Definition. A **sequence** is a list of order numbers

$$\{a_1, a_2, a_3, a_4, \dots, a_n, \dots\},$$

which is denoted by

$$\{a_n\}_{n=1}^{\infty}, \quad \text{or} \quad \{a_n\} \quad \text{for short.}$$

Example (1).

$$\{2n-1\}_{n=1}^{\infty}, \qquad a_n = 2n-1, \qquad \{1, 3, 5, 7, 9, \dots, 2n-1, \dots\}$$

Example (2).

$$\left\{\frac{(-1)^n n}{e^n}\right\}_{n=1}^{\infty}, \qquad a_n = \frac{(-1)^n n}{e^n}, \qquad \left\{-\frac{1}{e}, \frac{2}{e^2}, -\frac{3}{e^3}, \dots, \frac{(-1)^n n}{e^n}, \dots\right\}$$

Example (3).

$$\left\{\sqrt{3n}\right\}_{n=1}^{\infty}, \qquad a_n = \sqrt{3n}, \qquad \{1, \sqrt{3}, 3, \sqrt{12}, \dots, \sqrt{3n}, \dots\}$$

Example 1. Find a formula for the general term of the sequence

$$\left\{\frac{4}{3}, \ -\frac{5}{9}, \ \frac{6}{27}, \ -\frac{7}{81}, \ \frac{8}{243}, \ \dots\right\}$$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$

$a_n = (-1)^{n+1} \dfrac{n+3}{3^n}$ is the general term.

Example 2. Some hard sequences with no formulas.

(1) The sequence $\{d_n\}$, where d_n is the n-th decimal of π.

$$\{1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, \cdots\}$$

(2) The sequence $\{p_n\}$, where p_n is the n-th prime number.

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, \cdots\}$$

(3) **Fibonacci sequence** $\{f_n\}$ is defined recursively by the conditions

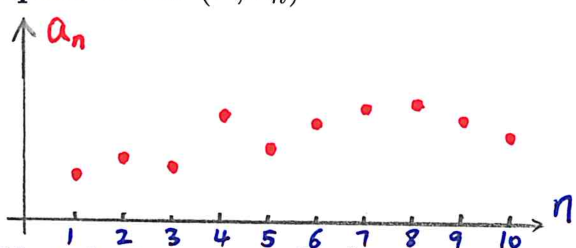
$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 3.$$

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots\}$$

Plotting a sequence as (n, a_n).

$y = f(x)$

$(x, f(x))$

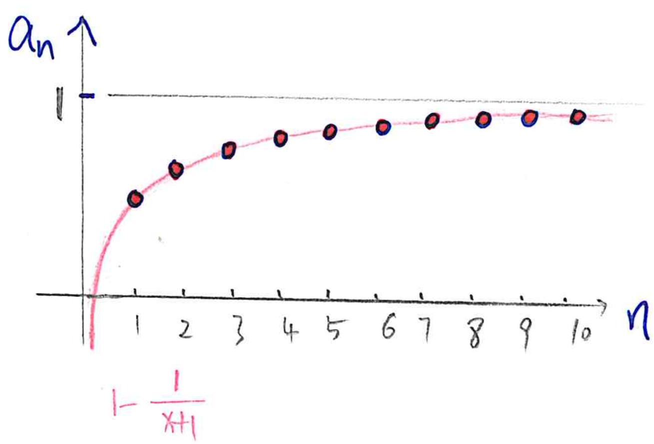


Think a_n as a function with variable $n \in$ Natural numbers.

Example 3. Plot the sequence $\{a_n\}$, where $a_n = \dfrac{n}{n+1}$.

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots\right\}$$

$$= \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1}$$



Definition. If a sequence $\{a_n\}$ has the limit L as $n \to \infty$, we write

$$\lim_{n\to\infty} a_n = L,$$

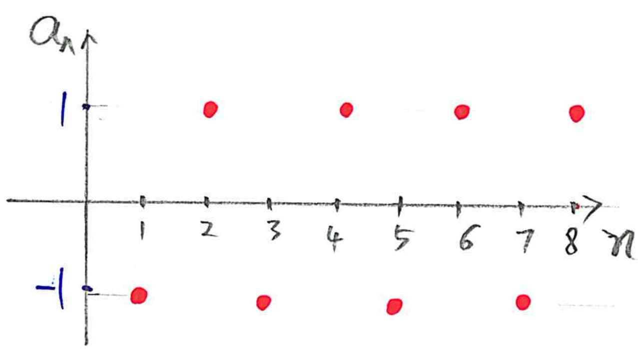
and say $\{a_n\}$ converges to L. If $\{a_n\}$ has no limit, we say $\{a_n\}$ diverges.

