Resonance varieties, Chen ranks and formality properties of finitely generated groups

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Abstract of Dissertation

Formality is a topological property that arises from the rational homotopy theory developed by Quillen and Sullivan in 70's. Roughly speaking, the rational homotopy type of a formal space is determined by its cohomology algebra. In this thesis, we explore the graded-formality, filtered-formality, and 1-formality of finitely-generated groups, by studying various Lie algebras over a field of characteristic 0 attached to such groups, including the associated graded Lie algebra, the holonomy Lie algebra, and the Malcev Lie algebra. We explain how these notions behave with respect to split injections, coproducts, direct products, and how they are inherited by solvable and nilpotent quotients.

We investigate the varied relationships among several algebraic and geometric invariants of finitely-generated groups, including the aforementioned Lie algebras, commutative differential graded algebras, Chen Lie algebras, Alexander-type invariants as well as resonance varieties and characteristic varieties. Significant results arise from the study of the interactions between theses objects, e.g., the tangent cone theorem of Dimca, Papadima and Suciu, and the Chen ranks formula conjectured by Suciu and proved by Cohen and Schenck.

For a finitely-presented group, we give an explicit formula for the cup product in low degrees, and an algorithm for computing the holonomy Lie algebra, using a Magnus expansion method. We also give a presentation for the Chen Lie algebra of a filtered-formal group, and discuss various approaches to computing the ranks of the graded objects under consideration.

We apply our techniques to several families of braid-like groups: the pure braid groups, the pure welded braid groups, the virtual pure braid groups, as well as their 'upper' variants. We also discuss several natural homomorphisms between these groups, and various ways to distinguish among them. We illustrate our approach with examples drawn from a variety of group-theoretic and topological contexts, such as 1-relator groups, finitely generated torsion-free nilpotent groups, link groups, and fundamental groups of Seifert fibered manifolds.

Table of Contents

List of Figures

Chapter 1

Introduction

The main focus of this thesis is on algebraic and geometric invariants of finitely generated groups, including commutative differential graded algebras, various graded or filtered Lie algebras, several modules over Laurent polynomial rings or polynomial rings, as well as two types of cohomology jump loci. We investigate the formality properties of finitely generated groups, which play an important role on studying the structure of these invariants and the relationships among them. We illustrate our approach with examples drawn from group theory, geometry and topology, including pure braid groups, pure virtual braids groups, pure welded braid groups, 1-relator groups, finitely generated torsion-free nilpotent groups, link groups, and fundamental groups of Seifert fibered manifolds. This thesis is based on my work with Alex Suciu in papers [143, 144, 145, 146].

1.1 Background and preliminaries

1.1.1 From groups to Lie algebras

Throughout, we will let G be a finitely generated group, and we will let $\mathbb{Q}$ be the field of rational numbers. Our main focus will be on several $\mathbb{Q}$-Lie algebras attached to such a group, and the way they all connect to each other.

By far the best known of these Lie algebras is the *associated graded Lie algebra*, $\mathrm{gr}(G;\mathbb{Q})$, introduced by P. Hall, W. Magnus, and E. Witt in the 1930s, cf. [102]. This is a finitely generated graded Lie algebra, whose graded pieces are the successive quotients of the lower central series of G (tensored with $\mathbb{Q}$), and whose Lie bracket is induced from the group commutator. The quintessential example is the free Lie algebra $\mathfrak{lie}(\mathbb{Q}^n)$, which is the associated graded Lie algebra of the free group on n generators, F_n. For a group G with a finite presentation, various approaches for finding a presentation for $\mathrm{gr}(G;\mathbb{Q})$ have been given by Lazard [89], Labute [83, 84], Falk–Randell [48], Anick [2], et al.

Closely related is the *holonomy Lie algebra*, $\mathfrak{h}(G;\mathbb{Q})$, introduced by T. Kohno in [81], building on work of K.-T. Chen [28], and further studied by Markl–Papadima [105] and Papadima–Suciu [117]. This is a quadratic Lie algebra, obtained as the quotient of the free Lie algebra on $H_1(G;\mathbb{Q})$ by the ideal generated by the image of the dual of the cup product map in degree 1. The holonomy Lie algebra comes equipped with a natural epimorphism $\Phi_G\colon \mathfrak{h}(G;\mathbb{Q}) \twoheadrightarrow \mathrm{gr}(G;\mathbb{Q})$, and can be viewed as the quadratic approximation to the associated graded Lie algebra of G. A presentation for the holonomy Lie algebra of a group with a finite commutator-relators presentation, was given by Papadima–Suciu [117], using the Magnus expansion. In this thesis, we further developed this method for any finitely presented group.

The most intricate of these Lie algebras (yet, in many ways, the most important) is the *Malcev Lie algebra*, $\mathfrak{m}(G;\mathbb{Q})$. As shown by A. Malcev in [104], every finitely generated, torsion-free nilpotent group N is the fundamental group of a nilmanifold, whose corresponding $\mathbb{Q}$-Lie algebra is $\mathfrak{m}(N;\mathbb{Q})$. Taking now the nilpotent quotients of G, we may define $\mathfrak{m}(G;\mathbb{Q})$ as the inverse limit of the resulting tower of nilpotent Lie algebras, $\mathfrak{m}(G/\Gamma_k G;\mathbb{Q})$. By construction, this is a complete, filtered Lie algebra. In two seminal papers, [132, 131], D. Quillen showed that $\mathfrak{m}(G;\mathbb{Q})$ is the set of all primitive elements in $\widehat{\mathbb{Q}G}$ (the completion of the group algebra of G with respect to the filtration by powers of the augmentation ideal), and that the associated graded Lie algebra of $\mathfrak{m}(G;\mathbb{Q})$ is isomorphic to $\mathrm{gr}(G;\mathbb{Q})$. In particu-

lar, $\mathrm{gr}(\mathfrak{m}(G;\mathbb{Q}))$ is generated in degree 1. If G admits a finite presentation, one can use this approach to find a presentation for the Malcev Lie algebra $\mathfrak{m}(G;\mathbb{Q})$, see Massuyeau [109] and Papadima [116].

1.1.2 Algebraic models and formality

Algebraic models play an important role in algebraic topology, especially in rational homotopy theory. They are also very useful in homological algebra, operad theory, and geometry. The de Rham complex provides a commutative differential graded algebra (CDGA) model for a smooth manifold. More generally, in his foundational paper on rational homotopy theory [148], D. Sullivan associated to each path-connected space X a CDGA model, $\mathcal{M}(X)$, called a 'minimal model', which can be viewed as an algebraic approximation to the space. Any CDGA which is quasi-isomorphic to the minimal model is an algebraic model of the space.

There is a close relationship between CDGA models of a space X and the Lie algebras attached to the fundamental group $G = \pi_1(X)$, provided that X is a connected CW-complex with finitely many 1-cells. As shown by Cenkl and Porter [25], Griffiths and Morgan [63] and Sullivan [148], the Lie algebra dual to the first stage of the minimal model is isomorphic to the Malcev Lie algebra $\mathfrak{m}(G;\mathbb{Q})$. The Chevalley–Eilenberg complex of the Malcev Lie algebra gives the first stage of the minimal model. As shown in [105] (see also Theorem 2.3.10), the dual Lie algebra of the 1-minimal model of $H^*(G;\mathbb{Q})$ is isomorphic to the degree completion of the holonomy Lie algebra of G.

Formality is an important property in rational homotopy theory. Roughly speaking, the rational homotopy type of a simply-connected formal space is determined by its cohomology algebra over $\mathbb{Q}$. In this thesis, we focus on developing the rational homotopy theory on non-simply-connected spaces. The space X is said to be *formal* if the commutative, graded differential algebra $\mathcal{M}(X)$ is quasi-isomorphic to the cohomology ring $H^*(X;\mathbb{Q})$, endowed with the zero differential. If there exists a CDGA morphism from the i-minimal model

$\mathcal{M}(X, i)$ to $H^*(X; \mathbb{Q})$ inducing isomorphisms in cohomology up to degree i and a monomorphism in degree $i+1$, then X is called *i-formal*. As shown in [39], the compact Kähler manifolds are formal spaces. The complements of complex hyperplane arrangements are also formal spaces [21]. The higher Heisenberg group $\mathcal{H}_n$ is $(n-1)$-formal, but not n-formal, see Măcinic [99].

All the Lie algebras and the algebraic models mentioned above can be defined over any field $\mathbb{K}$ of characteristic 0. As is well-known, a space X with finite Betti numbers is formal over $\mathbb{Q}$ if and only it is formal over $\mathbb{K}$. This foundational result was proved independently and in various degrees of generality by Halperin and Stasheff [67], Neisendorfer and Miller [115], and Sullivan [148].

A finitely generated group G is said to be 1-*formal* (over $\mathbb{Q}$) if it has a classifying space $X = K(G, 1)$ which is 1-formal. The 1-formality of X depends only on the fundamental group $G = \pi_1(X)$. Hence, the study of the various Lie algebras attached to the fundamental group of a space provides a fruitful way to look at the formality problem. Indeed, the group G is 1-formal if and only if the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{Q})$ is isomorphic to the rational holonomy Lie algebra of G, completed with respect to the lower central series (LCS) filtration.

1.1.3 Cohomology jumping loci

The cohomology jumping loci associated to a connected CW-complex X of finite-type are essential tools in this thesis, including resonance varieties and characteristic varieties. The resonance varieties of X, which originated from the study of complements of hyperplane arrangements by M. Falk in [50], are homogeneous subvarieties of $H^1(X; \mathbb{C})$. More generally, the resonance varieties of a complex CDGA (A^*, d) with finite dimensional A^1, recently studied by Dimca, Papadima, Suciu, and others, [13, 125, 43, 100, 142], are defined by

$$\mathcal{R}^i_k(A, d) = \{a \in A^1 \mid \dim(H^i(A; \delta_a)) \geq k\}, \quad \text{where } \delta_a(u) = d(u) + a \cdot u \text{ for } u \in A^i. \quad (1.1)$$

The *resonance varieties* of X are the resonance varieties of $H^*(X;\mathbb{C})$ with zero differential, denoted by $\mathcal{R}^i_k(X)$. The *characteristic varieties* of X are the jumping loci for cohomology with coefficients in rank 1 local systems,

$$\mathcal{V}^i_k(X) = \{\rho \in \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \mid \dim(H_i(X;\mathbb{C}_\rho)) \geq k\}. \tag{1.2}$$

If a group G admits a finite-type classifying space $K(G,1)$, the jump loci of the group G are defined in terms of the jump loci of the corresponding classifying space. The cohomology algebra $H^*(G,\mathbb{C})$ may be turned into a family of cochain complexes parametrized by the affine space $H^1(G,\mathbb{C})$, from which one may define the *resonance varieties* of the group, $\mathcal{R}^i_d(G)$, as the loci where the cohomology of those cochain complexes jumps. The resonance varieties and characteristic varieties of some classes of such groups have been studied. The degree one resonance varieties of right-angled Artin groups were described explicitly in [118, Theorem 5.5]. A complete description of the resonance varieties and the characteristic varieties, in all degrees, is described in [119, Theorem 3.8] for toric complexes associated to arbitrary finite simplicial complexes. Results relating to the cohomology jump loci of the complement of a complex hyperplane arrangement can be found in [50, 33, 94].

Best understood are the degree 1 cohomology jump loci, $\mathcal{R}_k(X) = \mathcal{R}^1_k(X)$ and $\mathcal{V}_k(X) = \mathcal{V}^1_k(X)$, which depend only on the fundamental group $G = \pi_1(X)$. In this case, the tangent cone to $\mathcal{V}_k(G)$ at the origin 1 is contained in $\mathcal{R}_k(G)$, see Libgober [93]. As shown in [42] by Dimca, Papadima and Suciu, these two types of varieties are closely related by the *Tangent Cone Theorem*: If G is 1-formal, then $\mathrm{TC}_1(\mathcal{V}_k(G)) = \mathcal{R}_k(G)$, and all irreducible components of $\mathcal{R}_k(G)$ are rationally defined linear subspaces of $H^1(G;\mathbb{C})$. This yields new and powerful obstructions for a finitely generated group G to be 1-formal.

