Recall that the Geometric Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

converges when $|x| < 1$. And when $|x| < 1$, we have

$$\boxed{\frac{1}{1-x}} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

More Generally, we want to express a function $f(x)$ as a power series:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad \text{for } x \in I.$$

Example 1. For $|x| < 1$

$$\boxed{\frac{1}{1+x}} = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

Example 2. For $|x| < 1$

$$\frac{x}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^{n+1} = x - x^2 + x^3 - x^4 + x^5 - x^6 + \cdots$$

Example 3. For $|x| < 1$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Find a power series representation for the function and Determine the interval of convergence.

Example 4. $f(x) = \dfrac{1}{2-x}$

$$f(x) = \frac{1}{2}\cdot\frac{1}{1-\left(\frac{x}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$$

interval of convergence: $\left|\frac{x}{2}\right| < 1$, $|x| < 2$

$(-2, 2)$ $\qquad -2 < x < 2$

Example 5. $f(x) = \dfrac{1}{x}$

$$f(x) = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

interval of convergence $|x-1| < 1$

$0 < x < 2$

or, $f(x) = \dfrac{1}{2+(x-2)} = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{2^{n+1}}$ $\qquad \left|\frac{x-2}{2}\right| < 1$ $\qquad 0 < x < 4$

Example 6. $f(x) = \dfrac{x}{1+9x^2}$

$$f(x) = \frac{x}{1+(3x)^2} = x\sum_{n=0}^{\infty}(-1)^n(3x)^{2n} = \sum_{n=0}^{\infty}(-1)^n\cdot 3^{2n}\cdot x^{2n+1}$$

interval of convergence $|(3x)^2| < 1$

$|3x| < 1$

$|x| < \frac{1}{3}$ $\qquad -\frac{1}{3} < x < \frac{1}{3}$

$\left(-\frac{1}{3}, \frac{1}{3}\right)$

If a function $f(x)$ can be written a power series:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad \text{for } |x-a| < R,$$

then, the derivative of $f(x)$ is

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

and the indefinite integral of $f(x)$ is

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} + C$$

Example 7. $f(x) = \dfrac{1}{(2-x)^2}$

Ex 4 $\quad \dfrac{1}{2-x} = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} \qquad -2<x<2$

$$\left(\frac{1}{2-x}\right)' = \frac{1}{(2-x)^2}$$

So $\quad f(x) = \dfrac{1}{(2-x)^2} = \displaystyle\sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{2^{n+1}}$

Example 7'. $f(x) = \dfrac{x}{(2-x)^2} = \displaystyle\sum_{n=1}^{\infty} \frac{n \cdot x^n}{2^{n+1}}$

Example 8. $f(x) = \ln(1+x)$

We know $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ $\quad |x|<1$

$$f(x) = \ln(1+x) = \int \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} (-1)^n \int x^n dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

Since $f(0) = \ln 1 = 0$, so $C = 0$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \overset{or}{=} x - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots$$

$$\overset{or}{=} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad -1<x<1$$

Example 9. $f(x) = x\ln(1+x^2)$

$$f(x) = x \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n} \qquad |x^2|<1$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{n} \qquad -1<x<1$$

$(-1, 1)$

or $f(x) = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{n+1}$$

Example 10. $f(x) = \ln(3-x)$

$$f(x) = \ln(3-x) = -\int \frac{1}{3-x}\,dx = -\frac{1}{3}\int \frac{1}{1-\left(\frac{x}{3}\right)}\,dx \qquad \left|\frac{x}{3}\right| < 1$$

$$= -\frac{1}{3}\int \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n dx = -\frac{1}{3}\int \sum_{n=0}^{\infty} \frac{x^n}{3^n}\,dx \qquad -3<x<3$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty} \frac{x^{n+1}}{3^n(n+1)} + C \qquad f(0)=\ln 3 \quad \text{So, } C=\ln 3$$

$$= \ln 3 - \sum_{n=0}^{\infty} \frac{x^{n+1}}{3^{n+1}(n+1)}$$

or $$= \ln 3 - \sum_{n=1}^{\infty} \frac{x^n}{3^n \cdot n}$$

Example 11*. $f(x) = x\ln(3-x^2)$

$$f(x) = x\left(\ln 3 - \sum_{n=1}^{\infty} \frac{x^{2n}}{3^n \cdot n}\right) \qquad \left|\frac{x^2}{3}\right| < 1$$

$$= x\ln 3 - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{3^n \cdot n} \qquad -\sqrt{3}<x<\sqrt{3}$$

Example 12*. $f(x) = \tan^{-1} x$

$$f(x) = \tan^{-1} x = \int \frac{1}{1+x^2}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx \qquad |x^2|<1 \qquad -1<x<1$$

$(-1,\ 1)$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C. \qquad f(0)=0 \quad \text{So } C=0$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$