Example 4. $\lim_{n\to\infty} 1/n = 0$, or $\{1/n\}$ converges to 0.

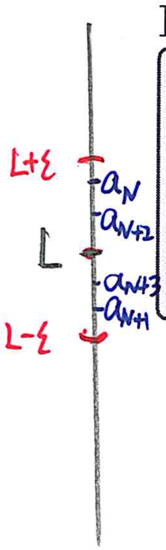
$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots\} \to 0$ if $n \to \infty$.

Example 5. The sequence $\{(-1)^n\}$ diverges.

$\{-1, 1, -1, 1, -1, 1, \cdots\}$



Precise Definition.



A sequence $\{a_n\}$ has **limit** L as $n \to \infty$,
if for every $\epsilon > 0$ **there is a corresponding integer** N **such that**

$$\text{if} \quad n > N, \text{ then } |a_n - L| < \epsilon.$$

$-\varepsilon < a_n - L < \varepsilon$

$L-\varepsilon < a_n < L+\varepsilon$

Theorem. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then,

$$\lim_{n\to\infty} f(n) = L.$$

Example 6. $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 0$, if $p > 0$.

$f(x) = \frac{1}{x^p}$ $\quad f(n) = a_n = \frac{1}{n^p}$

$\lim_{x\to\infty} f(x) = 0$ if $p > 0$.

So $\lim_{n\to\infty} \frac{1}{n^p} = 0$ if $p > 0$.

More over

$$\lim_{n\to\infty} \frac{1}{n^p} = \begin{cases} 0 & \text{if } p>0 \\ 1 & \text{if } p=0 \\ \infty & \text{if } p<0 \end{cases}$$

Example 7. $\displaystyle\lim_{n\to\infty} r^n = 0$, if $|r| < 1$.

$f(x) = r^x$ $\quad f(n) = r^n$

$\lim_{x\to\infty} f(x) = 0$ if $|r| < 1$

So $\lim_{n\to\infty} r^n = 0$ if $|r| = 1$

More over

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=1 \\ \infty & \text{if } r>1 \\ \text{divergent} & \text{if } r \le -1 \end{cases}$$

Definition. A sequence $\{a_n\}$ has **limit** ∞ as $n \to \infty$, denoted as $\lim_{n\to\infty} a_n = \infty$, if for every positive number M there is an integer N such that

$$\text{if } n > N, \text{ then } a_n > M.$$

$\frac{0}{0}$ or $\frac{\infty}{\infty}$ L'Hospital's Rule!

The Limit Laws for functions hold for the limits of sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent and k is a constant number, then

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ka_n = k\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_nb_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

$$\lim_{n\to\infty}(a_n)^p = (\lim_{n\to\infty} a_n)^p \text{ if } p>0 \text{ and } a_n>0.$$

Correct: $\frac{2}{\infty}=0$ $\frac{0}{2}=0$

Wrong: $\frac{2}{0}$

The Squeeze Theorem: If $a_n \le b_n \le c_n$ for $n \ge N$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then

$$\lim_{n\to\infty} b_n = L$$

Theorem. If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Example 8. Find $\lim_{n\to\infty}\frac{5n}{n+1}$ $= \lim_{n\to\infty}\frac{5}{1+\frac{1}{n}} = \frac{5}{1+\lim_{n\to\infty}\frac{1}{n}} = 5$

Example 9. Find $\lim_{n\to\infty}\frac{\ln n}{2n}$ $=0$

$$\lim_{x\to\infty}\frac{\ln x}{2x} = \lim_{x\to\infty}\frac{1/x}{2} = 0$$

L'Hospital's Rule.

Ex 9 Find $\lim_{n\to\infty}(-1)^n\frac{\ln n}{2n}$.

$=0$

Theorem. If $\lim_{n\to\infty} a_n = L$ and the function $f(x)$ is continuous at $x = L$ then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Example 10. Find $\lim\limits_{n\to\infty} \cos(\pi/n)$.

$$= \cos\left(\lim_{n\to\infty}\left(\frac{\pi}{n}\right)\right)$$

$$= \cos 0$$

$$= 1$$

Example 11. Find $\lim\limits_{n\to\infty} \tan(2^{-n})$.