1.1.4 Alexander invariants

The Alexander invariants, originating from the study the Alexander polynomials of knots and links by J.W. Alexander in [1], play an important role in investigating resonance varieties,

characteristic varieties and Chen ranks.

Let X be a connected CW-complex, with fundamental group G. Let $X' \to X$ be the Galois cover corresponding to the commutator subgroup $G' \subset G$. Then the Alexander invariant of X is defined by $B(G) := H_1(X'; \mathbb{C})$ and the action of $G_{\mathrm{ab}} = G/G'$ corresponds to the action in homology of the group of covering transformations. A useful algebraic interpretation for the Alexander invariant is given by $B(G) = G'/G''$, with the action of G_{ab} given by conjugation. In [106], W.S. Massey gave a presentation for the Alexander invariant for the complement of links. In [33, 32], Cohen and Suciu further developed Massey's method and gave an explicit presentation for the Alexander invariant of the complement of a complex hyperplane arrangement using Fox derivatives.

The work of E. Hironaka [71] shows that that the degree 1 characteristic varieties of a finitely generated group G coincide with the support varieties of its Alexander invariant, at least away from the origin. A more general statement, valid in arbitrary degrees, was recently proved in [122]

In a similar fashion, we define the infinitesimal Alexander invariant of a finitely generated, graded Lie algebra $\mathfrak{g}$ to be the graded $\mathrm{Sym}(\mathfrak{g}_1)$-module $\mathfrak{B}(\mathfrak{g}) = \mathfrak{g}'/\mathfrak{g}''$. Suppose G is a finitely presented, commutator-relators group. As shown in [110], for each $k \geq 1$, the resonance variety $\mathcal{R}_k(G)$ coincides, at least away from the origin $0 \in H^1(G; \mathbb{C})$, with the support variety of the annihilator of d-th exterior power of the infinitesimal Alexander invariant; that is,

$$\mathcal{R}_k(G) = \mathrm{Supp}\Big(\bigwedge^k \mathfrak{B}(\mathfrak{h}(G))\Big) := V\Big(\mathrm{Ann}\Big(\bigwedge^k \mathfrak{B}(\mathfrak{h}(G))\Big)\Big). \tag{1.3}$$

1.1.5 Chen Lie algebras

K.T. Chen in [27] studied the lower central series (LCS) quotients of the maximal metabelian quotient G/G'', of a finitely generated group G, which were called *Chen groups* by Murasugi in [114], who used them to study the Milnor invariants of links. The associated graded Lie

algebras $\operatorname{gr}(G/G'';\mathbb{C})$ of the maximal metabelian quotient are called the *Chen Lie algebras*. The LCS ranks of the Chen groups are called the *Chen ranks*, and are denoted by $\theta_k(G)$. Chen computed the Chen ranks $\theta_k(F_n)$ by introducing a path integral technique associated to a free group F_n with n generators. Let G be the fundamental group of the complement of a link of n components with connected linking graph. As conjectured by Murasugi [114] and proved by Massey–Traldi [108] and Labute [85], the link group G has the same LCS ranks ϕ_k and the same Chen ranks θ_k as the free group F_{n-1}, for all $k > 1$. Furthermore, G has the same Chen Lie algebra as F_{n-1} (see [117]).

As shown by Massey in [106], the Chen ranks of G can be computed from the Alexander invariant $B(G)$. More precisely, if we view this abelian group as a module over $\mathbb{C}[G_{\mathrm{ab}}]$, then filter it by powers of the augmentation ideal, and take the associated graded module, $\operatorname{gr}(B(G))$, viewed as a module over the symmetric algebra $S = \operatorname{Sym}(H \otimes \mathbb{C})$, we have that $\theta_{k+2}(G) = \dim \operatorname{gr}_k(B(G))$ for all $k \geq 0$. Similarly, the Chen ranks of a finitely generated, graded Lie algebra $\mathfrak{g}$ are defined to be $\theta_k(\mathfrak{g}) = \dim(\mathfrak{g}/\mathfrak{g}'')_k$, and we show that $\theta_{k+2}(\mathfrak{g}) = \dim \mathfrak{B}(\mathfrak{g})_k$ for all $k \geq 0$.

In [31], Cohen and Suciu developed Massey's method and computed the Chen ranks of the pure braid groups P_n by finding the Gröbner basis of the corresponding Alexander invariants. Generalizing a theorem in [117], we connect the Chen Lie algebra and the associated graded Lie algebra of a filtered-formal group. If G is a filtered-formal group, then the Chen Lie algebra $\operatorname{gr}(G/G'';\mathbb{Q})$ is isomorphic to $\operatorname{gr}(G;\mathbb{Q})/\operatorname{gr}(G;\mathbb{Q})''$.

There is a close relationship between Chen ranks and resonance varieties. In [140], Suciu conjectured that the Chen ranks of an arrangement group G are given by

$$\theta_k(G) = \sum_{m \geq 2} h_m \cdot \theta_k(F_m), \qquad \text{for } k \gg 0, \tag{1.4}$$

where h_m is the number of m-dimensional components of $\mathcal{R}_1(G)$, and F_m is the free group with m generators. Recently, D. Cohen and Schenck [34] showed that, for a finitely presented, commutator-relators 1-formal group G, the Chen ranks formula holds, provided the

components of $\mathcal{R}_1(G)$ are zero-isotropic, projectively disjoint, and reduced as schemes.

1.1.6 Summary of algebraic and geometric invariants

We summarize all the algebraic and geometric invariants described above using diagram 1.1 in the next page. The following questions are interesting and important to investigate.

- How to compute or describe these invariants for a finitely presented group?
- How to compute the dimensions or ranks of these graded Lie algebras or modules in each degree.
- When do the dotted arrows exist? When are the various maps isomorphisms?
- How do these invariants behave with respect to products, coproducts, semidirect products, inclusions, projections, etc.?
- What are the relationships among these invariants?
- Use these invariants to study several families of important spaces and groups.

Research Diagram:

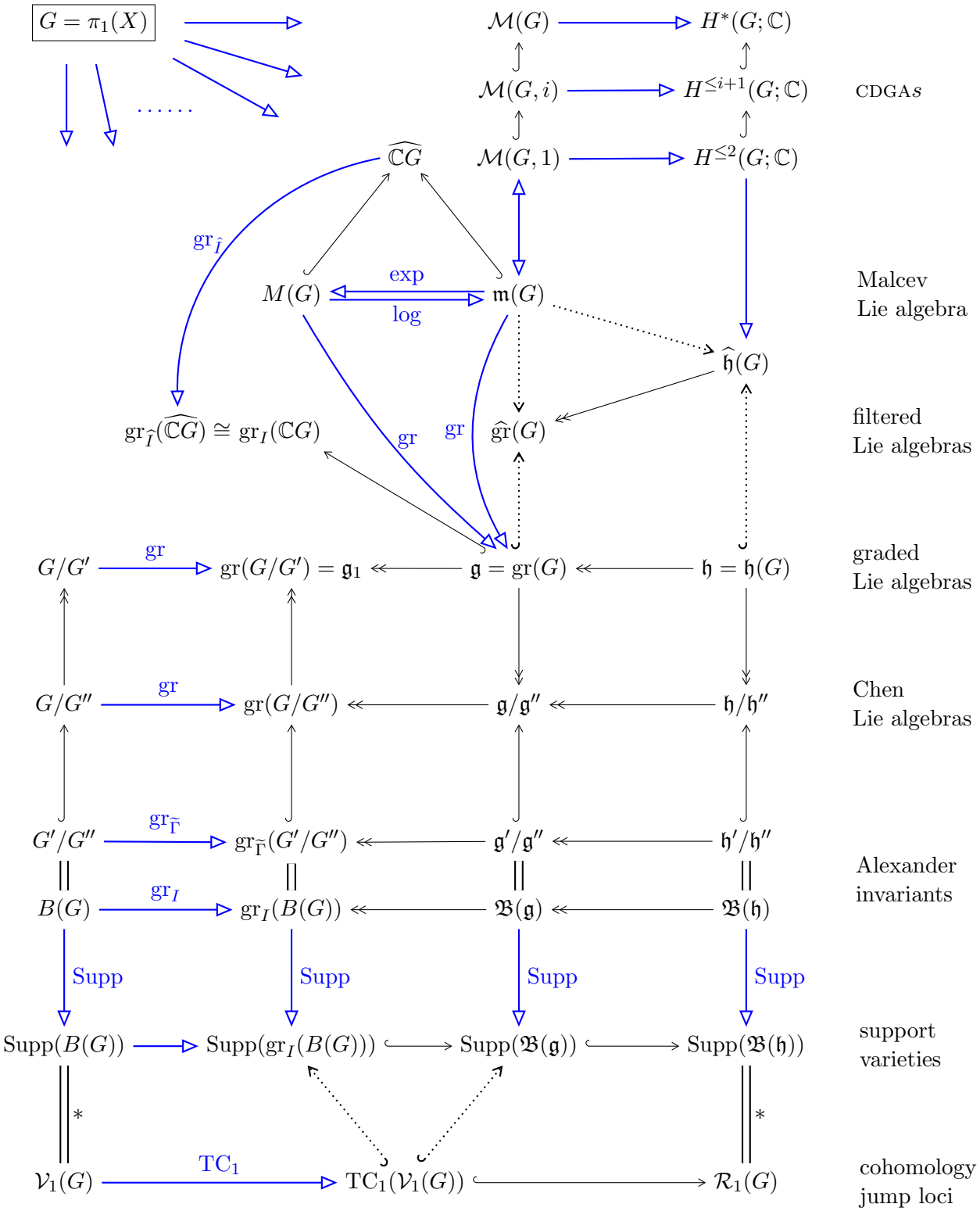


Figure 1.1: Algebraic and geometric invariants of $G = \pi_1(X)$.

Note: 1. The ground field is $\mathbb{C}$. 2. The open triangle arrows are functors. $*$: away from origin.

1.1.7 Pure braid groups and their relatives

The techniques described above have a large range of applicability in a variety of examples in group theory, algebraic geometry, low-dimensional topology and geometry. We start with the braid groups and their relatives, which have showed their importance in several important fields of mathematics as well as in physics.

Pure braid groups

Let F_n be the free group on generators $x_1, \dots, x_n$, and let $\mathrm{Aut}(F_n)$ be its automorphism group. Magnus [101] showed that the map $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ which sends an automorphism to the induced map on the abelianization $(F_n)^{\mathrm{ab}} = \mathbb{Z}^n$ is surjective, with kernel denoted by IA_n. An automorphism of F_n is called a *permutation-conjugacy*, if it sends each generator x_i to a conjugate of $x_{\tau(i)}$, for some permutation $\tau \in S_n$.

The classical Artin braid group B_n is the subgroup of $\mathrm{Aut}(F_n)$ consisting of those permutation-conjugacy automorphisms which fix the word $x_1 \cdots x_n \in F_n$, see [18, 69]. The kernel of the canonical projection from B_n to the symmetric group S_n is the pure braid group P_n on n strings, whose classifying space is $\mathrm{Conf}_n(\mathbb{C})$, the configuration space of n ordered points in the complex plane. The cohomology algebras of the pure braid groups are computed by Arnold in [3].

