Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

An (infinite) **series** is the sum of a sequence $\{a_n\}$, that is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

The sum symbol is denoted by $\sum a_n$ for short.

Example 1. Find the series $\sum a_n$, where $a_n = \frac{3}{10^n}$.

$$\sum a_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots$$

$$= 0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

$$= 0.3333\cdots$$

$$= \frac{1}{3}$$

We define a new sequence $\{s_n\}_{n=1}^{\infty}$ by partial sums

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = L$, then we say that $\sum a_n$ is **convergent** and write

$$\sum_{n=1}^{\infty} a_n = L.$$

The number L is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, we say that the series is **divergent**.

Example 2. Find the series $\sum b_n$, where $b_n = \frac{1}{2^{n-1}}$. $= \left(\frac{1}{2}\right)^{n-1}$

$$S_n = 1 + \underbrace{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}}$$

$$\frac{1}{2} S_n = \underbrace{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}} + \frac{1}{2^n}$$

$$S_n - \frac{1}{2} S_n = 1 - \frac{1}{2^n}$$

$$S_n = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-\frac{1}{2}} = 2 \qquad \text{So } \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2$$

• Let $r = \frac{1}{2}$, Example 2 gives $\sum_{n=1}^{\infty} (r)^{n-1} = \frac{1}{1-r}$

Example 3 (**Geometric Series** $\sum ar^{n-1}$). The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| \leq 1$ and

$$\sum ar^{n-1} = \frac{a}{1-r} \qquad \text{if } |r| < 1.$$

If $|r| > 1$, then the geometric series $\sum ar^{n-1}$ is divergent.

Example 4. Whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} 3^n(-4)^{1-n}$$

$$a_n = 3^n(-4)^{1-n} = \frac{3^n}{(-4)^{n-1}} = 3\left(-\frac{3}{4}\right)^{n-1}$$

So $\sum_{n=1}^{\infty} a_n$ is a geometric series with $a=3$ and $r=-\frac{3}{4}$

$|r|<1$

So $\sum a_n$ is convergent and $\sum a_n = \frac{a}{1-r} = \frac{12}{7}$

Example 5. Express $2.\overline{45}$ as a ratio of integers.

$$2.\overline{45} = 2 + \frac{45}{10^2} + \frac{45}{10^4} + \frac{45}{10^6} + \cdots$$

$$= 2 + \sum_{n=1}^{\infty} 45 \cdot \left(\frac{1}{10^2}\right)^n$$

$$= 2 + \sum_{n=1}^{\infty} \frac{45}{10^2}\left(\frac{1}{10^2}\right)^{n-1}$$

Geometric series

$a = \frac{45}{10^2}$ $r = \frac{1}{10^2}$

$|r|<1$

$$= 2 + \frac{45/100}{1-\frac{1}{100}}$$

$$= 2 + \frac{5}{11} = \frac{27}{11}$$

Example 6. Find the sum of the series

$$2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81} - \frac{2}{243} + \cdots = \sum_{n=1}^{\infty} 2\left(\frac{-1}{3}\right)^{n-1}$$

a geometric series $\sum_{n=1}^{\infty} a r^{n-1}$ with $a=2$ $r=\frac{-1}{3}$

$|r|<1$ So the series converges

and the sum is $\frac{a}{1-r} = \frac{2}{1+\frac{1}{3}} = \frac{3}{2}$

Example 7. Whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right)$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Example (8.) Whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} 3^n (2)^{1-n}$$

$a_n = 3^n (2)^{1-n} = \frac{3^n}{2^{n-1}} = 3\left(\frac{3}{2}\right)^{n-1}$

So $\sum a_n$ is a geometric series with $a=3$, $r=\frac{3}{2}$

$|r|>1$, So $\sum a_n$ is divergent.

Example (9.) Express $1.\overline{56}$ as a fraction.

$1.\overline{56} = 1 + \frac{56}{10^2} + \frac{56}{10^4} + \frac{56}{10^6} + \cdots$

$= 1 + \sum_{n=1}^{\infty} 56 \left(\frac{1}{10^2}\right)^n$

$= 1 + \sum_{n=1}^{\infty} \frac{56}{10^2} \left(\frac{1}{10^2}\right)^{n-1}$ $a=\frac{56}{10^2}$ $r=\frac{1}{10^2}$

$= 1 + \frac{56/100}{1-\frac{1}{100}}$ $|r|<1$

$= \frac{155}{99}$

Theorem (Divergence Test)
If the series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Equivalently,

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.

Example 10. Whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{7^n}$$

$$a_n = \frac{3^{2n}}{7^n} = \frac{9^n}{7^n} = \left(\frac{9}{7}\right)^n$$

So $\lim_{n\to\infty} a_n = \infty$

So $\sum a_n$ is divergent.

Example 11. Whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^2-5}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2-1}{3n^2-5} = \lim_{n\to\infty} \frac{2-1/n^2}{3-5/n^2} = \frac{2}{3}$$

So $\sum a_n$ is divergent.

Example 12. Whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}(-1)^n$$

$\lim_{n\to\infty}(-1)^n$ diverges, so it is not zero.

So $\sum_{n=1}^{\infty}(-1)^n$ diverges.

Example 15. Whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}2^{1/n}$$

$\lim_{n\to\infty}2^{1/n} = 2^{\lim_{n\to\infty}\frac{1}{n}} = 2^0 = 1$

So $\sum_{n=1}^{\infty}2^{1/n}$ diverges.

Warning: The converse of the Divergence Test is not true.

Example 14. The Harmonic Series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Although $\lim_{n\to\infty}\frac{1}{n} = 0$, $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Theorem. If $\sum a_n$ and $\sum b_n$ are convergent, then

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

and

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Example 15. Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{2}{n(n+1)} + \frac{1}{3^n}\right)$$

Since $\sum \frac{2}{n(n+1)}$ and $\sum \frac{1}{3^n}$ converge,

we have $\sum\left(\frac{2}{n(n+1)} + \frac{1}{3^n}\right)$ converges

and $\sum\left(\frac{2}{n(n+1)} + \frac{1}{3^n}\right) = \sum \frac{2}{n(n+1)} + \sum \frac{1}{3^n}$

$= 2\sum \frac{1}{n(n+1)} + \sum \frac{1}{3}\left(\frac{1}{3}\right)^{n-1}$.

$= 2 + \frac{\frac{1}{3}}{1-\frac{1}{3}}$

$= 2.5$