§11.2 Series

Definition

An (infinite) **series** is the sum of a sequence $\{a_n\}$, that is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

The sum symbol is denoted by $\sum a_n$ for short.

A new sequence of Partial Sums

Definition

We define a new sequence $\{s_n\}_{n=1}^{\infty}$ by partial sums

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = L$, then we say that $\sum a_n$ is **convergent** and write

$$\sum_{n=1}^{\infty} = L.$$

The number L is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, we say that the series is **divergent**.

Example

Example 3 (Geometric Series $\sum ar^{n-1}$). The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and

$$\sum ar^{n-1} = \frac{a}{1-r} \qquad \text{if } \; |r| < 1.$$

If $|r| \geq 1$, then the geometric series $\sum ar^{n-1}$ is divergent.

Theorem (Divergence Test)

If the series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Equivalently,

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.

Warning:

The **converse** of the Divergence Test is **not** true.

Example

Example 14. The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.