Recall: If $F'(x) = f(x)$, then $F(x)$ is an antiderivative of $f(x)$. The **indefinite integral** (general antiderivative) of $f(x)$ is

$$\boxed{\int f(x)\,dx = F(x) + C}$$

where C is a arbitrary constant number.

Example 1. Find the indefinite integral $\int 4\sqrt{x} + 3\cos 2x\,dx$.

$$\int 4x^{\frac{1}{2}} + 3\cos 2x\,dx = 4\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3}{2}\sin 2x + C$$

$$= \frac{8}{3}x^{3/2} + \frac{3}{2}\sin 2x + C$$

§5.5. The Substitution Rule

Recall the chain rule for derivative:

$$\big(F(u(x))\big)' = F'(u(x))u'(x).$$

Example 2. (1) Compute the derivative of $F(x) = \sin(2x^3)$.

$$F'(x) = \left(\cos(2x^3)\right)\cdot 6x^2$$

(2) Compute the indefinite integral $\int 6x^2\cos(2x^3)\,dx$

$$\int 6x^2\cos(2x^3)\,dx = \sin(2x^3) + C$$

Take indefinite integral for both sides of the chain rule formula.

$$\int (F(u(x)))'\,dx = \int F'(u(x))u'(x)\,dx.$$

Suppose $F'(u) = f(u)$. Simplify both sides,

$$F(u(x)) + C = \int F'(u(x))u'(x)\,dx.$$

Then, we have the **Substitution Rule** for indefinite integral:

$$\int f(u(x))u'(x)\,dx = F(u(x)) + C$$

where $F'(u) = f(u)$.

- Choose $u(x)$ such that the integral may be rewritten in the desired form $f(u(x))u'(x)$. Find the composition function.

Example 3. Find the indefinite integral $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Let $u(x) = \sqrt{x} = x^{\frac{1}{2}}$. Then, $du = u'\,dx = \frac{1}{2}x^{-\frac{1}{2}}\,dx$.

so, $dx = 2x^{\frac{1}{2}}\,du$.

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\sin u}{\sqrt{x}}\,2x^{\frac{1}{2}}\,du = 2\int \sin u\,du$$

$$= -2\cos u + C = -2\cos\sqrt{x} + C$$

Example 4. Find the indefinite integral $\int \frac{4x^3 - x}{\sqrt{2x^4 - x^2 + 5}} dx$

Let $u(x) = 2x^4 - x^2 + 5$.

Then $du = (8x^3 - 2x)\,dx$. So $dx = \frac{1}{8x^3-2x}\,du$.

$$\int \frac{4x^3-x}{\sqrt{2x^4-x^2+5}}\,dx = \int \frac{4x^3-x}{\sqrt{u}} \cdot \frac{1}{8x^3-2x}\,du = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = (2x^4-x^2+5)^{\frac{1}{2}} + C$$

$$\overset{\text{or}}{=} \sqrt{2x^4-x^2+5} + C$$

Example 5. Find the indefinite integral $\int e^{x^2+2} x\, dx$

Let $u(x) = x^2+2$.

Then $du = 2x\,dx$. So $dx = \frac{1}{2x}\,du$

$$\int e^{x^2+2} x\,dx = \int e^u x \cdot \frac{1}{2x}\,du = \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2+2} + C$$

Example 6. Find the indefinite integral $\int \tan x \, dx$ $= \int \frac{\sin x}{\cos x} dx$

Let $u(x) = \cos x$ Then $du = -\sin x \, dx$

So $dx = -\frac{1}{\sin x} du$

$$\int \tan x \, dx = \int \frac{\sin x}{u} \cdot \left(-\frac{1}{\sin x}\right) du = -\int \frac{1}{u} du$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

Example 7. Find the indefinite integral $\int \big(\sin(1 - e^x)\big) e^x \, dx$

Let $u(x) = 1 - e^x$

Then $du = -e^x dx$. So $dx = -\frac{1}{e^x} du$

$$\int (\sin(1-e^x)) e^x \, dx = \int (\sin u) \, e^x \left(-\frac{1}{e^x}\right) du$$

$$= -\int \sin u \, du = -(-\cos u) + C$$

$$= \cos(1-e^x) + C$$

method 1

Example 8. Compute the definite integral $\int_1^2 \frac{x}{\sqrt{x^2+3}}\, dx$

1. Find antiderivative using substitution.
2. Then use Fundamental Theorem of Calculus.

Then, we have the **Substitution Rule** for definite integral:

$$\int_a^b f(u(x))u'(x)\, dx = F(u(x))|_a^b = F(u(b)) - F(u(a))$$

where $F'(u) = f(u)$.

