Recall that the Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

converges when $|r| < 1$, and diverges when $|r| \geq 1$.

More generally, a **power series** is

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where $c_n \in \mathbb{R}$ are constant numbers called coefficients and x is a variable.

A power series may converge for some $x \in \mathbb{R}$ of and diverge for some other $x \in \mathbb{R}$.

More generally, a **power series centered at** a is

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Example 1. For what values of x is the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x^{n+1}|}{(n+1)!} \frac{n!}{|x^n|} = \frac{|x|}{n+1} \to 0 \quad \text{when } n \to \infty$$

So $\sum \frac{x^n}{n!}$ is convergent for any $x \in \mathbb{R}$.

By ratio test

Theorem

For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only 3 possibilities:

(1) The series converges only when $x = a$

(2) The series converges for all x. $\in \mathbb{R}$

(3) There is a number $R > 0$ such that the series converges when $|x-a| < R$ and diverges when $|x-a| > R$.

$a-R < x < a+R$

$a-R$ a $a+R$

The number R in case (3) is called the **radius of convergence** of the power series.

The set of all x for which series converges is called the **interval of convergence**. I

In case (1), we think $R = 0$ and the interval of convergence is $I = \{0\}$.

In case (2), we think $R = \infty$ and the interval of convergence is $I = \mathbb{R}$.

In case (3), the interval of convergence is $a - R < x < a + R$.

Example 2. Find the radius of convergence and interval of convergence of the geometric series $\sum_{n=0}^{\infty} x^n$.

$\sum_{n=0}^{\infty} x^n$ is convergent when $|x|<1$, and divergent when $|x| \geq 1$

So, the radius of convergence $R=1$

and the interval of convergence is $(-1, 1)$

or $-1 \leq x \leq 1$.

Ex3' : $\sum \frac{(x-3)^n}{n}$. $R=1$. $[2, 4)$

Example 3. For what values of x is the series $\sum_{n=0} nx^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)|x^{n+1}|}{n\,|x^n|} = \frac{(n+1)}{n}|x| \longrightarrow |x| \text{ when } n\to\infty.$$

So By ratio test, $\sum_{n=0}^{\infty} n x^n$ is convergent when $|x|<1$.
and divergent when $|x|>1$.

when $|x|=1$ $|a_n| = n \to \infty$. So, $\sum_{n=0}^{\infty} n x^n$ is divergent when $|x| \geq 1$.

Example 4. Find the radius of convergence and interval of convergence of the series $\sum_{n=0} n!x^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\, x^{n+1}}{n!\, x^n} = (n+1)|x| \longrightarrow \infty \text{ when } n\to\infty \text{ if } |x|\neq 0.$$

So $\sum_{n=0}^{\infty} n!\, x^n$ is divergent when $x\neq 0$. By ratio test

- radius of convergence $R=0$
- interval of convergence $\{0\}$

Example 5. Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n2^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x^{n+1}|}{(n+1)2^{n+1}} \cdot \frac{n 2^n}{|x^n|} = \frac{n \cdot |x|}{(n+1)2} \to \frac{|x|}{2} \text{ when } n\to\infty \quad \text{By ratio test}$$

So $\sum_{n=0}^{\infty} \frac{x^n}{n \cdot 2^n}$ is absolutely convergent when $\frac{|x|}{2} < 1$. So $|x| < 2$.
is divergent when $|x| > 2$.

when $x = 2$, $\sum_{n=0}^{\infty} \frac{1}{n}$ is divergent.

when $x = -2$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ is convergent. $-2 \leq x < 2$.

So, radius of convergence $R = 2$. interval of convergence is $[-2, 2)$

Example 6. Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n (x-2)^n}{\sqrt[4]{n}}$. By ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}|x-2|^{n+1}}{\sqrt[4]{n+1}} \cdot \frac{\sqrt[4]{n}}{3^n |x-2|^n} = 3\sqrt[4]{\frac{n}{n+1}} \cdot |x-2| \to 3|x-2| \text{ when } n\to\infty$$

So, the series is absolutely convergent when $3|x-2| < 1$, $|x-2| < \frac{1}{3}$

when $x-2 = \frac{1}{3}$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$ is convergent.

when $x-2 = -\frac{1}{3}$, $\sum_{n=0}^{\infty} \frac{1}{\sqrt[4]{n}}$ is divergent.

So, the radius of convergence $R = \frac{1}{3}$

The interval of convergence is $-\frac{1}{3} < x-2 \leq \frac{1}{3}$ or $\frac{5}{3} < x \leq \frac{7}{3}$.

Example 7. Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{n!(x+2)^n}{5^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\,|x+2|^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n!\,|x+2|^n} = \frac{(n+1)}{5}|x+2| \to \infty \text{ when } n\to\infty.$$

So the series is only convergent when $x+2=0$. By ratio test

- The radius of convergence $R=0$.
- The interval of convergence is $\{0\}$

Example 8. Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{n^2+1}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}|x-3|^{n+1}}{((n+1)^2+1)} \cdot \frac{(n^2+1)}{2^n|x-3|^n} = 2|x-3|\left(\frac{n^2+1}{n^2+2n+2}\right) = 2|x-3|\frac{1+1/n^2}{1+\frac{2}{n}+\frac{2}{n^2}}$$

So $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|x-3|$

By ratio test So the series is convergent when $2|x-3|<1$. $|x-3|<\frac{1}{2}$.

When $|x-3|=\frac{1}{2}$, the series $\sum_{n=0}^{\infty}\frac{1}{n^2+1}$ is convergent by comparison test.

So . The radius of convergence $R=\frac{1}{2}$

The interval of convergence is $\cdot\, |x-3|\leq\frac{1}{2}$

or $\frac{5}{2}\leq x\leq\frac{7}{2}$ or $[\frac{5}{2},\frac{7}{2}]$

Example 9. Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{n(x+4)^n}{n^3-2}$. By ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)|x+4|^{n+1}}{(n+1)^3-2} \cdot \frac{n^3-2}{n|x+4|^n} = \frac{(n+1)(n^3-2)}{((n+1)^3-2)n} \cdot |x+4| \to |x+4| \text{ when } n\to\infty$$

So the series is convergent when $|x+4|<1$. and the radius of convergence $R=1$.

when $|x+4|=1$, $|a_n| = \frac{n}{n^3-2}$, So $\sum|a_n|$ is absolutely convergent by limit comparison test. with $\frac{1}{n^2}$

So the interval of convergence is $|x+4|\leq 1$

or $-5\leq x\leq -3$ or $[-5, -3]$

Example 10. Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-2)^n(x-5)^{n+2}}{\sqrt{n}}$. By ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2)^{n+1}|x-5|^{n+3}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(2)^n|x-5|^{n+2}} = 2\sqrt{\frac{n}{n+1}}\,|x-5| \to 2|x-5| \text{ when } x\to\infty$$

So, $\sum a_n$ is convergent when $2|x-5|<1$, $|x-5|<\frac{1}{2}$

The radius of convergence is $R=\frac{1}{2}$

When $x-5=\frac{1}{2}$, $a_n = \frac{1}{4}\frac{(-1)^n}{\sqrt{n}}$ So $\sum a_n$ is convergent.

When $x-5=-\frac{1}{2}$, $a_n = \frac{1}{4}\frac{1}{\sqrt{n}}$, So $\sum a_n$ is divergent.

So The interval of convergence is $-\frac{1}{2}<x-5\leq\frac{1}{2}$.

or $\frac{9}{2}<x\leq\frac{11}{2}$ or $(\frac{9}{2}, \frac{11}{2}]$