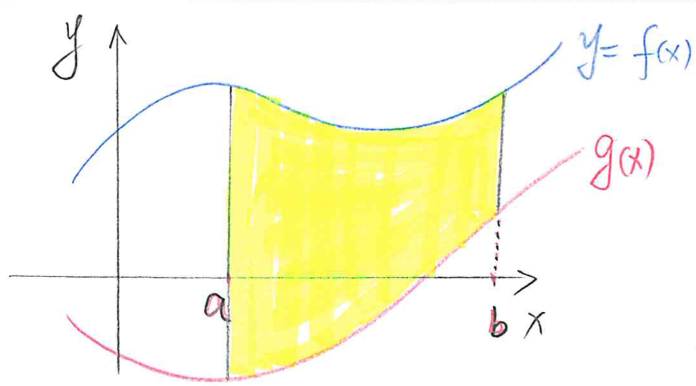
We can use definite integral $\int_a^b f(x)\,dx$ to calculate area of region under the graph of a **positive** function $f(x)$ and above the x-axis.

Now, we use integral to calculate area of region that lies between the graphs of two continuous functions $f(x)$ and $g(x)$.

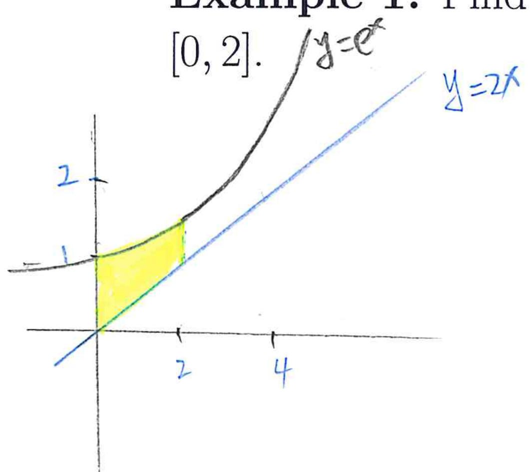
> If $\boxed{f(x) \geq g(x)}$ in the interval $[a, b]$, the **area** A of the region bounded by the curves $f(x)$, $g(x)$ and the lines $x = a$, $x = b$ can be computed by
>
> $$A = \int_a^b [f(x) - g(x)]\,dx$$



Graphs

- lines $y=ax+b$
- parabolas $y=ax^2+bx+c$
- $y=\sin x$ • $y=x^3$ • $y=\sqrt{x}$
- $y=\cos x$ • $y=e^x$ • $y=\ln x$

Example 1. Find the area between $y = e^x$ and $y = 2x$ bounded by $[0, 2]$.



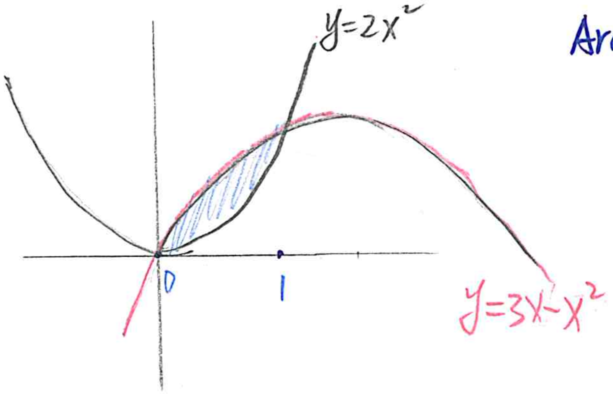
$$\text{Area} = \int_0^2 e^x - 2x\,dx$$

$$= e^x - x^2 \Big|_0^2$$

$$= e^2 - 4 - (e^0 - 0)$$

$$= e^2 - 5$$

Example 2. Sketch the region bounded by $y = 2x^2$ and $y = 3x - x^2$, and find the area.



intersection $2x^2 = 3x - x^2$

$3x^2 - 3x = 0$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ or 1

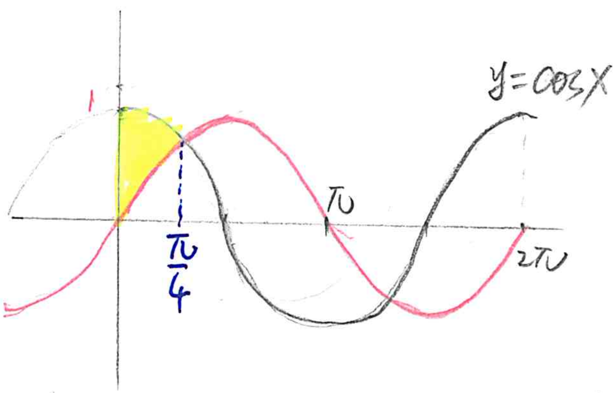
$$\text{Area} = \int_0^1 3x - x^2 - 2x^2 \, dx$$

$$= 3\int_0^1 x - x^2 \, dx$$

$$= 3\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= \frac{1}{2}$$

Example 3. Sketch the region enclosed by $y = \sin x$ and $y = \cos x$ over $[0, \pi/4]$, and find the area.