Pure welded braid groups

The set of all permutation-conjugacy automorphisms forms a subgroup of $\mathrm{Aut}(F_n)$, known as the welded braid group wB_n, cf. [4, 6, 9, 15, 54]. As shown in [54], wB_n is isomorphic to a group of generalised braids with the classical crossing and the welded crossing in Figure 1.3. The pure welded braid group, $wP_n = wB_n \cap \mathrm{IA}_n$, is generated by the Magnus automorphisms α_{ij} $(1 \leq i \neq j \leq n)$, which send x_i to $x_j x_i x_j^{-1}$ and leave invariant the remaining generators of F_n. The subgroup generated by the automorphisms α_{ij} with $i < j$ is called the *upper*

pure welded braid group wP_n^+. McCool gave presentations for wP_n and wP_n^+ in [112], (also known as the McCool groups). As shown in [20], the pure welded braid group wP_n (wP_n^+) is the fundamental group of the space of configurations of "parallel rings" (of unequal size). The cohomology algebra of wP_n was given in [75], while the cohomology algebra of wP_n^+ was given in [29].

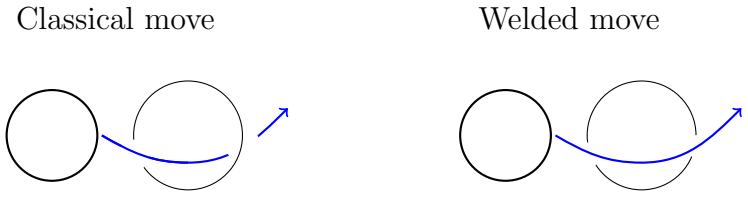


Figure 1.2: Untwisted flying rings.

Pure virtual braid groups

Another class of braid-like groups are the virtual braid groups vB_n, which were introduced in [78] and further studied in [6, 9, 79, 7, 76]. As shown by Kamada in [76], any virtual link can be constructed as the closure of a virtual braid, which is unique up to certain Reidemeister-type moves. In this thesis, we will be mostly interested in the kernel of the canonical epimorphism $vB_n \to S_n$, called the *pure virtual braid group*, vP_n, and a certain subgroup of this group, vP_n^+, which we call the *upper pure virtual braid group*. A presentation for vP_n and vP_n^+ was given by Bardakov in [7], Whether or not the virtual (pure) braid groups are subgroups of $\mathrm{Aut}(F_n)$ is still an open question [7, 62]. The groups vP_n and vP_n^+ were also independently studied in [11, 91] as groups arising from the Yang-Baxter equations. Classifying spaces for these groups can be constructed by taking quotients of permutahedra by suitable actions of the symmetric groups.

The three types of braid crossings mentioned above are depicted in Figure 1.3.

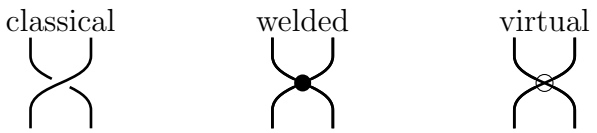


Figure 1.3: Braid crossings.

Pure braid groups on surfaces

Another important class of braid-like groups are the pure braid groups on compact Riemann surfaces Σ_g of genus g, denoted by $P_{g,n}$. Much work has been done for this class of groups, see [16, 13, 23, 65], but still there are several unsolved problems. In our future work, we will compute the resonance varieties of $P_{g,n}$, the resonance varieties of the CDGA model of $P_{g,n}$, and the Chen ranks of $P_{g,n}$ and explore their relationship.

The groups mentioned so far fit into the diagram from Figure 1.4. A related diagram can be found in [4].

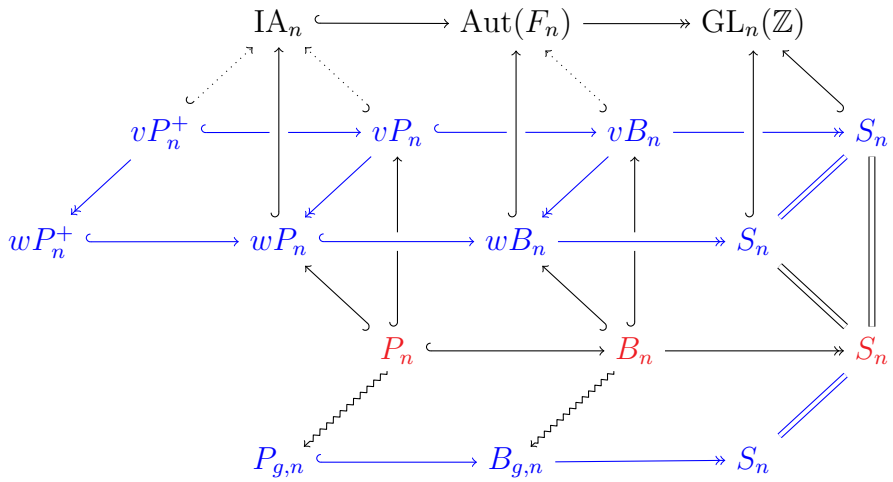


Figure 1.4: Braid-like groups and automorphism groups of free groups.

This work was motivated in good part by the papers [11, 23] of Etingof et al. on the triangular and quasi-triangular groups, also known as the (upper) pure virtual braid groups. In §7, we apply the techniques developed here to study the formality properties of such

groups. Related results for the pure welded braid groups and other braid-like groups will be given in §7, §8 and §9.4.

1.2 Summary of main results

1.2.1 Graded formality and filtered formality

We find it useful to separate the 1-formality property of a group G into two complementary properties: graded formality and filtered formality. More precisely, we say that G is *graded-formal* (over $\mathbb{Q}$) if the associated graded Lie algebra $\mathrm{gr}(G;\mathbb{Q})$ is isomorphic, as a graded Lie algebra, to the holonomy Lie algebra $\mathfrak{h}(G;\mathbb{Q})$. Likewise, we say that G is *filtered-formal* (over $\mathbb{Q}$) if the Malcev Lie algebra $\mathfrak{m} = \mathfrak{m}(G;\mathbb{Q})$ is isomorphic, as a filtered Lie algebra, to the completion of its associated graded Lie algebra, $\widehat{\mathrm{gr}}(\mathfrak{m})$, where both $\mathfrak{m}$ and $\widehat{\mathrm{gr}}(\mathfrak{m})$ are endowed with the respective inverse limit filtrations. As we show in Proposition 3.3.4, the group G is 1-formal if and only if it is both graded-formal and filtered-formal.

All four possible combinations of these formality properties occur:

1. Examples of 1-formal groups include finitely generated free groups and free abelian groups (more generally, right-angled Artin groups), groups with first Betti number equal to 0 or 1, fundamental groups of compact Kähler manifolds, and fundamental groups of complements of complex algebraic hypersurfaces.

2. There are many torsion-free, nilpotent groups (Example 3.3.6, or, more generally, 9.1.7) as well as link groups (Examples 9.2.15 and 9.2.16) which are filtered-formal, but not graded-formal.

3. There are also finitely presented groups, such as those from Examples 3.3.7, 9.2.6, and 9.2.17 which are graded-formal but not filtered-formal.

4. Finally, there are groups which enjoy none of these formality properties. Indeed, if G_1 is one of the groups from (2) and G_2 is one of the groups from (3), then Theorem 1.2.3 below shows that the product $G_1 \times G_2$ and the free product $G_1 * G_2$ are neither graded-formal, nor filtered-formal.

For a finite-dimensional, nilpotent Lie algebra $\mathfrak{m}$ over the field $\mathbb{Q}$, the filtered-formality of such a Lie algebra coincides with the notions of 'Carnot', 'naturally graded', 'homogeneous' and 'quasi-cyclic' which appear in [37, 77, 92].

Recently, D. Bar-Natan has explored the Taylor expansion of any ring R, see [5]. In the case $R = \mathbb{Q}G$, the existence of a Taylor expansion of R is equivalent to saying that G is filtered-formal. The existence of a quadratic Taylor expansion of R is equivalent to the 1-formality of G.

1.2.2 Minimal model and formality

We start by reviewing in §2.1–§2.2 some basic notions pertaining to filtered and graded Lie algebras, as well as the notions of quadratic and Koszul algebras, while in §2.3, we analyze in detail the relationship between the 1-minimal model $\mathcal{M}(A, 1)$ and the dual Lie algebra $\mathfrak{L}(A)$ of a differential graded $\mathbb{Q}$-algebra (A, d). The reason for doing this is a result of Sullivan [148], which gives a functorial isomorphism of pronilpotent Lie algebras, $\mathfrak{L}(A) \cong \mathfrak{m}(G; \mathbb{Q})$, provided $\mathcal{M}(A, 1)$ is a 1-minimal model for a finitely generated group G.

Of particular interest is the case when A is a connected, graded commutative algebra with $\dim(A^1) < \infty$, endowed with the differential $d = 0$. In Theorem 2.3.10, we show that $\mathfrak{L}(A)$ is isomorphic (as a complete, filtered Lie algebra) to the degree completion of the holonomy Lie algebra of A. In the case when $A^{\leq 2} = H^{\leq 2}(G; \mathbb{Q})$ for some finitely generated group G, this result recovers the aforementioned characterization of the 1-formality property of G. Both filtered-formality and graded-formality can be interpreted using the language of minimal models. More precisely, in §3.2.3, we show the following theorem.

Theorem 1.2.1. *A finitely generated group G is filtered-formal over $\mathbb{Q}$ if and only if the canonical 1-minimal model $\mathcal{M}(G;\mathbb{Q})$ is filtered-isomorphic to a 1-minimal model $\mathcal{M}$ with positive Hirsch weights.*

The above theorem was suggested to us by R. Porter. He also noted that G is graded-formal if and only if the 1-minimal model of G is isomorphic to the 1-minimal model of $H^*(G;\mathbb{Q})$ as bigraded algebras. From this point of view, the work of Morgan [113] implies that the fundamental groups of complex smooth algebraic varieties are filtered-formal.

1.2.3 Propagation of formality

We investigate the way in which the various formality notions for groups behave with respect to split injections, coproducts, and direct products. Our first result in this direction is a combination of Theorem 3.1.17 and 3.3.10, and can be stated as follows.

Theorem 1.2.2. *Let G be a finitely generated group, and let $K \leq G$ be a subgroup. Suppose there is a split monomorphism $\iota\colon K \to G$. Then:*

1. *If G is graded-formal, then K is also graded-formal.*
2. *If G is filtered-formal, then K is also filtered-formal.*
3. *If G is 1-formal, then K is also 1-formal.*

In particular, if a semi-direct product $G_1 \rtimes G_2$ has one of the above formality properties, then G_2 also has that property; in general, though, G_1 will not, as illustrated in Example 3.1.19.

As shown by Dimca et al. [42], both the product and the coproduct of two 1-formal groups is again 1-formal. Also, as shown by Plantiko [127], the product and coproduct of two graded-formal groups is again graded-formal. We sharpen these results in the next theorem, which is a combination of Propositions 3.1.21 and 3.3.12.

Theorem 1.2.3. *Let G_1 and G_2 be two finitely generated groups. The following conditions are equivalent.*

1. *G_1 and G_2 are graded-formal (respectively, filtered-formal, or 1-formal).*
2. *$G_1 * G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).*
3. *$G_1 \times G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).*

Both Theorem 1.2.2 and 1.2.3 can be used to decide the formality properties of new groups from those of known groups. In general, though, even when both G_1 and G_2 are 1-formal, we cannot conclude that an arbitrary semi-direct product $G_1 \rtimes G_2$ is 1-formal (see Example 9.2.17).

The various formality properties are not necessarily inherited by quotient groups. However, as we shall see in Theorem 1.2.5 and Theorem 3.3.8, respectively, filtered formality is passed on to the derived quotients and to the nilpotent quotients of a group.

