Taylor Series

Suppose a function $f(x)$ can be written a power series:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad \text{for } |x-a| < R,$$

Then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This is called **Taylor series** of $f(x)$ at $x = a$.

It is called **Maclaurin series** if $a = 0$.

Example 1. $a=0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

The radius of convergence $R = \infty$.

Example 2. $a=0$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The radius of convergence $R = \infty$.

Example 3. $a=0$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

The radius of convergence $R = \infty$.

Example 4.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

The radius of convergence $R = 1$.

Find the Maclaurin series or Taylor series for the following functions.

Example 5. $f(x) = xe^{3x}$

$$f(x) = x \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{3^n \cdot x^{n+1}}{n!} \qquad R=\infty$$

Example 6. $f(x) = x^4$ at $a = 1$

$f(x) = x^4 \quad f(a) = 1$

$f'(x) = 4x^3 \quad f'(a) = 4$

$f^{(2)}(x) = 12x^2 \quad f^{(2)}(a) = 12$

$f^{(3)}(x) = 24x \quad f^{(3)}(a) = 24$

$f^{(4)}(x) = 24 \quad f^{(4)}(a) = 24$

$f^{(n)}(x) = 0$ if $n \geq 5$

$$f(x) = C_0 (x-1)^0 + C_1(x-1) + C_2(x-1)^2 + C_3(x-1)^3 + C_4(x-1)^4$$

$$= 1 + 4(x-1) + \frac{12}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

$$= 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4.$$

$R = \infty$

Example 7. $f(x) = \sin \pi x$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin(\pi x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\pi x)^{2n+1}}{(2n+1)!} \qquad R=\infty$$

$$\text{or } = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1} x^{2n+1}}{(2n+1)!}$$

Example 8. $f(x) = \ln x$ at $a = 1$.

method 1: We know $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ $\quad -1 < x \leq 1$

So $\ln(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$ $\quad -1 < x-1 \leq 1$

$0 < x \leq 2$

method 2: $\ln(x) = C_0 + C_1(x-1) + C_2(x-1)^2 + C_3(x-1)^3 + \cdots$

$f'(x) = x^{-1}$ $\quad f'(1) = 1$ $\quad C_0 = f(1) = 0$

$f''(x) = -x^{-2}$ $\quad f''(1) = -1$ $\quad C_1 = \frac{f'(1)}{1} = 1$

$f^{(3)}(x) = 2x^{-3}$ $\quad f^{(3)}(1) = 2$ $\quad C_2 = \frac{f''(1)}{2!} = -\frac{1}{2}$

$\vdots$ $\quad C_3 = \frac{f^{(3)}(1)}{3!} = \frac{1}{3}$

$f^{(n)} = (-1)^{n-1}(n-1)!\, x^{-n}$ $\quad f^{(n)}(1) = (-1)^{n-1}(n-1)!$

$f^{(n+1)} = (-1)^n n!\, x^{-n-1}$ $\quad f^{(n+1)}(1) = (-1)^n n!$ $\quad C_{n+1} = \frac{f^{(n+1)}(1)}{(n+1)!} = (-1)^n \frac{1}{n+1}$

Show series converges and **Find the sum**.

Example 9. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $R=\infty$

So $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$

Example 10. $\sum_{n=0}^{\infty} (-1)^n \frac{(\pi/3)^{2n}}{(2n)!}$

recall $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $R=\infty$

So, $\sum_{n=0}^{\infty} (-1)^n \frac{(\pi/3)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Example 11. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{3^{2n+1}(2n+1)!}$ $= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!}$

recall $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

So $\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

Example 12. $\displaystyle\sum_{n=0}^{\infty} \frac{(\ln 5)^n}{n!}$

$$\sum_{n=0}^{\infty} \frac{(\ln 5)^n}{n!} = e^{\ln 5} = 5$$

Example 13. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n}$

recall $\displaystyle \ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $\quad |x|<1$ $\quad -1 < x \leq 1$

When $x=2$, the series is divergent!

Example 14. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \left(-\frac{2}{3}\right)^n}{n} = -\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(-\frac{2}{3}\right)^n}{n}}$

So $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n3^n} = -\ln\left(-\frac{2}{3}+1\right) = -\ln\left(\frac{1}{3}\right) = -(\ln 1 - \ln 3)$

$= \ln 3.$

Example 15. $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{n3^n} = \sum_{n=1}^{\infty} \frac{2^n}{n3^n} - \frac{2}{3} = \ln 3 - \frac{2}{3}$