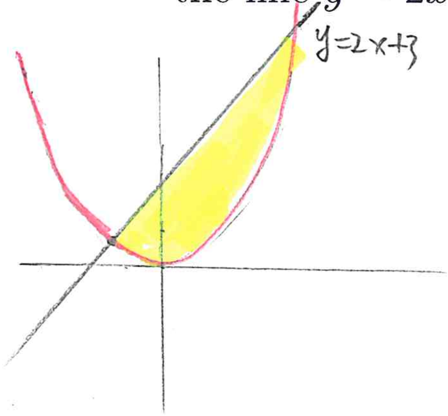
$$\text{Area} = \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx$$

$$= \sin x + \cos x \Big|_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1)$$

$$= \sqrt{2} - 1$$

Example 4. Sketch the region enclosed by the parabola $y = x^2$ and the line $y = 2x + 3$, and find the area.



intersection

$x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = -1$ or 3

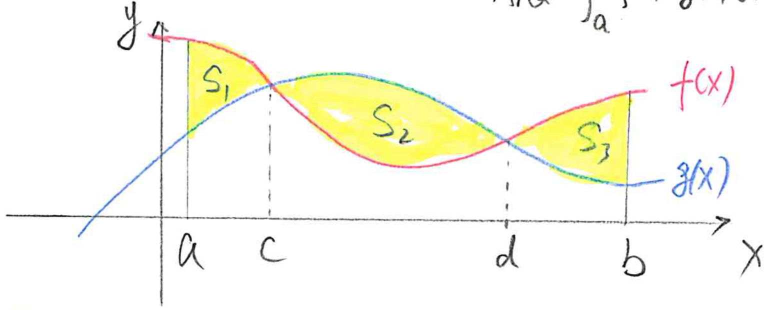
$$\text{Area} = \int_{-1}^{3} 2x + 3 - x^2 \, dx$$

$$= x^2 + 3x - \frac{x^3}{3}\Big|_{-1}^{3}$$

$$= \frac{32}{3}$$

If we don't have the assumption $f(x) \geq g(x)$, how should we do?
We break the region in several parts.

$$\text{Area} = \int_a^c f(x)-g(x)\,dx + \int_c^d g(x)-f(x)\,dx + \int_d^b f(x)-g(x)\,dx$$

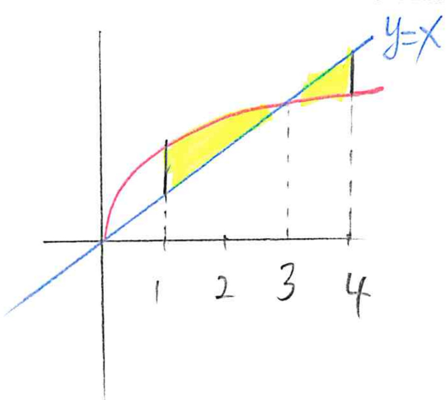


> The **area** A of the region bounded by the curves $f(x)$, $g(x)$ and the lines $x = a$, $x = b$ can be computed by
>
> $$A = \int_a^b |f(x) - g(x)|\,dx$$

Here $|f(x) - g(x)|$ is the absolute value defined by

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } f(x) \leq g(x) \end{cases}$$

Example 5. Sketch the region bounded by $y = x$ and $y = \sqrt{3x}$ over $[1, 4]$, and write down the integral for calculating the area. (Do not evaluate the integral)



$$\text{Area} = \int_1^3 \sqrt{3x} - x\,dx + \int_3^4 x - \sqrt{3x}\,dx$$

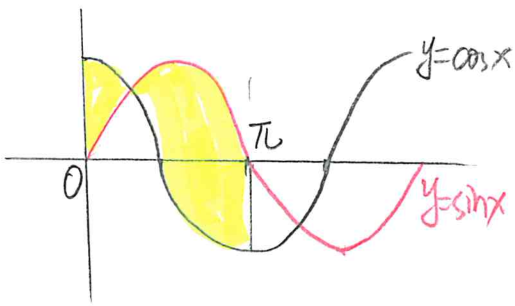
intersection

$x = \sqrt{3x}$

$x^2 = 3x$

$x = 0$ or 3

Example 6. Sketch the region enclosed by $y = \sin x$ and $y = \cos x$ over $[0, \pi]$, and find the area.