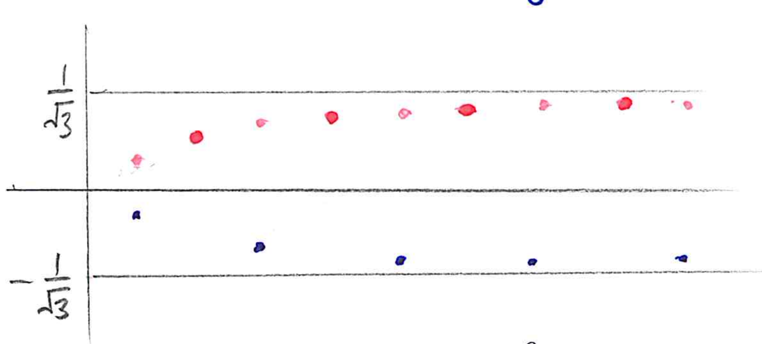
$$= \tan\left(\lim_{n\to\infty}\left(\frac{1}{2}\right)^n\right)$$

$$= \tan(0) = 0$$

Example 12. Find $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt{3n^2+2}}$.

$$= \lim_{n\to\infty} \frac{1}{\sqrt{3+\frac{2}{n^2}}} = \frac{1}{\sqrt{3+\lim\limits_{n\to\infty} 3/n^2}} = \frac{1}{\sqrt{3}}$$

Example 13. Find $\lim\limits_{n\to\infty} \dfrac{(-1)^n n}{\sqrt{3n^2+2}}$. divergent.



Example 14. Find $\lim_{n\to\infty} \frac{\sin n}{2n}$. $=0$

$$\lim_{x\to\infty} \frac{\sin x}{2x} = 0$$

NO L'Hospital's Rule !

Example 15. Find $\lim_{n\to\infty} 2ne^{-n}$. $=0$

$$\lim_{x\to\infty} 2xe^{-x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

L'Hospital's Rule

Example 16. Find $\lim_{n\to\infty} \frac{2n+10}{n^2-4}$.

$$= \lim_{n\to\infty} \frac{\frac{2}{n}+\frac{10}{n^2}}{1-\frac{4}{n^2}} = \frac{0+0}{1-0} = 0$$

Example 17. Find $\lim_{n\to\infty} 2 - \frac{2^n}{3^{n-1}}$.

$$= 2 - \lim_{n\to\infty} \left(\frac{2}{3}\right)^n \cdot 3$$

$$= 2 - 0$$

$$= 2$$

Squeeze theorem.

Example 18. Find $\lim_{n\to\infty} \frac{n!}{n^n}$.

$$a_n = \frac{n!}{n^n} = \frac{n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot 3\cdot 2\cdot 1}{n\quad n\quad n\quad\quad n\quad n\quad n}$$

$$0 \le a_n \le \frac{1}{n}$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$ So $\lim_{n\to\infty} a_n = 0$

Definition

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is

$$a_1 < a_2 < a_3 < \cdots .$$

A sequence $\{a_n\}$ is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$, that is

$$a_1 > a_2 > a_3 > \cdots .$$

A sequence $\{a_n\}$ is called **monotonic** if it is either increasing or decreasing.

Definition

A sequence $\{a_n\}$ is called **bounded above** if there is a number M such that

$$a_n \leq M \text{ for all } n \geq 1$$

A sequence $\{a_n\}$ is called **bounded below** if there is a number m such that

$$m \leq a_n \text{ for all } n \geq 1.$$

A sequence $\{a_n\}$ is called **bounded sequence** if it is bounded above and below.

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

Example* 19 Find the limit of the sequence $\{a_n\}$ defined by the recurrence relation $a_1 = 2$ and $a_{n+1} = \frac{1}{2}(a_n + 6)$ for $n = 1, 2, 3, \ldots$

Step 1 ① $\{a_n\}$ is increasing.

Step 2. ② $0 \leq a_n \leq 6$. bounded.

So $\{a_n\} \longrightarrow L$.

Step 3. Take $\lim_{n\to\infty}$ on both sides

$$a_{n+1} = \frac{1}{2}(a_n + 6)$$

we have $L = \frac{1}{2}(L + 6)$

$$L = 6$$