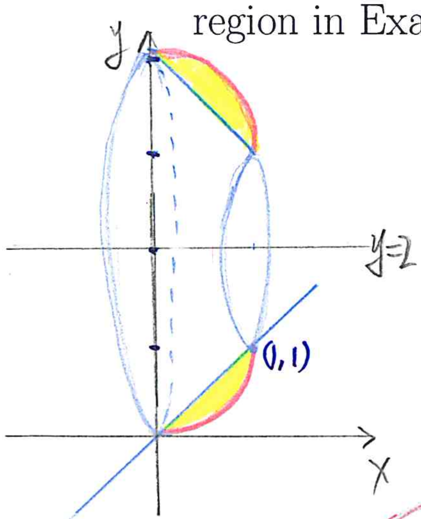
$$A(x) = \pi x^2 - \pi(x^3)^2 = \pi(x^2 - x^6)$$

$$\text{Volume} = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^6)\,dx = \pi\left(\frac{x^3}{3} - \frac{x^7}{7}\right)\Big|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{7}\right) = \frac{4}{21}\pi$$



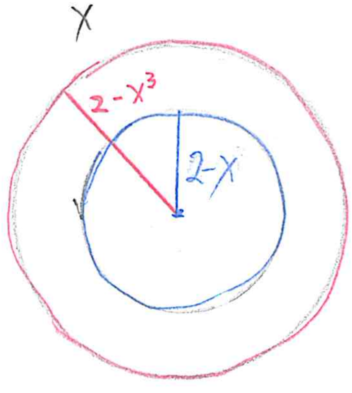


Example 5. Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.



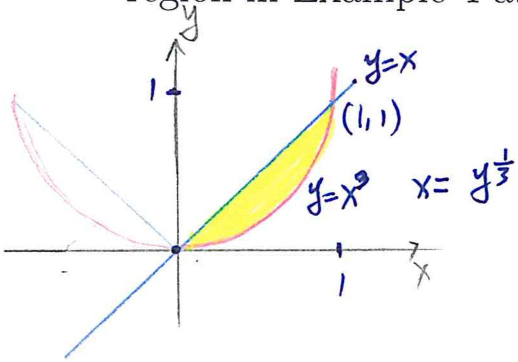
$$A(x) = \pi(2-x^3)^2 - \pi(2-x)^2 = \pi(x^6 - 4x^3 - x^2 + 4x)$$

$$\text{Volume} = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^6 - 4x^3 - x^2 + 4x)\,dx = \pi\left(\frac{x^7}{7} - x^4 - \frac{x^3}{3} + 2x^2\right)\Big|_0^1 = \pi\left(\frac{1}{7} - 1 - \frac{1}{3} + 2\right) = \frac{11}{21}\pi$$



cross-section

Example 6. Find the volume of the solid obtained by rotating the region in Example 4 about y-axis.



$$A(y) = \pi\left(y^{\frac{1}{3}}\right)^2 - \pi\left(y^2\right)$$

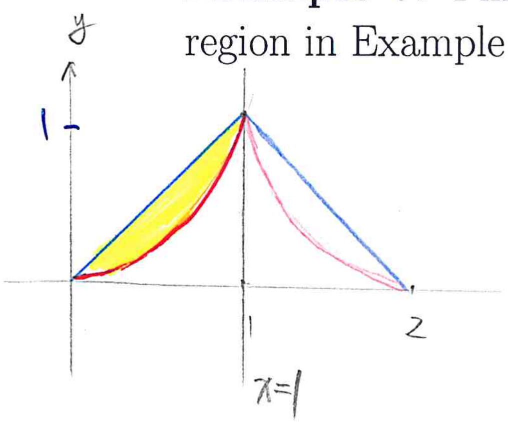
$$\text{Volume} = \int_0^1 A(y)\,dy$$

$$= \int_0^1 \pi\left(y^{\frac{2}{3}} - y^2\right) dy$$

$$= \pi\left(\frac{3}{5}y^{\frac{5}{3}} - \frac{y^3}{3}\right)\Big|_0^1$$

$$= \frac{4}{15}\pi$$

Example 7. Find the volume of the solid obtained by rotating the region in Example 4 about the line $x = 1$.



$$A(y) = \pi(1-y)^2 - \pi\left(1 - y^{\frac{1}{3}}\right)^2$$

$$\text{Volume} = \int_0^1 A(y)\,dy$$

$$= \int_0^1 \pi\left(y^2 - 2y - y^{\frac{2}{3}} + 2y^{\frac{1}{3}}\right) dy$$

$$= \pi\left(\frac{y^3}{3} - y^2 - \frac{3}{5}y^{\frac{5}{3}} + \frac{3}{2}y^{\frac{4}{3}}\right)\Big|_0^1$$

$$= \pi\left(\frac{1}{3} - 1 - \frac{3}{5} + \frac{3}{2}\right)$$

$$= \frac{7}{30}\pi$$