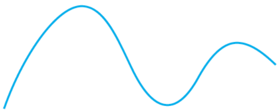
§8.1 Arc Length

We know how to calculate the length of a line segment between $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ by Pythagorean Theorem:

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Question: How to calculate the length of a curve C defined by equation $y = f(x)$, with $a \leq x \leq b$?



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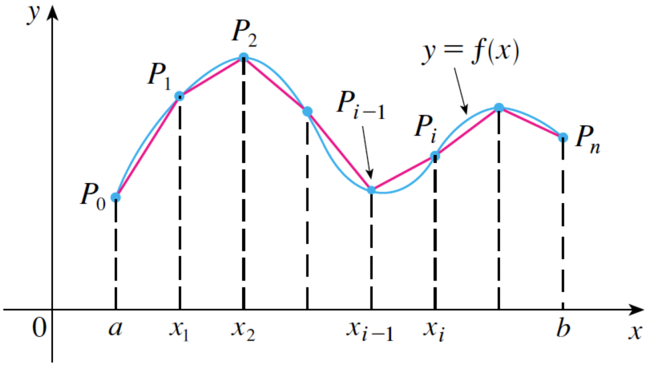

The length L of C is approximately the sum of the lengths of line segments

$$L \approx \sum_{i=1}^{n} |P_{i-1}P_i|.$$

$L \approx \sum_{i=1}^{n} |P_{i-1}P_i|$.Here,

$$\begin{aligned}
|P_{i-1}P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\
&= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\
&= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i \\
&\approx \sqrt{1 + [f'(x_i)]^2}\ \Delta x_i
\end{aligned}$$

The more line segments, the better approximating. To make it precise, we take the limit:

$$\begin{aligned}
L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i)]^2}\Delta x_i \\
&= \int_a^b \sqrt{1 + [f'(x)]^2}\ dx
\end{aligned}$$

Theorem (The Arc Length Formula)

If $f'(x)$ is continuous on $[a, b]$, then the length of the curve $y = f(x)$ on $[a, b]$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Similarly,

Theorem

If $x = g'(y)$ *is continuous on* $c \leq y \leq d$*, then the length of the curve* $x = g(y)$ *on* $c \leq y \leq d$ *is*

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

It is useful to define a function $s(x)$ measuring the arc length of a curve from starting point $t = a$ to any other point $t = x$ on the curve C.

The Arc Length Function

If $y = f'(t)$ is continuous on $[a, b]$, then the the Arc Length **Function** is defined as

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

Here, x is the variable for the arc length function.
The Fundamental Theorem of Calculus gives

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Take squares both sides, It can also be written as

$$(ds)^2 = (dx)^2 + (dy)^2$$