1.2.4 Presentations

In §4.1 to §4.3 we analyze the presentations of the various Lie algebras attached to a finitely presented group G. Some of the motivation and techniques come from the work of Labute [83, 84] and Anick [2], who gave presentations for the associated graded Lie algebra $\mathrm{gr}(G;\mathbb{Q})$, provided G has a 'mild' presentation.

Our main interest, though, is in finding presentations for the holonomy Lie algebra $\mathfrak{h}(G;\mathbb{Q})$ and its solvable quotients. In the special case when G is a commutator-relators group, such presentations were given in [117]. To generalize these results to arbitrary finitely presented groups, we first compute the cup product map $\cup\colon H^1(G;\mathbb{Q}) \wedge H^1(G;\mathbb{Q}) \to H^2(G;\mathbb{Q})$, using Fox Calculus and Magnus expansion techniques modelled on the approach of Fenn and Sjerve from [55]. The next result is a summary of Proposition 4.1.9 and Theorems 4.2.6, 4.3.1, and 4.3.5.

Theorem 1.2.4. *Let G be a group with finite presentation $\langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$, and let $b = \dim H_1(G; \mathbb{Q})$.*

1. *There exists a group $\widetilde{G}$ with echelon presentation $\langle x_1, \dots, x_n \mid w_1, \dots, w_m \rangle$ such that $\mathfrak{h}(G; \mathbb{Q}) \cong \mathfrak{h}(\widetilde{G}; \mathbb{Q})$.*

2. *The holonomy Lie algebra $\mathfrak{h}(G; \mathbb{Q})$ is the quotient of the free $\mathbb{Q}$-Lie algebra with generators $\mathbf{y} = \{y_1, \dots, y_b\}$ in degree 1 by the ideal I generated by $\kappa_2(w_{n-b+1}), \dots, \kappa_2(w_m)$, where κ_2 is determined by the Magnus expansion for $\widetilde{G}$.*

3. *The solvable quotient $\mathfrak{h}(G; \mathbb{Q})/\mathfrak{h}(G; \mathbb{Q})^{(i)}$ is isomorphic to $\mathfrak{lie}(\mathbf{y})/(I + \mathfrak{lie}^{(i)}(\mathbf{y}))$.*

Here we say that G has an 'echelon presentation' if the augmented Jacobian matrix of Fox derivatives of this presentation is in row-echelon form. Theorem 1.2.4 yields an algorithm for finding a presentation for the holonomy Lie algebra of a finitely presented group, and thus, a presentation for the associated graded Lie algebra of a finitely presented, graded-formal group.

1.2.5 Chen Lie algebras and Alexander invariants

In §6.1, we investigate some of the relationships between the lower central series and the derived series of a finitely generated group, on one hand, and the derived series of the corresponding Lie algebras, on the other hand.

In [27], Chen studied the lower central series quotients of the maximal metabelian quotient of a finitely generated free group, and computed their graded ranks. More generally, following Papadima and Suciu [117], we may define the *i-th Chen Lie algebras* of a group G as the associated graded Lie algebras of its solvable quotients, $\operatorname{gr}(G/G^{(i)}; \mathbb{Q})$. Our next theorem (which combines Theorem 6.1.5 and Corollary 6.1.7) sharpens and extends the main result of [117].

Theorem 1.2.5. *Let G be a finitely generated group. For each $i \geq 2$, the quotient map $G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded $\mathbb{Q}$-Lie algebras,*

$$\Psi_G^{(i)} \colon \operatorname{gr}(G;\mathbb{Q})/\operatorname{gr}(G;\mathbb{Q})^{(i)} \twoheadrightarrow \operatorname{gr}(G/G^{(i)};\mathbb{Q}) \,.$$

Moreover,

1. *If G is a filtered-formal group, then each solvable quotient $G/G^{(i)}$ is also filtered-formal, and the map $\Psi_G^{(i)}$ is an isomorphism.*
2. *If G is a 1-formal group, then $\mathfrak{h}(G;\mathbb{Q})/\mathfrak{h}(G;\mathbb{Q})^{(i)} \cong \operatorname{gr}(G/G^{(i)};\mathbb{Q})$.*

Given a finitely presented group G, the solvable quotients $G/G^{(i)}$ need not be finitely presented. Thus, finding presentations for the Chen Lie algebra $\operatorname{gr}(G/G^{(i)})$ can be an arduous task. Nevertheless, Theorem 1.2.5 provides a method for finding such presentations, under suitable formality assumptions. The theorem can also be used as an obstruction to 1-formality.

Our next main result (a combination of Propositions 5.2.2, 6.2.2, and 6.2.3), relates the various Alexander-type invariants associated to a group, as follows.

Theorem 1.2.6. *Let G be a finitely generated group with abelianization H, and set $S = \operatorname{Sym}(H \otimes \mathbb{C})$. There exists then surjective morphisms of graded S-modules,*

$$\mathfrak{B}(\mathfrak{h}(G)) \xrightarrow{\psi} \mathfrak{B}(\operatorname{gr}(G)) \xrightarrow{\varphi} \operatorname{gr}(B(G)) \,. \tag{1.5}$$

Moreover, if G is graded-formal, then ψ is an isomorphism, and if G is filtered-formal, then φ is an isomorphism.

This result yields the following inequalities between the various types of Chen ranks associated to a finitely generated group G:

$$\theta_k(\mathfrak{h}(G)) \geq \theta_k(\operatorname{gr}(G)) \geq \theta_k(G), \tag{1.6}$$

with the first inequality holding as equality if G is graded-formal, and the second inequality holding as equality if G is filtered-formal.

1.2.6 Resonance varieties and Chen ranks

Recall from §1.1.3 that the *resonance varieties* of a group G with finite-type cohomology algebra, $\mathcal{R}^i_d(G)$, are defined as the loci where the cohomology of those cochain complexes jumps by (1.1), where $A = H^*(G;\mathbb{C})$ and the differential d is zero. We study in §5.1.4 the behavior of resonance under products and coproducts, obtaining formulas which generalize those from [122], see Propositions 5.1.3 and 5.1.4.

The *Chen ranks formula* (1.4) reveals a close relationship between the first resonance variety and the Chen ranks of G. Recall that Cohen and Schenck [34] showed that, for a finitely presented, commutator-relators 1-formal group G, the Chen ranks formula holds, provided the components of $\mathcal{R}_1(G)$ are zero-isotropic, projectively disjoint, and reduced as schemes. With the help of Theorem 1.2.4, in Proposition 6.3.2, we show that the theorem of Cohen and Schenck is still true without the "commutator-relators" assumption.

In §6.3, we analyze the Chen ranks formula in a wider setting, with a view towards comparing the Chen ranks and the resonance varieties of the pure virtual braid groups. We start by noting that formula (1.4) may hold even for non-1-formal groups, such as the fundamental groups of complements of suitably chosen arrangements of planes in $\mathbb{R}^4$.

Next, we look at the way the Chen ranks formula behaves well with respect to products and coproducts of groups. The conclusion may be summarized as follows.

Proposition 1.2.1. *If both G_1 and G_2 satisfy the Chen ranks formula (1.4), then $G_1 \times G_2$ also satisfies the Chen ranks formula, but $G_1 * G_2$ may not.*

1.2.7 The pure virtual braids

Bardakov gave in [7] a presentation for the pure virtual braid group vP_n, much simpler than the usual presentation of the pure braid group P_n. As shown in [11], there exits a monomorphism from P_n to vP_n. Moreover, there are split injections $vP_n \to vP_{n+1}$, $vP_n^+ \to vP_{n+1}^+$ and $vP_n^+ \to vP_n$.

The pure braid group P_n has center $\mathbb{Z}$, so there is a decomposition $P_n \cong \overline{P}_n \times \mathbb{Z}$. Using a decomposition of vP_3 given by Bardakov, Mikhailov, Vershinin, and Wu in [10], we show that $vP_3 \cong \overline{P}_4 * \mathbb{Z}$. In this context, it is worth noting that the center of vP_n is trivial for $n \geq 2$, and the center of vP_n^+ is trivial for $n \geq 3$, with one possible exception; see Dies and Nicas [40].

Labute [84] and Anick [2] defined the notion of a 'mild' presentation for a group. If G admits such a presentation, then a presentation for the (complex) associated graded Lie algebra $\mathrm{gr}(G)$ can be obtained from the classical Magnus expansion. In general, though, finding a presentation for this Lie algebra is an onerous task. In §9.2.3, we prove the following result.

Proposition 1.2.2. *The pure braid groups P_n and the pure virtual braid groups vP_n and vP_n^+ admit mild presentations if and only $n \leq 3$.*

Nevertheless, for each n, explicit presentations for the associated graded Lie algebra of P_n (as well as vP_n and vP_n^+) were given in [80, 48] (respectively in [11, 91]). All these associated graded Lie algebras are isomorphic to the holonomy Lie algebras of the corresponding groups, i.e., $\mathrm{gr}(P_n) \cong \mathfrak{h}(P_n)$, $\mathrm{gr}(vP_n) \cong \mathfrak{h}(vP_n)$ and $\mathrm{gr}(vP_n^+) \cong \mathfrak{h}(vP_n^+)$. Furthermore, the universal enveloping algebras of these graded Lie algebras are Koszul algebras.

Using Koszul duality and some combinatorial manipulations, we find that the LCS ranks of the groups $G_n = P_n$, vP_n, or vP_n^+ are given by

$$\phi_k(G_n) = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) \left[\sum_{m_1+2m_2+\cdots+nm_n=d} (-1)^{s_n} d(m!) \prod_{j=1}^{n} \frac{(b_{n,n-j})^{m_j}}{(m_j)!} \right], \tag{1.7}$$

where m_j are non-negative integers, $s_n = \sum_{i=1}^{[n/2]} m_{2i}$, $m = \sum_{i=1}^{n} m_i - 1$, and μ is the Möbius function, while $b_{n,j}$ are the (unsigned) Stirling numbers of the first kind (for $G_n = P_n$), the Lah numbers (for $G_n = vP_n$), or the Stirling numbers of the second kind (for $G_n = vP_n^+$).

The work of Bartholdi et al. [11] and Lee [91] mentioned above shows that the pure virtual braid group vP_n and its subgroup vP_n^+ are graded-formal, for all n. Furthermore,

Bartholdi, Enriquez, Etingof, and Rains state that the groups vP_n and vP_n^+ are not 1-formal for $n \geq 4$, and sketch a proof of this claim. One of the aims of this thesis (indeed, the original motivation for this work) is to provide a detailed proof of this fact.

Theorem 1.2.7. *The groups vP_n and vP_n^+ are both 1-formal if $n \leq 3$, and they are both non-1-formal (and thus, not filtered formal) if $n \geq 4$.*

From Propositions 3.1.21 and 3.3.12, the 1-formality property of groups is preserved under split injections and (co)products. Consequently, the fact that we have split injections between the various pure virtual braid groups allows us to reduce the proof of Theorem 1.2.7 to verifying the 1-formality of vP_3 and the non-1-formality of vP_4^+. To prove the first statement, we use the free product decomposition $vP_3 \cong \mathbb{Z} * \overline{P}_4$. For the second statement, we compute the resonance variety $\mathcal{R}_1^1(vP_4^+)$, and use the geometry of this variety, together with the Tangent Cone Theorem from [42] to reach the desired conclusion.

1.2.8 The pure welded braids

Recall from §1.1.7 that wP_n are the pure welded braid groups (McCool groups), with the subgroups wP_n^+ upper pure welded braid groups (upper McCool groups).

In [38], D. Cohen showed that

$$\mathcal{R}_1(wP_n) = \bigcup_{1\leq i<j\leq n} C_{ij} \cup \bigcup_{1\leq i<j<k\leq n} C_{ijk} \tag{1.8}$$

where C_{ij} and C_{ijk} are certain linear subspaces of of $H^1(wP_n;\mathbb{C})$ of dimension 2 and 3, respectively.

