§7.4

- A **polynomial** is a function with variable x of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, \cdots, a_1, a_0$ are real numbers.

- If $a_n \neq 0$, we say the **degree** of $p(x)$ is n.

- A ratio of polynomials $f(x) = \dfrac{p(x)}{q(x)}$ is called a **rational function**.

- If the degree $\deg(p(x)) \geq \deg(q(x))$, we say that the rational function $f(x) = \dfrac{p(x)}{q(x)}$ is **improper**.

- On the other side, if the degree $\deg(p(x)) < \deg(q(x))$, we say that the rational function $f(x) = \dfrac{p(x)}{q(x)}$ is **proper**.

We can simplify an improper rational function $f(x) = \frac{p(x)}{q(x)}$ as the sum of a polynomial and a proper rational function:

$$f(x) = s(x) + \frac{r(x)}{q(x)}$$

such that $\deg(r(x)) < \deg(q(x))$.

In this section, the title is §*7.4 Integration of Rational Functions (by Partial Fractions.)*

The integration of a polynomial function $s(x)$ is easy. So, we only focus on solving Integration of Proper Rational Functions $\frac{r(x)}{q(x)}$.

We can always express a *simple* rational function $f(x) = \dfrac{r(x)}{q(x)}$ as a sum of simpler fractions, called **partial fractions:**

$$\text{Type (I):} \quad \frac{A}{(ax+b)^m} \quad \text{or} \quad \text{Type (II):} \quad \frac{Ax+B}{(ax^2+bx+c)^n}$$

In Type (II), $b^2 - 4ac < 0$.

Case 1. The sum only involves the first type of partial fractions.

Partial fraction technique: Express a *simple* rational function $f(x) = \frac{r(x)}{q(x)}$ as a sum of **partial fractions.**

1. Any polynomial can be decomposed as the products of powers of linear functions $(ax + b)^m$ and powers of quadratic polynomials $(ax^2 + bx + c)^n$.
2. Then we can suppose the partial fraction decomposition is the sum of

$$\text{Type (I):} \quad \frac{A}{(ax+b)^m} \quad \text{or} \quad \text{Type (II):} \quad \frac{Bx+C}{(ax^2+bx+c)^n}$$

Then, solve all A, B, C in the decomposition.

Case 2. The sum involves partial fractions of Type (II).

We need to use the result

$$\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

or the substitution by $x = a \tan\theta$.