§11.1 Sequences and Limits

Definition

A **sequence** is a list of order numbers

$$\{a_1, a_2, a_3, a_4, \ldots, a_n, \ldots\},$$

which is denoted by

$$\{a_n\}_{n=1}^{\infty}, \quad \text{or} \quad \{a_n\} \quad \text{for short.}$$

Example

(1). $\{a_n\}_{n=1}^{\infty}$ is $\{2n-1\}$

(2). $\{b_n\}_{n=1}^{\infty}$ is $\left\{\dfrac{(-1)^n n}{e^n}\right\}$

(3). $\{c_n\}_{n=1}^{\infty}$ is $\left\{\sqrt{3n}\right\}$

Definition (Limits)

If a sequence $\{a_n\}$ has the limit L as $n \to \infty$, we write

$$\lim_{n\to\infty} a_n = L,$$

and say $\{a_n\}$ **converges** to L.

If $\{a_n\}$ has no limit, we say $\{a_n\}$ **diverges**.

Theorem

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then,

$$\lim_{n\to\infty} f(n) = L.$$

Definition (Limit L (Precise Definition.))

A sequence $\{a_n\}$ has **limit** L as $n \to \infty$,
if for every $\epsilon > 0$ there is a corresponding integer N such that

$$\textbf{if} \quad n > N, \ \textbf{then} \ |a_n - L| < \epsilon.$$

Definition (Limit ∞ (Precise Definition.))

A sequence $\{a_n\}$ has **limit** ∞ as $n \to \infty$, denoted as $\lim_{n\to\infty} a_n = \infty$, **if for every positive number M there is an integer N such that**

$$\textbf{if } n > N, \ \textbf{then } a_n > M.$$

The Limit Laws for functions hold for the limits of sequences

Theorem (Limit Laws)

If $\{a_n\}$ and $\{b_n\}$ are convergent and k is a constant number, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} k a_n = k \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

$$\lim_{n\to\infty}(a_n)^p = (\lim_{n\to\infty} a_n)^p \text{ if } p > 0 \text{ and } a_n > 0.$$

Theorem

The Squeeze Theorem: If $a_n \leq b_n \leq c_n$ for $n \geq N$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then

$$\lim_{n\to\infty} b_n = L$$

Theorem

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Theorem

Theorem. If $\lim_{n\to\infty} a_n = L$ and the function $f(x)$ is continuous at $x = L$ then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Definition (Monotonic)

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is

$$a_1 < a_2 < a_3 < \cdots .$$

A sequence $\{a_n\}$ is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$, that is

$$a_1 > a_2 > a_3 > \cdots .$$

A sequence $\{a_n\}$ is called **monotonic** if it is either increasing **or** decreasing.

Definition (Bounded)

A sequence $\{a_n\}$ is called **bounded above** if there is a number M such that

$$a_n \leq M \text{ for all } n \geq 1$$

A sequence $\{a_n\}$ is called **bounded below** if there is a number m such that

$$m \leq a_n \text{ for all } n \geq 1.$$

A sequence $\{a_n\}$ is called **bounded sequence** if it is bounded above **and** below.

Monotonic Sequence Theorem

Every *bounded*, *monotonic* sequence is convergent.