Example (1). Is the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent or divergent?

$\lim_{n\to\infty} \frac{1}{n^2} = 0$

Divergent Test fails. No conclusion.

Example (2). Is the series $\sum_{n=1}^{\infty} \frac{1}{n}$ convergent or divergent?

$\lim_{n\to\infty} \frac{1}{n} = 0$ Harmonic series.

Divergent Test fails. No conclusion.

Example (3). Is the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ convergent or divergent?

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Divergent Test fails. No conclusion.

Suppose $f(x)$ is **continuous**, **positive** and **decreasing** on $[1, \infty]$ and $a_n = f(n)$.

The Integral Test

The series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. §7.8

Warning: $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\, dx$ are convergent to different values!

Example 1. (Key Example)

Determine the value of p such that the **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

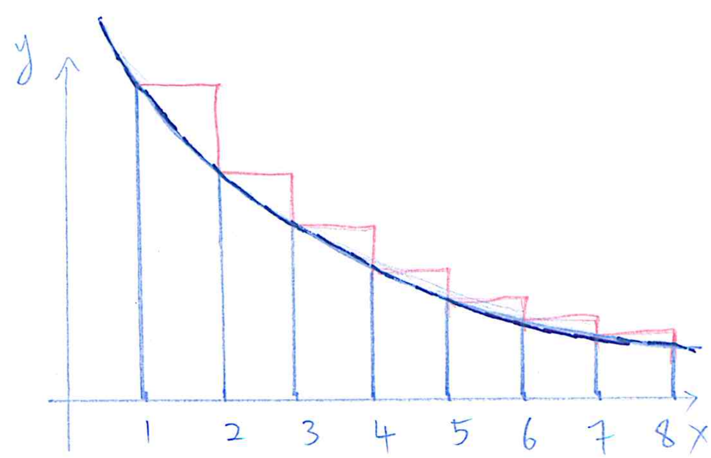
Use the Integral Test: Calculate $\int_1^{\infty} \frac{1}{x^p}\, dx$.

From Example 4 in §7.8.

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^t = \lim_{t\to\infty} \frac{1}{1-p}\left(\frac{1}{t^{p-1}} - 1\right) = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \text{divergent} & \text{if } p<1 \end{cases}$$

(p=1) $$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{t\to\infty} \ln x\Big|_1^t = \lim_{t\to\infty} (\ln t - \ln 1) = \infty \quad \text{if } p=1.$$

Ex(1) p=2



The Integral Test

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $p \leq 1$.

Determine whether the series converges:

Example 2. $\sum_{n=1}^{\infty} \frac{3}{(2n+1)^2}$ divergent test fails!

- $f(x) = \frac{3}{(2x+1)^2} = 3(2x+1)^{-2}$
- $f'(x) = 3(-2)(2x+1)^{-3} \cdot 2 < 0$ when $x \geq 1$. So $f(x)$ is decreasing.
- $\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t 3(2x+1)^{-2}dx$

$$= \lim_{t\to\infty} \frac{3}{2} \frac{(2x+1)^{-1}}{-1}\Big|_1^t$$

$$= \lim_{t\to\infty} \left(-\frac{3}{2}\frac{1}{2t+1} - \frac{-3}{2(3)}\right) = \frac{1}{2} \text{ convergent.}$$

Example 3. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $\lim_{n\to\infty} \frac{\ln n}{n} = 0$. divergent test fails.

- $f(x) = \frac{\ln x}{x}$ $f'(x) = ((\ln x)(x^{-1}))' = (x^{-1})(x^{-1}) + (\ln x)(-x^{-2}) = x^{-2}(1-\ln x) < 0$ when $x > e$

So $f(x)$ is decreasing.

- $\int_1^\infty \frac{\ln x}{x}dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Big|_1^t = \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty$ divergent

Let $u = \ln x$, $du = \frac{1}{x}dx$ $\int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$

Determine whether the series converges:

Example 4*. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

- $f(x) = \frac{1}{x^2+1}$ is decreasing when $x \geq 1$. since x^2+1 is increasing. when $n \geq 1$.

- $\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1}x \Big|_1^t = \lim_{t\to\infty} \left(\tan^{-1}t - \frac{\pi}{4}\right)$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

convergent.

Example 5. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. divergent test fails.

- $f(x) = \frac{x}{x^2+1}$ $f'(x) = \frac{1}{x^2+1} + x(-1)(x^2+1)^{-2}\cdot 2x = \frac{1-x^2}{(x^2+1)^2} < 0$ when $x \geq 1$

decreasing.

- $\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \frac{1}{2}\ln(x^2+1)\Big|_1^t = \lim_{t\to\infty}\left(\frac{1}{2}\ln(t^2+1) - \frac{1}{2}\ln 2\right) = \infty$

divergent.

Let $u = x^2+1$

$du = 2x\,dx$

$dx = \frac{1}{2x}\,du$

$\int \frac{x}{x^2+1}\,dx = \int \frac{x}{u}\frac{1}{2x}\,du$

$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln(x^2+1) + C$

Determine whether the series converges:

Example 6. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Divergent Test fails since $\lim_{n\to\infty} \frac{1}{n\sqrt{\ln n}} = 0$

- $f(x) = \frac{1}{x\sqrt{\ln x}}$ decreasing since $x\sqrt{\ln x}$ is increasing when $x \geq 1$.

- $\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{t\to\infty} 2\sqrt{\ln x}\Big|_1^t = \lim_{t\to\infty} 2\sqrt{\ln t} = \infty$

divergent.

- Let $u = \ln x$
  $du = \frac{1}{x}\,dx$
  $dx = x\,du$

$$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int \frac{1}{x\sqrt{u}}\,x\,du = \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} + C$$
$$= 2\sqrt{\ln x} + C.$$

Example 7. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$ divergent

$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}} = 1 \neq 0$ divergent test works!

Suppose $f(x)$ is **continuous**, **positive** and **decreasing** on $[1, \infty]$ and $a_n = f(n)$.
For $f(x)$ as in Integral Test, we have remainder estimate:

Remainder Estimate for the Integral Test*

If we denote the remainder of the sequence as $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_{n}^{\infty} f(x)dx$$

Example 8*. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using the first 10 terms. Estimate the error.

$$\int_n^\infty \frac{1}{x^2}dx = \lim_{t\to\infty} -x^{-1}\Big|_n^t = \lim_{t\to\infty}\left(-\frac{1}{t}-\left(-\frac{1}{n}\right)\right) = \frac{1}{n}$$

$n=10$ $\qquad \frac{1}{11} \le R_n \le \frac{1}{10}$

$$S_{10} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{10^2} = 1.55$$

$$S \approx S_{10}+R_{10} \approx 1.55 \pm 0.1$$

In fact $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64$

Compare to $\int_1^\infty \frac{1}{x^2}dx = 1$