Let $\sum_{n=1}^{\infty} a_n$ be a series.

We consider a new series of absolute values:

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + |a_4| + |a_5| + \cdots$$

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Example 1. Is $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$ absolutely convergent?

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is the p-series with $p=2$. So it is convergent.

So $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$ is absolutely convergent.

Example 2. Is $\sum_{n=1}^{\infty} (-1)^{n}\frac{1}{n}$ absolutely convergent?

$\sum (-1)^n \frac{1}{n}$ is converrgent.

$\sum \frac{1}{n}$ is not convergent.

So $\sum (-1)^n \frac{1}{n}$ is not absolutely convergent.

A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Theorem

If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Determine whether the series absolutely convergent:

Example 3. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

$$\left|\frac{\sin n}{n^2}\right| = \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$$

$\sum \frac{1}{n^2}$ is convergent.

So $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$ is convergent.

So $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is absolutely convergent

Example 4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+2}$

$$|b_n| = \frac{n}{n^2+2}$$

Limit Comparison Test:

$$\frac{\frac{n}{n^2+2}}{\frac{1}{n}} = \frac{n^2}{n^2+2} = \frac{1}{1+2/n^2} \to 1 \text{ when } n \to \infty$$

So $\sum \frac{n}{n^2+2}$ is divergent, since $\sum \frac{1}{n}$ is divergent.

So. NOT absolutely divergent.

The following test is very useful and powerful.

Theorem (The Ratio Test)

Given a series $\sum a_n$, and suppose

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

(1.) If $L < 1$, then the series $\sum a_n$ is absolutely convergent.

(2.) If $L > 1$ or $L = \infty$, then the series $\sum a_n$ is divergent.

(3.) If $L = 1$, then the Ratio Test is *inconclusive.*

Determine whether the series is absolutely convergent.

Example 5. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{5^n}$ By ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{5^{n+1}}\cdot\frac{5^n}{n^3} = \frac{1}{5}\cdot\left(\frac{n+1}{n}\right)^3 = \frac{1}{5}\left(\frac{1+\frac{1}{n}}{1}\right)^3 \longrightarrow \frac{1}{5} \text{ when } n\to\infty$$

So $\sum\frac{n^3}{5^n}$ is absolutely convergent.

Example 6. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^3+5}$ By ratio test

conditionally convergent. by Alternating series test
and limit comparison test with $\frac{1}{n}$

- The Ratio Test does not work.

$$\frac{(n+1)^2}{(n+1)^3+5}\cdot\frac{n^3+5}{n^2} \longrightarrow 1 \text{ when } n\to\infty.$$

Use Ratio Test to determine whether the series is absolutely convergent.

Example 7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

$\sum \frac{1}{n^3}$ is convergent.

Ratio Test does not work!

Example 8. $\sum_{n=1}^{\infty} \frac{(-1.1)^n}{n^3}$ By ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1.1^{n+1}}{(n+1)^3} \cdot \frac{n^3}{1.1^n} = 1.1\left(\frac{n}{n+1}\right)^3 = 1.1\left(\frac{1}{1+1/n}\right) \to 1.1 \overset{>1}{} \text{ when } n\to\infty$$

So $\sum \frac{(-1.1)^n}{n^3}$ is not absolutely convergent.

Example 9. $\sum_{n=1}^{\infty} n^2(2/3)^n$ By ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2 (2/3)^{n+1}}{n^2 (2/3)^n} = \frac{2}{3}\left(\frac{n+1}{n}\right)^2 = \frac{2}{3}\left(\frac{1+1/n}{1}\right)^2 \to \frac{2}{3} \overset{<1}{} \text{ when } n\to\infty$$

So $\sum_{n=1}^{\infty} n^2 (2/3)^n$ is absolutely convergent.

Example 10. $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ By ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \frac{10}{n+1} \to 0 \quad \text{when } n \to \infty$$

So $\sum \frac{10^n}{n!}$ is absolutely convergent.

Example 11. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ By ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x^{n+1}|}{(n+1)!} \cdot \frac{n!}{|x^n|} = \frac{|x|}{n+1} \to 0 \quad \text{when } n \to \infty$$

So $\sum \frac{x^n}{n!}$ is absolutely convergent for any $x \in \mathbb{R}$.

Example 12. $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ By ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} = \frac{n+1}{100} \to \infty \quad \text{when } n \to \infty$$

So $\sum \frac{n!}{100^n}$ is divergent.

The following test is convenient to apply when n-th powers occur.

Theorem (The Root Test)*

Given a series $\sum a_n$, and suppose

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

(1.) If $L < 1$, then the series $\sum a_n$ is absolutely convergent.

(2.) If $L > 1$ or $L = \infty$, then the series $\sum a_n$ is divergent.

(3.) If $L = 1$, then the Ratio Test is *inconclusive*.

Notice that $|a_n|^{1/n} = \sqrt[n]{a_n}$.

Use Root Test to determine whether the series is absolutely convergent.

Example 14*. $\sum_{n=1}^{\infty} \left(\frac{n+4}{3n-2}\right)^n$

$$|a_n|^{\frac{1}{n}} = \frac{n+4}{3n-2} \longrightarrow \frac{1}{3} \quad \text{when } n\to\infty$$

So the series is absolutely convergent.