Type I: Infinite integrals

Definition (improper integral of Type I)

(1.) If $\int_a^t f(x)\,dx$ exists for all $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

(2.) If $\int_t^b f(x)\,dx$ exists for all $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

The improper integral is called **convergent** if the limit exists (as a finite number) and **divergent** if the limit does not exist.

(3.) If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

Type II: discontinuous integrals

Definition (improper integral of Type II)

(1.) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\, dx = \lim_{t \to b-} \int_a^t f(x)\, dx$$

(2.) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\, dx = \lim_{t \to a+} \int_t^b f(x)\, dx$$

The improper integral is called **convergent** if the limit exists (as a finite number) and **divergent** if the limit does not exist.

(3.) If f is a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ convergent, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Theorem (Comparison Theorem)

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for x on $[a, \infty]$.

(1) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(2) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.