In this thesis, we pursue this line of inquiry by analyzing the resonance varieties and the Chen Lie algebras of the upper McCool groups.

Theorem 1.2.8 (Theorem 8.4.2)**.** *The resonance varieties of the upper McCool groups are given by*

$$\mathcal{R}_1(wP_n^+) = \bigcup_{2\leq j<i\leq n} L_{ij},$$

where $L_{i,j}$ is the j-dimensional linear subspace of $H^1(wP_n^+;\mathbb{C}) = \mathbb{C}^{\binom{n}{2}}$ defined by the equations

$$\begin{cases} x_{i,l} + x_{j,l} = 0 & \text{for } 1 \leq l \leq j-1, \\ x_{i,l} = 0 & \text{for } j+1 \leq l \leq i-1, \\ x_{s,t} = 0 & \text{for } s \neq i,\ s \neq j, \text{ and } 1 \leq t < s. \end{cases}$$

Comparing the resonance varieties of wP_n with those of wP_n^+, we obtain the following corollary.

Corollary 1.2.3 (Corollary 7.1.2)**.** *There is no epimorphism from wP_n^+ to wP_n for $n \geq 4$. In particular, the inclusion $\iota\colon wP_n^+ \to wP_n$ admits no splitting for $n \geq 4$.*

In [34], Cohen and Schenck showed the full McCool groups satisfy the Chen ranks formula (1.4), from which they deduce that

$$\theta_k(wP_n) = (k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}. \tag{1.9}$$

Rather surprisingly, it turns out that the resonance varieties $\mathcal{R}_1(wP_n^+)$ no longer obey the hypothesis of [34], and, in fact, the Chen ranks of wP_n^+ no longer obey formula (1.4). Nevertheless, using a different approach, based on a refinement of the Gröbner basis algorithm from [31], we find a closed formula for those Chen ranks.

Theorem 1.2.9 (Theorem 8.5.5)**.** *The Chen ranks of the upper McCool groups, $\theta_k = \theta_k(wP_n^+)$, are given by $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, $\theta_3 = 2\binom{n+1}{4}$, and*

$$\theta_k = \binom{n+k-2}{k+1} + \theta_{k-1} = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}$$

for $k \geq 4$.

Both P_n and wP_n^+ are iterated semidirect products of the form $F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$. A question from [29] asks whether or not the groups P_n and wP_n^+ are isomorphic. We already know that for $n \leq 3$ the answer is yes. As a quick application of this result, we obtain the following corollary, which answers this question for all n.

Corollary 1.2.4 (Corollary 8.5.6). *For each $n \geq 4$, the pure braid group P_n, the upper McCool group wP_n^+, and the direct product $\Pi_n := \prod_{i=1}^{n-1} F_i$ are all non-isomorphic, although they all do have the same LCS ranks and the same Betti numbers.*

The fact that $P_n \ncong \Pi_n$ for $n \geq 4$ was already established in [31], also using the Chen ranks. The novelty here is the distinction between wP_n^+ and the other two groups.

1.2.9 Hilbert series

An important aspect in the study of the graded Lie algebras attached to a finitely generated group G is the computation of the Hilbert series of these objects. If $\mathfrak{g}$ is such a graded Lie algebra, and $U(\mathfrak{g})$ is its universal enveloping algebra, the Poincaré–Birkhoff–Witt theorem expresses the graded ranks of $\mathfrak{g}$ in terms of the Hilbert series of $U(\mathfrak{g})$.

In favorable situations, which oftentimes involve the formality notions discussed above, this approach permits us to determine the LCS ranks $\phi_i(G) = \dim \operatorname{gr}_i(G;\mathbb{Q})$ or the Chen ranks $\theta_i(G) = \dim \operatorname{gr}_i(G/G'';\mathbb{Q})$, as well as the holonomy versions of these ranks, $\bar{\phi}_i(G) = \dim \mathfrak{h}_i(G;\mathbb{Q})$ and $\bar{\theta}_i(G) = \dim \mathfrak{h}_i(G;\mathbb{Q})/\mathfrak{h}_i''(G;\mathbb{Q})$. In this context, the isomorphisms provided by Theorem 1.2.5, as well as the presentations provided by Theorem 1.2.4 prove to be valuable tools.

Using these techniques, we compute in §9.2 the ranks $\bar{\phi}_i(G)$ and $\bar{\theta}_i(G)$ for one-relator groups G, while in §9.3 we compute the whole set of ranks for the fundamental groups of closed, orientable Seifert manifolds.

1.2.10 Nilpotent groups

Our techniques apply especially well to the class of finitely generated, torsion-free nilpotent groups. Carlson and Toledo [24] studied the 1-formality properties of such groups, while Plantiko [127] gave a sufficient conditions for such groups to be non-graded-formal. For nilpotent Lie algebras, the notion of filtered-formality has been studied by Leger [92], Cor-

nulier [37], Kasuya [77], and others. In particular, Cornulier [37] proves that the systolic growth of a finitely generated nilpotent group G is asymptotically equivalent to its growth if and only if the Malcev Lie algebra $\mathfrak{m}(G;\mathbb{Q})$ is filtered-formal (or, 'Carnot'), while Kasuya [77] shows that the variety of flat connections on a filtered-formal (or, 'naturally graded'), n-step nilpotent Lie algebra $\mathfrak{g}$ has a singularity at the origin cut out by polynomials of degree at most $n+1$.

We investigate in §9.1 the filtered formality of nilpotent groups, and the way this property interacts with other properties of these groups. The next result combines Theorem 9.1.3 and Proposition 9.1.8.

Theorem 1.2.10. *Let G be a finitely generated, torsion-free nilpotent group.*

1. *Suppose G is a 2-step nilpotent group with torsion-free abelianization. Then G is filtered-formal.*

2. *Suppose G is filtered-formal. Then the universal enveloping algebra $U(\mathrm{gr}(G;\mathbb{Q}))$ is Koszul if and only if G is abelian.*

As mentioned previously, nilpotent quotients of finitely generated filtered-formal groups are filtered-formal. In particular, each n-step, free nilpotent group $F/\Gamma_n F$ is filtered-formal. A classical example is the unipotent group $U_n(\mathbb{Z})$, which is known to be filtered-formal by Lambe and Priddy [88], but not graded-formal for $n \geq 3$.

1.2.11 Further applications

We illustrate our approach with several other classes of finitely presented groups. We first look at 1-relator groups, whose associated graded Lie algebras were first determined by Labute in [83]. We give in §9.2.1– §9.2.4 presentations for the holonomy Lie algebra and the Chen Lie algebras of a 1-relator group, compute the respective Hilbert series, and discuss the formality properties of these groups.

It has been known since the pioneering work of W. Massey [107] that fundamental groups of link complements are not always 1-formal. In fact, as shown by Hain in [64], such groups need not be graded-formal. However, as shown by Anick [2], Berceanu–Papadima [14], and Papadima–Suciu [117], if the linking graph is connected, then the link group is graded-formal. Building on work of Dimca et al. [42], we give in §9.2.5 an example of a link group which is graded-formal, yet not filtered-formal.

We end in §9.3 with a detailed study of fundamental groups of (orientable) Seifert fibered manifolds from a rational homotopy viewpoint. Let M be such a manifold. Using Theorem 1.2.4, we find an explicit presentation for the holonomy Lie algebra of $\pi_1(M)$. On the other hand, using the minimal model of M (as described by Putinar in [130]), we find a presentation for the Malcev Lie algebra $\mathfrak{m}(\pi_1(M);\mathbb{Q})$, and we use this information to derive a presentation for $\operatorname{gr}(\pi_1(M);\mathbb{Q})$. As an application, we show that Seifert manifold groups are filtered-formal, and determine precisely which ones are graded-formal.

In future work, we will investigate the many and varied connections between the characteristic and resonance varieties of spaces and CDGA models, and we will explore the relationship between the ranks of the Chen Lie algebras and the dimensions of the resonance varieties of the CDGA models of groups. We briefly state these projects in the context of pure braid groups on Riemann surfaces §9.4 and picture groups from quiver representations §9.5, etc.

We use Macaulay 2, GAP and Mathematica to carry out computations in this thesis.

Chapter 2

Finitely generated Lie algebras and formality properties

In this chapter, we first review some definitions and properties relating graded Lie algebras and filtered Lie algebras, and prove some lemmas which will be useful for the rest of this thesis. In particular, we investigate the formality properties of a complete filtered Lie algebra. We then study several relationships between these Lie algebras and associative algebras, focusing on the notion of quadraticity and Koszul properties. At last, we study the minimal model and the (partial) formality properties of a differential graded algebra, which are very important notions in rational homotopy theory. In this thesis, we focus on developing the theory of the non-simply-connected rational homotopy theory. This chapter is based on the work in my paper [143] with Alex Suciu.

2.1 Filtered and graded Lie algebras

In this section we study the interactions between filtered Lie algebras, their completions, and their associated graded Lie algebras, mainly as they relate to the notion of filtered formality.

2.1.1 Graded Lie algebras

We start by reviewing some standard material on Lie algebras, following the exposition from the works of Ekedahl and Merkulov [47], Polishchuk and Positselski [128], Quillen [131], and Serre [136].

Fix a ground field $\mathbb{Q}$ of characteristic 0. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{Q}$, i.e., a $\mathbb{Q}$-vector space $\mathfrak{g}$ endowed with a bilinear operation $[\,,\,]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying the Lie identities. We say that $\mathfrak{g}$ is a *graded Lie algebra* if $\mathfrak{g}$ decomposes as $\mathfrak{g}=\bigoplus_{i\geq 1}\mathfrak{g}_i$ and the Lie bracket sends $\mathfrak{g}_i\times\mathfrak{g}_j$ to $\mathfrak{g}_{i+j}$, for all i and j. A morphism of graded Lie algebras is a $\mathbb{Q}$-linear map $\varphi\colon \mathfrak{g}\to\mathfrak{h}$ which preserves the Lie brackets and the degrees.

The most basic example of a graded Lie algebra is constructed as follows. Let V a $\mathbb{Q}$-vector space. The tensor algebra $T(V)$ has a natural Hopf algebra structure, with comultiplication Δ and counit ε the algebra maps given by $\Delta(v)=v\otimes 1+1\otimes v$ and $\varepsilon(v)=0$, for $v\in V$. The *free Lie algebra* on V is the set of primitive elements, i.e.,

$$\mathfrak{lie}(V)=\{x\in T(V)\mid \Delta(x)=x\otimes 1+1\otimes x\}, \tag{2.1}$$

with Lie bracket $[x,y]=x\otimes y-y\otimes x$ and grading induced from $T(V)$.

Now suppose all elements of V are assigned degree 1 in $T(V)$. Then the inclusion $\iota\colon \mathfrak{lie}(V)\to T(V)$ identifies $\mathfrak{lie}_1(V)$ with $T_1(V)=V$. Furthermore, ι maps $\mathfrak{lie}_2(V)$ to $T_2(V)=V\otimes V$ by sending $[v,w]$ to $v\otimes w-w\otimes v$ for each $v,w\in V$; we thus may identify $\mathfrak{lie}_2(V)\cong V\wedge V$ by sending $[v,w]$ to $v\wedge w$.

A Lie algebra $\mathfrak{g}$ is said to be *finitely generated* if there is an epimorphism $\varphi\colon \mathfrak{lie}(\mathbb{Q}^n)\to\mathfrak{g}$ for some $n\geq 1$. If, moreover, the Lie ideal $\mathfrak{r}=\ker\varphi$ is finitely generated as a Lie algebra, then $\mathfrak{g}$ is called *finitely presented*.