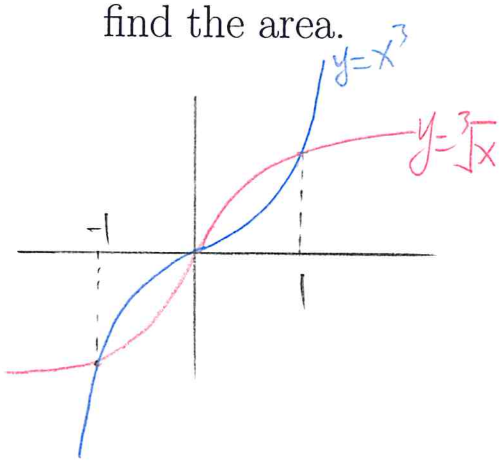
$$\text{Area} = \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \, dx$$

$$= \sin x + \cos x \Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\pi}$$

$$= \sqrt{2} - 1 + (-(-1) - 0) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= 2\sqrt{2}$$

Example 7. Sketch the region bounded by $y = x^3$ and $y = \sqrt[3]{x}$, and find the area.



$$\text{Area} = \int_{-1}^{0} x^3 - x^{\frac{1}{3}} \, dx + \int_0^1 x^{\frac{1}{3}} - x^3 \, dx$$

intersection

$x^3 = \sqrt[3]{x}$

$x^9 = x$

$x^8 = 1$

$x = 1$ or -1

$$= \frac{x^4}{4} - \frac{3}{4}x^{\frac{4}{3}} \Big|_{-1}^{0} + \frac{3}{4}x^{\frac{4}{3}} - \frac{x^4}{4}\Big|_0^1$$

$$= -\left(\frac{1}{4} - \frac{3}{4}\right) + \left(\frac{3}{4} - \frac{1}{4}\right)$$

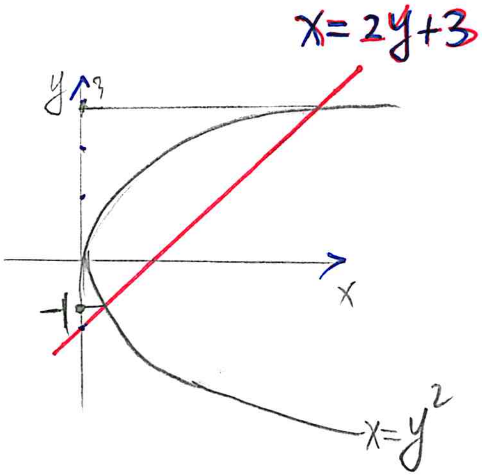
$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

Same principle holds for finding areas between $x = f(y)$ and $y = g(y)$.

> If $f(y) \geq g(y)$ in the interval $[c, d]$, the **area** A of the region bounded by the curves $f(y)$, $g(y)$ and the lines $y = c$, $y = d$ can be computed by
>
> $$A = \int_c^d [f(y) - g(y)]\, dy$$

Example 8. Sketch the region enclosed by the parabola $x = y^2$ and the line $y = x/2 - 3/2$, and find the area.



$$\text{Area} = \int_{-1}^{3} 2y+3-y^2\, dy$$

$$= y^2+3y-\frac{y^3}{3}\Big|_{-1}^{3}$$

$$= \frac{32}{3}$$

intersection

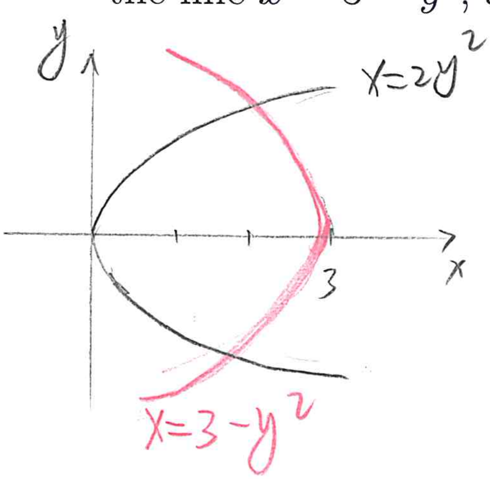
$y^2 = 2y+3$

$y^2-2y-3=0$

$(y-3)(y+1)=0$

$y=3$ or -1

Example 9. Sketch the region enclosed by the parabola $x = 2y^2$ and the line $x = 3 - y^2$, and find the area.



$$\text{Area} = \int_{-1}^{1} 3-y^2-2y^2\, dy$$

$$= \int_{-1}^{1} 3-3y^2\, dy$$

$$= 3y-\frac{3}{3}y^3\Big|_{-1}^{1}$$

$$= 4$$

intersection

$2y^2 = 3-y^2$

$3y^2 = 3$

$y^2 = 1$

$y = -1$ or 1