§11.6 Absolute Convergence and the Ratio and Root Tests

Let $\sum_{n=1}^{\infty} a_n$ be a series.
We consider a new series of absolute values:

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + |a_4| + |a_5| + \cdots$$

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem

If a series $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, then it is convergent.*

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

The Ratio Test

The following test is very useful and powerful.

Theorem (The Ratio Test)

Given a series $\sum a_n$, and suppose

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

(1.) *If $L < 1$, then the series $\sum a_n$ is absolutely convergent.*

(2.) *If $L > 1$ or $L = \infty$, then the series $\sum a_n$ is divergent.*

(3.) *If $L = 1$, then the Ratio Test is inconclusive.*

The Root Test*

The following test is convenient to apply when n-th powers occur.

Theorem (The Root Test)

Given a series $\sum a_n$, and suppose

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

(1.) *If $L < 1$, then the series $\sum a_n$ is absolutely convergent.*

(2.) *If $L > 1$ or $L = \infty$, then the series $\sum a_n$ is divergent.*

(3.) *If $L = 1$, then the Ratio Test is inconclusive.*