If $\mathfrak{g}$ is finitely generated and all the generators $x_1,\dots,x_n\in\mathfrak{lie}(\mathbb{Q}^n)$ can be chosen to have degree 1, then we say $\mathfrak{g}$ is *generated in degree* 1. If, moreover, the Lie ideal $\mathfrak{r}$ is homogeneous, then $\mathfrak{g}$ is a graded Lie algebra. In particular, if $\mathfrak{r}$ is generated in degree 2, then we say the graded Lie algebra $\mathfrak{g}$ is a *quadratic Lie algebra*.

2.1.2 Filtrations

We will be very much interested in this work in Lie algebras endowed with a filtration, usually but not always enjoying an extra 'multiplicative' property. At the most basic level, a *filtration* $\mathcal{F}$ on a Lie algebra $\mathfrak{g}$ is a nested sequence of Lie ideals, $\mathfrak{g} = \mathcal{F}_1\mathfrak{g} \supset \mathcal{F}_2\mathfrak{g} \supset \cdots$.

A well-known such filtration is the *derived series*, $\mathcal{F}_i\mathfrak{g} = \mathfrak{g}^{(i-1)}$, defined by $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$ for $i \geq 1$. The derived series is preserved by Lie algebra maps. The quotient Lie algebras $\mathfrak{g}/\mathfrak{g}^{(i)}$ are solvable; moreover, if $\mathfrak{g}$ is a graded Lie algebra, all these solvable quotients inherit a graded Lie algebra structure. The next lemma (which will be used in §4.3.2) follows straight from the definitions, using the standard isomorphism theorems.

Lemma 2.1.1. *Let $\mathfrak{g} = \mathfrak{lie}(V)/\mathfrak{r}$ be a finitely generated Lie algebra. Then $\mathfrak{g}/\mathfrak{g}^{(i)} \cong \mathfrak{lie}(V)/(\mathfrak{r} + \mathfrak{lie}(V)^{(i)})$. Furthermore, if $\mathfrak{r}$ is a homogeneous ideal, then this is an isomorphism of graded Lie algebras.*

The existence of a filtration $\mathcal{F}$ on a Lie algebra $\mathfrak{g}$ makes $\mathfrak{g}$ into a topological vector space, by defining a basis of open neighborhoods of an element $x \in \mathfrak{g}$ to be $\{x + \mathcal{F}_k\mathfrak{g}\}_{k\in\mathbb{N}}$. The fact that each basis neighborhood $\mathcal{F}_k\mathfrak{g}$ is a Lie subalgebra implies that the Lie bracket map $[\,,\,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is continuous; thus, $\mathfrak{g}$ is, in fact, a topological Lie algebra. We say that $\mathfrak{g}$ is *complete* (respectively, *separated*) if the underlying topological vector space enjoys those properties.

Given an ideal $\mathfrak{a} \subset \mathfrak{g}$, there is an induced filtration on it, given by $\mathcal{F}_k\mathfrak{a} = \mathcal{F}_k\mathfrak{g} \cap \mathfrak{a}$. Likewise, the quotient Lie algebra, $\mathfrak{g}/\mathfrak{a}$, has a naturally induced filtration with terms $\mathcal{F}_k\mathfrak{g}/\mathcal{F}_k\mathfrak{a}$. Let $\overline{\mathfrak{a}}$ be the closure of $\mathfrak{a}$ in the filtration topology. Then $\overline{\mathfrak{a}}$ is a closed ideal of $\mathfrak{g}$. Moreover, by the continuity of the Lie bracket, we have that

$$\overline{[\overline{\mathfrak{a}}, \overline{\mathfrak{r}}]} = \overline{[\mathfrak{a}, \mathfrak{r}]}. \tag{2.2}$$

Finally, if $\mathfrak{g}$ is complete (or separated), then $\mathfrak{g}/\overline{\mathfrak{a}}$ is also complete (or separated).

2.1.3 Completions

For each $j \geq k$, there is a canonical projection $\mathfrak{g}/\mathcal{F}_j\mathfrak{g} \to \mathfrak{g}/\mathcal{F}_k\mathfrak{g}$, compatible with the projections from $\mathfrak{g}$ to its quotient Lie algebras $\mathfrak{g}/\mathcal{F}_k\mathfrak{g}$. The *completion* of the Lie algebra $\mathfrak{g}$ with respect to the filtration $\mathcal{F}$ is defined as the limit of this inverse system, i.e.,

$$\widehat{\mathfrak{g}} := \varprojlim_k \mathfrak{g}/\mathcal{F}_k\mathfrak{g} = \Big\{(g_1, g_2, \dots) \in \prod_{i=1}^{\infty} \mathfrak{g}/\mathcal{F}_i\mathfrak{g} \;\Big|\; g_j \equiv g_k \bmod \mathcal{F}_k\mathfrak{g} \text{ for all } j > k\Big\}. \tag{2.3}$$

Using the fact that $\mathcal{F}_k(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, It is readily seen that $\widehat{\mathfrak{g}}$ is a Lie algebra, with Lie bracket defined componentwise. Furthermore, $\widehat{\mathfrak{g}}$ has a natural inverse limit filtration, $\widehat{\mathcal{F}}$, given by

$$\widehat{\mathcal{F}}_k\widehat{\mathfrak{g}} := \widehat{\mathcal{F}_k\mathfrak{g}} = \varprojlim_{i \geq k} \mathcal{F}_k\mathfrak{g}/\mathcal{F}_i\mathfrak{g} = \{(g_1, g_2, \dots) \in \widehat{\mathfrak{g}} \mid g_i = 0 \text{ for all } i < k\}. \tag{2.4}$$

Note that $\widehat{\mathcal{F}}_k\widehat{\mathfrak{g}} = \overline{\mathcal{F}_k\mathfrak{g}}$, and so each term of the filtration $\widehat{\mathcal{F}}$ is a closed Lie ideal of $\widehat{\mathfrak{g}}$. Furthermore, the Lie algebra $\widehat{\mathfrak{g}}$, endowed with this filtration, is both complete and separated.

Let $\alpha\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ be the canonical map to the completion. Then α is a morphism of Lie algebras, preserving the respective filtrations. Clearly, $\ker(\alpha) = \bigcap_{k\geq 1} \mathcal{F}_k\mathfrak{g}$. Hence, α is injective if and only if $\mathfrak{g}$ is separated. Furthermore, α is bijective if and only if $\mathfrak{g}$ is complete and separated.

2.1.4 Filtered Lie algebras

A *filtered Lie algebra* (over the field $\mathbb{Q}$) is a Lie algebra $\mathfrak{g}$ endowed with a $\mathbb{Q}$-vector filtration $\{\mathcal{F}_k\mathfrak{g}\}_{k\geq 1}$ satisfying the 'multiplicativity' condition

$$[\mathcal{F}_r\mathfrak{g}, \mathcal{F}_s\mathfrak{g}] \subseteq \mathcal{F}_{r+s}\mathfrak{g} \tag{2.5}$$

for all $r, s \geq 1$. Obviously, this condition implies that each subspace $\mathcal{F}_k\mathfrak{g}$ is a Lie ideal, and so, in particular, $\mathcal{F}$ is a Lie algebra filtration. Let

$$\operatorname{gr}^{\mathcal{F}}(\mathfrak{g}) := \bigoplus_{k\geq 1} \mathcal{F}_k\mathfrak{g}/\mathcal{F}_{k+1}\mathfrak{g}. \tag{2.6}$$

be the associated graded vector space to the filtration $\mathcal{F}$ on $\mathfrak{g}$. Condition (2.5) implies that the Lie bracket map on $\mathfrak{g}$ descends to a map $[\,,\,]\colon \operatorname{gr}^{\mathcal{F}}(\mathfrak{g}) \times \operatorname{gr}^{\mathcal{F}}(\mathfrak{g}) \to \operatorname{gr}^{\mathcal{F}}(\mathfrak{g})$, which makes $\operatorname{gr}^{\mathcal{F}}(\mathfrak{g})$ into a graded Lie algebra, with graded pieces given by decomposition (2.6).

A morphism of filtered Lie algebras is a linear map $\phi\colon \mathfrak{g} \to \mathfrak{h}$ preserving Lie brackets and the given filtrations, $\mathcal{F}$ and $\mathcal{G}$. Such a morphism induces a morphism of associated graded Lie algebras, $\operatorname{gr}(\phi)\colon \operatorname{gr}^{\mathcal{F}}(\mathfrak{g}) \to \operatorname{gr}^{\mathcal{G}}(\mathfrak{h})$.

If $\mathfrak{g}$ is a filtered Lie algebra, then its completion, $\widehat{\mathfrak{g}}$, is again a filtered Lie algebra. Indeed, if $\mathcal{F}$ is the given multiplicative filtration on $\mathfrak{g}$, and $\widehat{\mathcal{F}}$ is the completed filtration on $\widehat{\mathfrak{g}}$, then $\widehat{\mathcal{F}}$ also satisfies property (2.5). Moreover, the canonical map to the completion, $\alpha\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$, is a morphism of filtered Lie algebras. It is readily seen that α induces isomorphisms

$$\mathfrak{g}/\mathcal{F}_k\mathfrak{g} \xrightarrow{\sim} \widehat{\mathfrak{g}}/\widehat{\mathcal{F}}_k\widehat{\mathfrak{g}}\,, \tag{2.7}$$

for each $k \geq 1$, see e.g. [47] From the 5-lemma, we obtain an isomorphism of graded Lie algebras,

$$\operatorname{gr}(\alpha)\colon \operatorname{gr}^{\mathcal{F}}(\mathfrak{g}) \xrightarrow{\sim} \operatorname{gr}^{\widehat{\mathcal{F}}}(\widehat{\mathfrak{g}})\,. \tag{2.8}$$

Lemma 2.1.2. *Let $\phi\colon \mathfrak{g} \to \mathfrak{h}$ be a morphism of complete, separated, filtered Lie algebras, and suppose $\operatorname{gr}(\phi)\colon \operatorname{gr}^{\mathcal{F}}(\mathfrak{g}) \to \operatorname{gr}^{\mathcal{G}}(\mathfrak{h})$ is an isomorphism. Then ϕ is also an isomorphism.*

Proof. The map ϕ induces morphisms $\phi_k\colon \mathfrak{g}/\mathcal{F}_k\mathfrak{g} \to \mathfrak{h}/\mathcal{G}_k\mathfrak{h}$ for all $k \geq 1$. By assumption, the homomorphisms $\operatorname{gr}_k(\phi)\colon \mathcal{F}_k\mathfrak{g}/\mathcal{F}_{k-1}\mathfrak{g} \to \mathcal{G}_k\mathfrak{h}/\mathcal{G}_{k-1}\mathfrak{h}$ are isomorphisms, for all $k > 1$. An easy induction on k shows that all maps ϕ_k are isomorphisms. Hence, the map $\hat{\phi}\colon \widehat{\mathfrak{g}} \to \widehat{\mathfrak{h}}$ is an isomorphism. By assumption, though, $\mathfrak{g} = \widehat{\mathfrak{g}}$ and $\mathfrak{h} = \widehat{\mathfrak{h}}$; hence $\phi = \hat{\phi}$, and we are done. □

Any Lie algebra $\mathfrak{g}$ comes equipped with a lower central series (LCS) filtration, $\{\Gamma_k(\mathfrak{g})\}_{k\geq 1}$, defined by $\Gamma_1(\mathfrak{g}) = \mathfrak{g}$ and $\Gamma_k(\mathfrak{g}) = [\Gamma_{k-1}(\mathfrak{g}), \mathfrak{g}]$ for $k \geq 2$. Clearly, this is a multiplicative filtration. Any other such filtration $\{\mathcal{F}_k(\mathfrak{g})\}_{k\leq 1}$ on $\mathfrak{g}$ is coarser than this filtration; that is, $\Gamma_k\mathfrak{g} \subseteq \mathcal{F}_k\mathfrak{g}$, for all $k \geq 1$. Any Lie algebra morphism $\phi\colon \mathfrak{g} \to \mathfrak{h}$ preserves LCS filtrations. Furthermore, the quotient Lie algebras $\mathfrak{g}/\Gamma_k\mathfrak{g}$ are nilpotent. For simplicity, we shall write

$\operatorname{gr}(\mathfrak{g}) := \operatorname{gr}^{\Gamma}(\mathfrak{g})$ for the associated graded Lie algebra and $\widehat{\mathfrak{g}}$ for the completion of $\mathfrak{g}$ with respect to the LCS filtration Γ. Furthermore, we shall take $\widehat{\Gamma}_k = \overline{\Gamma}_k$ as the canonical filtration on $\widehat{\mathfrak{g}}$.

