Let $\sum b_n$ be a positive series, that is $b_n > 0$ for all n.

Then $\sum(-1)^n b_n$ or $\sum(-1)^{n+1} b_n$ is called an **alternating series**.

- Its terms are alternately positive and negative.

For example, the following alternating harmonic series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Let $\sum b_n$ be a **positive** series.

Theorem. (Alternating Series Test)

$b_{n+1} < b_n$ for all n.

If b_n is **decreasing** and $\lim_{n\to\infty} b_n = 0$, then the alternating series

$$\sum(-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

is convergent.

The sequence b_n is decreasing: $b_{n+1} \le b_n$ for all n.

Example 1. Is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ convergent or divergent?

① $b_n = \frac{1}{n} > 0$

② b_n is decreasing

③ $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$

So The alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is convergent.

Example 2. Is $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$ convergent or divergent?

① $b_n = \frac{1}{\sqrt{n}} > 0$

② b_n is decreasing

③ $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

So $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$ is convergent.

Determine whether the series converges:

Example 3. $\sum_{n=1}^{\infty}(-1)^n\frac{2}{\sqrt{3n-1}}$

① $b_n = \frac{2}{\sqrt{3n-1}} > 0$

② b_n is decreasing.

$$f(x) = \frac{2}{\sqrt{3x-1}}. \quad f'(x) = 2\left(-\tfrac{1}{2}\right)\cdot(3x-1)^{-\frac{3}{2}}\cdot 3 < 0 \text{ when } x>1$$

③ $\lim_{n\to\infty}\frac{2}{\sqrt{3n-1}} = 0.$

So The alternating series $\sum_{n=1}^{\infty}(-1)^n\frac{2}{\sqrt{3n-1}}$ is convergent.

Example 4. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+2}$

① $b_n = \frac{n}{n^2+2} > 0$

② b_n is decreasing

$$f(x) = \frac{x}{x^2+2} \quad f'(x) = \frac{x^2+2-x(2x)}{(x^2+2)^2} < 0 \text{ when } x>2$$

③ $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n+2/n} = 0$

So, the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+2}$ is convergent.

Example 5. $\sum_{n=1}^{\infty}(-1)^n\frac{3n^2-1}{5n^2+6}$

① $b_n = \frac{n}{n^2+2} > 0$

② . . .

③ $\lim_{n\to\infty}\frac{3n^2-1}{5n^2+6} = \lim_{n\to\infty}\frac{3-1/n^2}{5+6/n^2} = \frac{3}{5} \neq 0.$

So $\sum(-1)^n\frac{3n^2-1}{5n^2+6}$ is divergent.

Determine whether the series converges:

Example 6. $\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{1}{n}\right)$ divergent.

$\lim_{n\to\infty} \cos(\frac{1}{n}) = \cos 0 = 1$.

Example 6. $\sum_{n=1}^{\infty} (-1)^n \sin(\frac{1}{n})$.

① $b_n = \sin(\frac{1}{n}) > 0$

② b_n is decreasing $\leftarrow$ $\boxed{f(x) = \sin(\frac{1}{x}) \quad f'(x) = \cos(\frac{1}{x})(-1)x^{-2} < 0}$

③ $\lim_{n\to\infty} \sin(\frac{1}{n}) = 0$. So $\sum(-1)^n \sin(\frac{1}{n})$ is convergent.

Let $\sum b_n$ be a **positive, decreasing** series such that $\lim_{n\to\infty} b_n = 0$. Let $s = \sum b_n$ and $s_n = \sum_{k=1}^{n} b_k$.

Remainder Estimate for Alternating Series*

If we denote the remainder of the sequence as $R_n = s - s_n$, then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Example 7. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ correct to 3 decimal places.

① $b_n = \frac{1}{n^3} > 0$.

② b_n is decreasing.

③ $\lim_{n\to\infty} \frac{1}{n^3} = 0$

So $\sum \frac{(-1)^n}{n^3}$ is convergent.

$b_{10} = \frac{1}{n^3} = 0.001$.

$S_{10} = -1 + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} \cdots + \frac{1}{10^3} \approx -0.901$