Recall the geometric series: $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent. to $\frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = 1$

Example 1. Is the series $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ convergent?

$$\frac{1}{2^n+n} < \frac{1}{2^n}$$

partial sum: $\sum_{i=1}^{n} \frac{1}{2^i+i} < \sum_{i=1}^{n} \frac{1}{2^i} < 1$

So, $S_n = \sum_{i=1}^{n} \frac{1}{2^i+i}$ is increasing and bounded above.

So, S_n is convergent.

So. $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ is convergent.

The Comparison Test

Suppose both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are **positive** series.

1. If $a_n \le b_n$ for all n, and $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
2. If $a_n \ge b_n$ for all n, and $\sum b_n$ is divergent, then $\sum a_n$ is divergent.

1. If a_n is smaller than convergent, then $\sum a_n$ is convergent.
2. If a_n is larger than divergent, then $\sum a_n$ is divergent

Determine whether the series converges:

Example 2. $\sum_{n=1}^{\infty} \frac{n}{n^3+n^2+3}$

① $\frac{n}{n^3+n^2+3} < \frac{n}{n^3} = \frac{1}{n^2}$

② The p-series $\sum \frac{1}{n^2}$ is convergent. ③ So $\sum \frac{n}{n^3+n^2+3}$ is convergent.

$(p=2)$ by comparison test.

Example 3. $\sum_{n=1}^{\infty} \frac{5+5^n}{2^n}$

① $\frac{5+5^n}{2^n} > \frac{5^n}{2^n} = \left(\frac{5}{2}\right)^n$

② The geometric $\sum \left(\frac{5}{2}\right)^n$ is divergent. ③ So $\sum \frac{5+5^n}{2^n}$ is divergent.

$r = \frac{5}{2} > 1$ by comparison test.

Example 4. $\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}$

- $\frac{1}{2^n\sqrt{n}} < \frac{1}{2^n}$
- The geometric series $\sum \frac{1}{2^n}$ is convergent.
- So, the series $\sum \frac{1}{2^n\sqrt{n}}$ is convergent. by comparison test.

Example 5. $\sum_{n=1}^{\infty} \frac{1}{2n-3}$

- $\frac{1}{2n-3} > \frac{1}{2n}$
- $\sum_{n=1}^{\infty} \frac{1}{2n}$ is divergent by the integral Test. $\int_1^{\infty} \frac{1}{2x}\,dx = \frac{1}{2}\lim_{t\to\infty} \ln t = \infty$
- So $\sum_{n=1}^{\infty} \frac{1}{2n-3}$ is divergent. by comparison test.

Suppose both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are **positive** series.

The Limit Comparison Test

If the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

is a positive finite number, **then**

$\sum a_n$ is convergent if and only if $\sum b_n$ is convergent.

Determine whether the series converges:

Example 6. $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ limit comparison test

① $\lim_{n\to\infty} \frac{\frac{1}{2n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+3} = \lim_{n\to\infty} \frac{1}{2+3/n} = \frac{1}{2}$

② Since $\sum \frac{1}{n}$ is divergent, ③ $\sum \frac{1}{2n+3}$ is divergent.

Example 7. $\sum_{n=1}^{\infty} \frac{1}{2n^2-3}$

① $\lim_{n\to\infty} \frac{\frac{1}{2n^2-3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{2n^2-3} = \lim_{n\to\infty} \frac{1}{2-3/n^2} = \frac{1}{2}$

② The p-series $\sum \frac{1}{n^2}$ is convergent.

③ The series $\sum \frac{1}{2n^2-3}$ is convergent. by limit comparison test

Determine whether the series converges:

Example 8. $\displaystyle\sum_{n=1}^{\infty} \frac{2+7^n}{\pi+3^n}$

① $$\frac{\frac{2+7^n}{\pi+3^n}}{\frac{7^n}{3^n}} = \frac{2+7^n}{\pi+3^n}\cdot\frac{3^n}{7^n} = \frac{2(3^n)+21^n}{\pi(7^n)+21^n} = \frac{2\left(\frac{1}{7}\right)^n+1}{\pi\left(\frac{1}{3}\right)^n+1} \to 1$$ when $n\to\infty$

② Since $\sum\left(\frac{7}{3}\right)^n$ is divergent.

③ The series $\sum \frac{2+7^n}{\pi+3^n}$ is divergent by limit comparison test

Example 9. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3-2n}{n^4+3n^2}$ limit comparison test

① $$\frac{\frac{n^3-2n}{n^4+3n^2}}{\frac{1}{n}} = \frac{(n^3-2n)\,n}{n^4+3n^2} = \frac{n^4-2n^2}{n^4+3n^2} = \frac{1-2/n^2}{1+3/n^2} \to 1$$ when $n\to\infty$

② The harmonic series $\sum\frac{1}{n}$ is divergent. ③ So $\displaystyle\sum_{n=1}^{\infty}\frac{n^3-2n}{n^4+3n^2}$ is divergent.

Example 10. $\displaystyle\sum_{n=1}^{\infty} \frac{3+4n^2}{(n^2-1)^3} = \sum_{n=1}^{\infty}\frac{4n^2+3}{n^6-3n^4+3n^2-1}$

① $$\frac{\frac{3+4n^2}{(n^2-1)^3}}{\frac{1}{n^4}} = \frac{4n^6+3n^4}{(n^2-1)^3} = \frac{4+(3/n^2)}{\left(1-\frac{1}{n^2}\right)^3} \to 4$$ when $n\to\infty$

② The P-series $\sum\frac{1}{n^4}$ is convergent

③ So the series $\sum\frac{3+4n^2}{(n^2-1)^3}$ is convergent by limit comparison test