Every graded Lie algebra, $\mathfrak{g} = \bigoplus_{i\geq 1} \mathfrak{g}_i$, has a canonical decreasing filtration induced by the grading, $\mathcal{F}_k\mathfrak{g} = \bigoplus_{i\geq k} \mathfrak{g}_i$. Moreover, if $\mathfrak{g}$ is generated in degree 1, then this filtration coincides with the LCS filtration $\Gamma_k(\mathfrak{g})$. In particular, the associated graded Lie algebra with respect to $\mathcal{F}$ coincides with $\mathfrak{g}$. In this case, the completion of $\mathfrak{g}$ with respect to the lower central series (or, degree) filtration is called the *degree completion* of $\mathfrak{g}$, and is simply denoted by $\widehat{\mathfrak{g}}$. It is readily seen that $\widehat{\mathfrak{g}} \cong \prod_{i\geq 1} \mathfrak{g}_i$. Therefore, the morphism $\alpha\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ is injective, and induces an isomorphism $\mathfrak{g} \cong \operatorname{gr}^{\widehat{\Gamma}}(\widehat{\mathfrak{g}})$. Moreover, if $\mathfrak{h}$ is a graded Lie subalgebra of $\mathfrak{g}$, then $\widehat{\mathfrak{h}} = \overline{\mathfrak{h}}$ and

$$\operatorname{gr}^{\widehat{\Gamma}}(\widehat{\mathfrak{h}}) = \mathfrak{h}. \tag{2.9}$$

Lemma 2.1.3. *If $\mathfrak{L}$ is a free Lie algebra generated in degree 1, and $\mathfrak{r}$ is a homogeneous ideal, then the projection $\pi\colon \mathfrak{L} \to \mathfrak{L}/\mathfrak{r}$ induces an isomorphism $\widehat{\mathfrak{L}}/\overline{\mathfrak{r}} \xrightarrow{\simeq} \widehat{\mathfrak{L}/\mathfrak{r}}$.*

Proof. Without loss of generality, we may assume $\mathfrak{r} \subset [\mathfrak{L}, \mathfrak{L}]$. The projection $\pi\colon \mathfrak{L} \to \mathfrak{L}/\mathfrak{r}$ extends to an epimorphism between the degree completions, $\hat{\pi}\colon \widehat{\mathfrak{L}} \to \widehat{\mathfrak{L}/\mathfrak{r}}$. This morphism takes the ideal generated by $\mathfrak{r}$ to 0; thus, by continuity, induces an epimorphism of complete, filtered Lie algebras, $\widehat{\mathfrak{L}}/\overline{\mathfrak{r}} \twoheadrightarrow \widehat{\mathfrak{L}/\mathfrak{r}}$. Taking the associated graded, we get an epimorphism $\operatorname{gr}(\hat{\pi})\colon \operatorname{gr}(\widehat{\mathfrak{L}}/\overline{\mathfrak{r}}) \twoheadrightarrow \operatorname{gr}(\widehat{\mathfrak{L}/\mathfrak{r}}) = \mathfrak{L}/\mathfrak{r}$. This epimorphism admits a splitting, induced by the maps $\Gamma_n\mathfrak{L} + \mathfrak{r} \to \widehat{\Gamma}_n\widehat{\mathfrak{L}} + \overline{\mathfrak{r}}$; thus, $\operatorname{gr}(\hat{\pi})$ is an isomorphism. The conclusion follows from Lemma 2.1.5. □

2.1.5 Filtered formality

We now consider in more detail the relationship between a filtered Lie algebra $\mathfrak{g}$ and the completion of its associated graded Lie algebra, $\widehat{\operatorname{gr}}(\mathfrak{g})$, with the inverse limit filtration. The following definition will play a key role in the sequel.

Definition 2.1.4. A complete, filtered Lie algebra $\mathfrak{g}$ is called *filtered-formal* if there is a filtered Lie algebra isomorphism $\mathfrak{g} \cong \widehat{\mathrm{gr}}(\mathfrak{g})$ which induces the identity on associated graded Lie algebras.

This notion appears in the work of Bezrukavnikov [16] and Hain [65], as well as in the work of Calaque–Enriquez–Etingof [23] under the name of 'formality', and in the work of Lee [91], under the name of 'weak-formality'. The reasons for our choice of terminology will become more apparent in §3.2.

It is easy to construct examples of Lie algebras enjoying this property. For instance, suppose $\mathfrak{m} = \widehat{\mathfrak{g}}$ is the completion of a finitely generated, graded Lie algebra $\mathfrak{g} = \bigoplus_{i \geq 1} \mathfrak{g}_i$; then $\mathfrak{m}$ is filtered-formal. Moreover, if $\mathfrak{g}$ has homogeneous presentation $\mathfrak{g} = \mathfrak{lie}(V)/\mathfrak{r}$, with V in degree 1, then, by Lemma 2.1.3, the complete, filtered Lie algebra $\mathfrak{m} = \prod_{i \geq 1} \mathfrak{g}_i$ has presentation $\mathfrak{m} = \widehat{\mathfrak{lie}}(V)/\overline{\mathfrak{r}}$.

Lemma 2.1.5. *Let $\mathfrak{g}$ be a complete, filtered Lie algebra, and let $\mathfrak{h}$ be a graded Lie algebra. If there is a Lie algebra isomorphism $\mathfrak{g} \cong \widehat{\mathfrak{h}}$ preserving filtrations, then $\mathfrak{g}$ is filtered-formal.*

Proof. By assumption, there exists a filtered Lie algebra isomorphism $\phi\colon \mathfrak{g} \to \widehat{\mathfrak{h}}$. The map ϕ induces a graded Lie algebra isomorphism, $\mathrm{gr}(\phi)\colon \mathrm{gr}(\mathfrak{g}) \to \mathfrak{h}$. In turn, the map $\psi := (\mathrm{gr}(\phi))^{-1}$ induces an isomorphism $\widehat{\psi}\colon \widehat{\mathfrak{h}} \to \widehat{\mathrm{gr}}(\mathfrak{g})$ of completed Lie algebras. Hence, the composition $\widehat{\psi} \circ \phi\colon \mathfrak{g} \to \widehat{\mathrm{gr}}(\mathfrak{g})$ is an isomorphism of filtered Lie algebras inducing the identity on $\mathrm{gr}(\mathfrak{g})$. □

Corollary 2.1.6. *Suppose $\mathfrak{m}$ is a filtered-formal Lie algebra. There exists then a graded Lie algebra $\mathfrak{g}$ such that $\mathfrak{m}$ is isomorphic to $\widehat{\mathfrak{g}} = \prod_{i \geq 1} \mathfrak{g}_i$.*

Let us also note for future use that filtered-formality is compatible with extension of scalars.

Lemma 2.1.7. *Suppose $\mathfrak{m}$ is a filtered-formal $\mathbb{Q}$-Lie algebra, and suppose $\mathbb{Q} \subset \mathbb{K}$ is a field extension. Then the $\mathbb{K}$-Lie algebra $\mathfrak{m} \otimes_{\mathbb{Q}} \mathbb{K}$ is also filtered-formal.*

Proof. Follows from the fact that completion commutes with tensor products. □

2.1.6 Products and coproducts

The category of Lie algebras admits both products and coproducts. We conclude this section by showing that filtered formality behaves well with respect to these operations.

Lemma 2.1.8. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be two filtered-formal Lie algebras. Then $\mathfrak{m} \times \mathfrak{n}$ is also filtered-formal.*

Proof. By assumption, there exist graded Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ such that $\mathfrak{m} \cong \widehat{\mathfrak{g}} = \prod_{i \geq 1} \mathfrak{g}_i$ and $\mathfrak{n} \cong \widehat{\mathfrak{h}} = \prod_{i \geq 1} \mathfrak{h}_i$. We then have

$$\mathfrak{m} \times \mathfrak{n} \cong \Big(\prod_{i \geq 1} \mathfrak{g}_i \Big) \times \Big(\prod_{i \geq 1} \mathfrak{h}_i \Big) = \prod_{i \geq 1} (\mathfrak{g}_i \times \mathfrak{h}_i) = \widehat{\mathfrak{g} \times \mathfrak{h}}. \tag{2.10}$$

Hence, $\mathfrak{m} \times \mathfrak{n}$ is filtered-formal. □

Now let $*$ denote the usual coproduct (or, free product) of Lie algebras, and let $\hat{*}$ be the coproduct in the category of complete, filtered Lie algebras. By definition,

$$\mathfrak{m} \mathbin{\hat{*}} \mathfrak{n} = \widehat{\mathfrak{m} * \mathfrak{n}} = \varprojlim_k (\mathfrak{m} * \mathfrak{n})/\Gamma_k(\mathfrak{m} * \mathfrak{n}). \tag{2.11}$$

We refer to Lazarev and Markl [90] for a detailed study of this notion.

Lemma 2.1.9. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be two filtered-formal Lie algebras. Then $\mathfrak{m} \mathbin{\hat{*}} \mathfrak{n}$ is also filtered-formal.*

Proof. As before, write $\mathfrak{m} = \widehat{\mathfrak{g}}$ and $\mathfrak{n} = \widehat{\mathfrak{h}}$, for some graded Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. The canonical inclusions, $\alpha\colon \mathfrak{g} \hookrightarrow \mathfrak{m}$ and $\beta\colon \mathfrak{h} \hookrightarrow \mathfrak{n}$, induce a monomorphism of filtered Lie algebras, $\widehat{\alpha * \beta}\colon \widehat{\mathfrak{g} * \mathfrak{h}} \to \widehat{\mathfrak{m} * \mathfrak{n}}$. Using [90, (9.3)], we infer that the induced morphism between associated graded Lie algebras, $\operatorname{gr}(\widehat{\alpha * \beta})\colon \operatorname{gr}(\widehat{\mathfrak{g} * \mathfrak{h}}) \to \operatorname{gr}(\widehat{\mathfrak{m} * \mathfrak{n}})$, is an isomorphism. Lemma 2.1.2 now implies that $\widehat{\alpha * \beta}$ is an isomorphism of filtered Lie algebras, thereby verifying the filtered-formality of $\mathfrak{m} \mathbin{\hat{*}} \mathfrak{n}$. □

2.2 Graded algebras and Koszul duality

The notions of graded and filtered algebras are defined completely analogously for an (associative) algebra A: the multiplication map is required to preserve the grading, respectively the filtration on A. In this section we discuss several relationships between Lie algebras and associative algebras, focussing on the notion of quadratic and Koszul algebras.