Let $u(x) = x^2+3$. Then $du = 2x\,dx$. So $dx = \frac{1}{2x}du$.

$$\int \frac{x}{\sqrt{x^2+3}}dx = \int \frac{x}{\sqrt{u}} \cdot \frac{1}{2x} du = \frac{1}{2}\int u^{-\frac{1}{2}} du = \frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= (x^2+3)^{\frac{1}{2}} + C$$

So $\int_1^2 \frac{x}{\sqrt{x^2+3}}dx = \sqrt{x^2+3}\,\Big|_1^2 = \sqrt{7} - \sqrt{4} = \sqrt{7} - 2$

method 2

Alternatively, we can use the formula:

$$\int_a^b f(u(x))u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)du$$

Let $u(x) = x^2+3$ Then $du = 2x\,dx$ So $dx = \frac{1}{2x}du$

$u(1) = 4 \quad u(2) = 7$

$$\int_1^2 \frac{x}{\sqrt{x^2+3}}dx = \int_4^7 \frac{x}{\sqrt{u}} \cdot \frac{1}{2x}du = \frac{1}{2}\int_4^7 u^{-\frac{1}{2}}du$$

$$= \frac{1}{2}\frac{u^{\frac{1}{2}}}{1/2}\Big|_4^7 = \sqrt{u}\Big|_4^7 = \sqrt{7} - 2$$

Example 9. Compute the definite integral $\int_0^{\pi/2} 3\sin^4 x \cos x \, dx$

Let $u(x) = \sin x$ Then $du = \cos x \, dx$. So $dx = \frac{1}{\cos x} du$

$u(0) = \sin 0 = 0 \qquad u(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$$\int_0^{\pi/2} 3\sin^4 x \cos x \, dx = \int_0^1 3u^4 \cos x \frac{1}{\cos x} du$$

$$= \int_0^1 3u^4 du = 3\frac{u^5}{5}\Big|_0^1 = \frac{3}{5}$$

Example 10. Compute the definite integral $\int_{-1}^{2} x^2\sqrt{x^3+1} \, dx$

Let $u(x) = x^3+1$. Then $du = 3x^2 dx$ So, $dx = \frac{1}{3x^2} du$

$u(-1) = 0 \qquad u(2) = 9$

$$\int_{-1}^{2} x^2\sqrt{x^3+1} \, dx = \int_0^9 x^2 u^{\frac{1}{2}} \frac{1}{3x^2} du$$

$$= \frac{1}{3}\int_0^9 u^{\frac{1}{2}} du$$

$$= \frac{1}{3} \cdot \frac{u^{\frac{3}{2}}}{3/2}\Big|_0^9$$

$$= \frac{2}{9}(9^{\frac{3}{2}}) = 6$$

Example 11. Compute the definite integral $\int_e^{e^2} \frac{1}{x(\ln x)^3}\, dx$

Let $u(x)=\ln x$. Then $du=\frac{1}{x}dx$ So $dx = x\,du$

$u(e)=\ln e = 1$ $u(e^2)=\ln e^2 = 2$

$$\int_e^{e^2} \frac{1}{x(\ln x)^3}\,dx = \int_1^2 \frac{1}{x\,u^3}\,x\,du = \int_1^2 u^{-3}\,du$$

$$= \frac{u^{-2}}{-2}\Big|_1^2 = -\frac{1}{2u^2}\Big|_1^2 = \left(-\frac{1}{8}\right)-\left(-\frac{1}{2}\right) = \frac{3}{8}$$

Example 12. Find the indefinite integral $\int x\sqrt{x+2}\,dx$

Let $u=x+2$ Then $du=dx$

$x=u-2$.

$$\int x\sqrt{x+2}\,dx = \int (u-2)\sqrt{u}\,du$$

$$= \int u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\,du$$

$$= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$