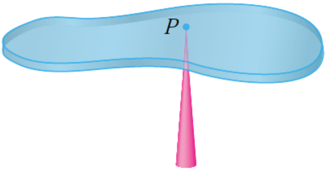
§8.3 Applications to Physics and Engineering

We already used definite integral to compute areas and volumes.

The general procedure of applications of definite integral is the following

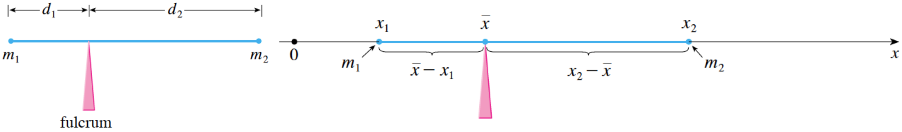
1. divide the problem into n small parts;
2. for each small part, we know how to find a formula for the problem;
3. then we take the sum and limit to get a definite integral.

Goal of this section: Find the point (fulcrum, or centroid) on which a *thin* plate of any given shape balances horizontally.



Moments and Centers of Mass

Case 1. We first look at an easy case: (only two mass points m_1 and m_2)



Archimedes' Law of the Lever tells us:

$$m_1 d_1 = m_2 d_2$$

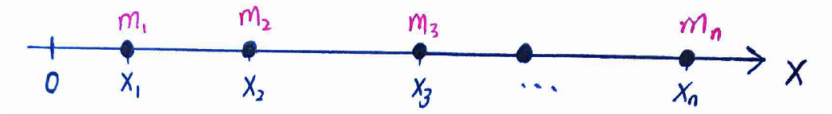
Look at it in x-axis with m_1 at x_1 and m_2 at x_2. The center of the mass is at $\bar{x}$.

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

The **center of the mass** is

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Case 2. Suppose we have n mass points $m_1, m_2, \ldots, m_n$ at points $x_1, x_2, \ldots, x_n$.

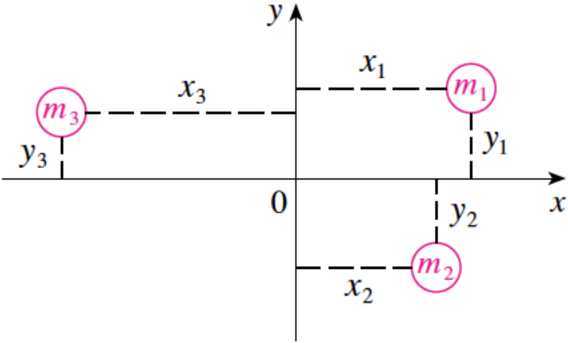


The **center of the mass** is

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

Here, $\sum_{i=1}^{n} m_i x_i$ is the **moment of the system about the origin**. The total mass of the system is $m = \sum_{i=1}^{n} m_i$.

Case 3. More generally, suppose we have n mass points $m_1, m_2, \ldots, m_n$ at points $(x_1, y_2), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane.



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The **moment of the system** **about the y-axis** is

$$M_y = \sum_{i=1}^{n} m_i x_i.$$

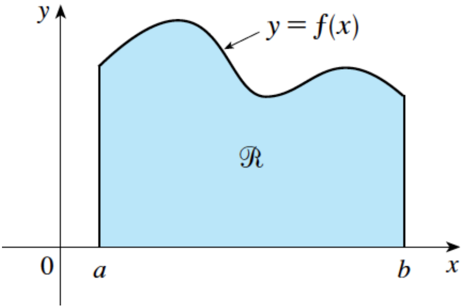
The **moment of the system** **about the x-axis** is

$$M_x = \sum_{i=1}^{n} m_i y_i.$$

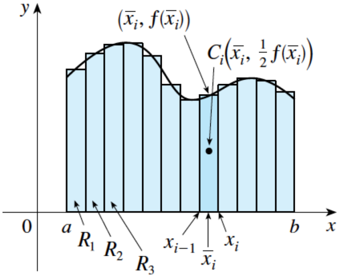
The coordinates $(\bar{x}, \bar{y})$ of the **center of the mass** are given by

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

- Up to now, we only considered **finite** number of mass points.
- We have not used the technique of Calculus yet.
- Now, let us solve our original problem:
  Consider a flat plate (called a **lamina**) with a fixed density ρ that on a region R of the plane.
- Calculate the center of mass of the plate (called the centroid of R).
- Suppose the region R is below a function $y = f(x)$ on the interval $[a, b]$.



We divide the interval $[a, b]$ into n subintervals, hence, divide the region R into n rectangles.



For each rectangle R_i, the **centroid** of R_i is $(\bar{x}_i, \bar{y}_i)$ where

$$\bar{y}_i = \frac{1}{2} f(\bar{x}_i).$$

The **area of the rectangle** R_i is $A_i = f(\bar{x}_i)\Delta x$ and so the **mass of** R_i is

$$m_i = \rho A_i = \rho f(\bar{x}_i)\Delta x.$$

Now, we think our problem as in Case 3: n mass points $m_1, \ldots, m_n$ at $(\bar{x}_1, \bar{y}_1), \ldots, (\bar{x}_n, \bar{y}_n)$.

The **moment of R about the y-axis** is approximated by

$$M_y \approx \sum_{i=1}^{n} m_i x_i = \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i) \Delta x.$$

The moment of R about the y-axis: is

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i) \Delta x = \rho \int_a^b x f(x) dx$$

The **moment of R about the x-axis** is approximated by

$$M_x \approx \sum_{i=1}^{n} m_i y_i = \sum_{i=1}^{n} \rho \frac{1}{2} [f(\bar{x}_i)]^2 \Delta x.$$

The moment of R about the x-axis: is

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \frac{1}{2} [f(\bar{x}_i)]^2 \Delta x = \frac{1}{2} \rho \int_a^b [f(x)]^2 dx$$

The **mass** of the plate R is the product of its density ρ and its area $A = \int_a^b f(x)dx$:

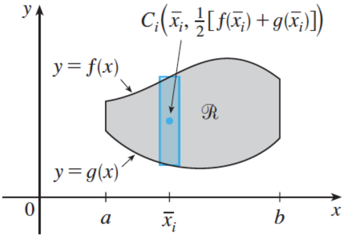
$$m = \rho A = \rho \int_a^b f(x)dx$$

The center of mass (**centroid** $(\bar{x}, \bar{y})$) of the plate R is

$$\bar{x} = \frac{M_y}{m} = \frac{1}{A}\int_a^b xf(x)dx \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{2A}\int_a^b [f(x)]^2 dx$$

Remark: The density ρ is cancelled in the formula of the centroid.

If the region R lies between two curves $y = f(x)$ and $y = g(x)$ for $f(x) \geq g(x)$,



then the center of mass (**centroid** $(\bar{x}, \bar{y})$) is

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]dx$$

$$\bar{y} = \frac{1}{2A}\int_a^b [f(x)]^2 - [g(x)]^2 dx$$

Here A is the area of R calculated by $A = \int_a^b [f(x) - g(x)]dx$.