2.2.1 Universal enveloping algebras

Given a Lie algebra $\mathfrak{g}$ over a field $\mathbb{Q}$ of characteristic 0, let $U(\mathfrak{g})$ be its universal enveloping algebra. This is the filtered algebra obtained as the quotient of the tensor algebra $T(\mathfrak{g})$ by the (two-sided) ideal I generated by all elements of the form $a\otimes b-b\otimes a-[a,b]$ with $a,b\in\mathfrak{g}$. By the Poincaré–Birkhoff–Witt theorem, the canonical map $\iota\colon \mathfrak{g}\to U(\mathfrak{g})$ is an injection, and the induced map, $\operatorname{Sym}(\mathfrak{g})\to \operatorname{gr}(U(\mathfrak{g}))$, is an isomorphism of graded (commutative) algebras.

Now suppose $\mathfrak{g}$ is a finitely generated, graded Lie algebra. Then $U(\mathfrak{g})$ is isomorphic (as a graded vector space) to a polynomial algebra in variables indexed by bases for the graded pieces of $\mathfrak{g}$, with degrees set accordingly. Hence, its Hilbert series is given by

$$\operatorname{Hilb}(U(\mathfrak{g}),t)=\prod_{i\geq 1}(1-t^i)^{-\dim(\mathfrak{g}_i)}. \tag{2.12}$$

For instance, if $\mathfrak{g}=\mathfrak{lie}(V)$ is the free Lie algebra on a finite-dimensional vector space V with all generators in degree 1, then $\dim(\mathfrak{g}_i)=\frac{1}{i}\sum_{d|i}\mu(d)\cdot n^{i/d}$, where $n=\dim V$ and $\mu\colon\mathbb{N}\to\{-1,0,1\}$ is the Möbius function.

Finally, suppose $\mathfrak{g}=\mathfrak{lie}(V)/\mathfrak{r}$ is a finitely presented, graded Lie algebra, with generators in degree 1 and relation ideal $\mathfrak{r}$ generated by homogeneous elements $g_1,\dots,g_m$. Then $U(\mathfrak{g})$ is the quotient of $T(V)$ by the two-sided ideal generated by $\iota(g_1),\dots,\iota(g_m)$, where $\iota\colon \mathfrak{lie}(V)\hookrightarrow T(V)$ is the canonical inclusion. In particular, if $\mathfrak{g}$ is a quadratic Lie algebra, then $U(\mathfrak{g})$ is a quadratic algebra.

2.2.2 Quadratic algebras

Now let A be a graded $\mathbb{Q}$-algebra. We will assume throughout that A is non-negatively graded, i.e., $A = \bigoplus_{i\geq 0} A_i$, and connected, i.e., $A_0 = \mathbb{Q}$. Every such algebra may be realized as the quotient of a tensor algebra $T(V)$ by a homogeneous, two-sided ideal I. We will further assume that $\dim V < \infty$.

An algebra A as above is said to be *quadratic* if $A_1 = V$ and the ideal I is generated in degree 2, i.e., $I = \langle I_2 \rangle$, where $I_2 = I \cap (V \otimes V)$. Given a quadratic algebra $A = T(V)/I$, identify $V^* \otimes V^* \cong (V \otimes V)^*$, and define the *quadratic dual* of A to be the algebra

$$A^! = T(V^*)/I^\perp, \tag{2.13}$$

where $I^\perp \subset T(V^*)$ is the ideal generated by the vector subspace $I_2^\perp := \{\alpha \in V^* \otimes V^* \mid \alpha(I_2) = 0\}$. Clearly, $A^!$ is again a quadratic algebra, and $(A^!)^! = A$.

For any graded algebra $A = T(V)/I$, we can define a quadrature closure $\bar{A} = T(V)/\langle I_2 \rangle$.

Proposition 2.2.1. *Let $\mathfrak{g}$ be a finitely generated graded Lie algebra generated in degree 1. There is then a unique, functorially defined quadratic Lie algebra, $\bar{\mathfrak{g}}$, such that $U(\bar{\mathfrak{g}}) = \overline{U(\mathfrak{g})}$.*

Proof. Suppose $\mathfrak{g}$ has presentation $\mathfrak{lie}(V)/\mathfrak{r}$. Then $U(\mathfrak{g})$ has a presentation $T(V)/(\iota(\mathfrak{r}))$. Set $\bar{\mathfrak{g}} = \mathfrak{lie}(V)/\langle \mathfrak{r}_2 \rangle$, where $\mathfrak{r}_2 = \mathfrak{r} \cap \mathfrak{lie}_2(V)$; then $U(\bar{\mathfrak{g}})$ has presentation $T(V)/\langle \iota(\mathfrak{r}_2) \rangle$. One can see that $\iota(\mathfrak{r}_2) = \iota(\mathfrak{r}) \cap V \otimes V$. □

A *commutative graded algebra* (for short, a CGA) is a graded $\mathbb{Q}$-algebra as above, which in addition is graded-commutative, i.e., if $a \in A_i$ and $b \in A_j$, then $ab = (-1)^{ij} ba$. If all generators of A are in degree 1, then A can be written as $A = \bigwedge(V)/J$, where $\bigwedge(V)$ is the exterior algebra on the $\mathbb{Q}$-vector space $V = A_1$, and J is a homogeneous ideal in $\bigwedge(V)$ with $J_1 = 0$. If, furthermore, J is generated in degree 2, then A is a quadratic CGA. The next lemma follows directly from the definitions.

Lemma 2.2.2. *Let $W \subset V \wedge V$ be a linear subspace, and let $A = \bigwedge(V)/\langle W\rangle$ be the corresponding quadratic cga. Then $A^! = T(V^*)/\langle \iota(W^\vee)\rangle$, where*

$$W^\vee := \{\alpha \in V^* \wedge V^* \mid \alpha(W) = 0\} = W^\perp \cap (V^* \wedge V^*), \tag{2.14}$$

and $\iota\colon V^* \wedge V^* \hookrightarrow V^* \otimes V^*$ is the inclusion map, given by $x \wedge y \mapsto x \otimes y - y \otimes x$.

For instance, if $A = \bigwedge(V)$, then $A^! = \operatorname{Sym}(V^*)$. Likewise, if $A = \bigwedge(V)/\langle V \wedge V\rangle = \mathbb{Q} \oplus V$, then $A^! = T(V^*)$.

2.2.3 Holonomy Lie algebras

Let A be a commutative graded algebra. Recall we are assuming that $A_0 = \mathbb{Q}$ and $\dim A_1 < \infty$. Because of graded-commutativity, the multiplication map $A_1 \otimes A_1 \to A_2$ factors through a linear map $\mu_A\colon A_1 \wedge A_1 \to A_2$. Dualizing this map, and identifying $(A_1 \wedge A_1)^* \cong A_1^* \wedge A_1^*$, we obtain a linear map,

$$\partial_A = (\mu_A)^*\colon A_2^* \to A_1^* \wedge A_1^*. \tag{2.15}$$

Finally, identify $A_1^* \wedge A_1^*$ with $\mathfrak{lie}_2(A_1^*)$ via the map $x \wedge y \mapsto [x, y]$.

Definition 2.2.3. The *holonomy Lie algebra* of A is the quotient

$$\mathfrak{h}(A) = \mathfrak{lie}(A_1^*)/\langle \operatorname{im} \partial_A\rangle \tag{2.16}$$

of the free Lie algebra on A_1^* by the ideal generated by the image of ∂_A under the above identification. Alternatively, using the notation from (2.14), we have that

$$\mathfrak{h}(A) = \mathfrak{lie}(A_1^*)/\langle \ker(\mu_A)^\vee\rangle. \tag{2.17}$$

By construction, $\mathfrak{h}(A)$ is a quadratic Lie algebra. Moreover, this construction is functorial: if $\varphi\colon A \to B$ is a morphism of CGAs as above, the induced map, $\mathfrak{lie}(\varphi_1^*)\colon \mathfrak{lie}(B_1^*) \to \mathfrak{lie}(A_1^*)$, factors through a morphism of graded Lie algebras, $\mathfrak{h}(\varphi)\colon \mathfrak{h}(B) \to \mathfrak{h}(A)$. Moreover, if φ is injective, then $\mathfrak{h}(\varphi)$ is surjective.

Clearly, the holonomy Lie algebra $\mathfrak{h}(A)$ depends only on the information encoded in the multiplication map $\mu_A\colon A_1 \wedge A_1 \to A_2$. More precisely, let $\bar{A}$ be the *quadratic closure* of A defined as

$$\bar{A} = \bigwedge(A_1)/\langle K \rangle, \tag{2.18}$$

where $K = \ker(\mu_A) \subset A_1 \wedge A_1$. Then $\bar{A}$ is a commutative, quadratic algebra, which comes equipped with a canonical homomorphism $q\colon \bar{A} \to A$, which is an isomorphism in degree 1 and a monomorphism in degree 2. It is readily verified that the induced morphism between holonomy Lie algebras, $\mathfrak{h}(A) \to \mathfrak{h}(\bar{A})$, is an isomorphism.

The following proposition is a slight generalization of a result of Papadima–Yuzvinsky [126, Lemma 4.1].

Proposition 2.2.4. *Let A be a commutative graded algebra. Then $U(\mathfrak{h}(A))$ is a quadratic algebra, and $U(\mathfrak{h}(A)) = \bar{A}^!$.*

Proof. By the above, $\bar{A} = \bigwedge(A_1)/\langle K \rangle$, where $K = \langle \ker(\mu_A) \rangle$. On the other hand, by (2.17) we have that $\mathfrak{h}(A) = \mathfrak{lie}(A_1^*)/\langle K^\vee \rangle$. Hence, by Lemma 2.2.2, $U(\mathfrak{h}(A)) = T(V^*)/\langle \iota(K^\vee) \rangle = \bar{A}^!$. □

Combining Propositions 2.2.1 and 2.2.4, we can see the relations between the quadratic closure of a Lie algebra and the holonomy Lie algebra.

Corollary 2.2.5. *Let $\mathfrak{g}$ be a finitely generated graded Lie algebra generated in degree 1. Then $\mathfrak{h}\left(\overline{U(\mathfrak{g})}^!\right) = \bar{\mathfrak{g}}$.*

Work of Löfwall [97, Theorem 1.1] yields another interpretation of the universal enveloping algebra of the holonomy Lie algebra.

Proposition 2.2.6 ([97])**.** *Let $\operatorname{Ext}_A^1(\mathbb{Q},\mathbb{Q}) = \bigoplus_{i\geq 0} \operatorname{Ext}_A^i(\mathbb{Q},\mathbb{Q})_i$ be the linear strand in the Yoneda algebra of A. Then $U(\mathfrak{h}(A)) \cong \operatorname{Ext}_A^1(\mathbb{Q},\mathbb{Q})$.*

In particular, the graded ranks of the holonomy Lie algebra $\mathfrak{h} = \mathfrak{h}(A)$ are given by $\prod_{n\geq 1}(1-t^n)^{\dim(\mathfrak{h}_n)} = \sum_{i\geq 0} b_{ii}t^i$, where $b_{ii} = \dim_{\mathbb{Q}} \operatorname{Ext}_A^i(\mathbb{Q},\mathbb{Q})_i$.

The next proposition shows that every quadratic Lie algebra can be realized as the holonomy Lie algebra of a (quadratic) algebra.

Proposition 2.2.7. *Let $\mathfrak{g}$ be a quadratic Lie algebra. There is then a commutative quadratic algebra A such that $\mathfrak{g} = \mathfrak{h}(A)$.*

Proof. By assumption, $\mathfrak{g}$ has a presentation of the form $\mathfrak{lie}(V)/\langle W\rangle$, where W is a linear subspace of $V \wedge V$. Define $A = \bigwedge(V^*)/\langle W^\vee\rangle$. Then, by (2.17),

$$\mathfrak{h}(A) = \mathfrak{lie}((V^*)^*)/\langle (W^\vee)^\vee\rangle = \mathfrak{lie}(V)/\langle W\rangle, \tag{2.19}$$

and this completes the proof. □

2.2.4 Koszul algebras

Any connected, graded algebra $A = \bigoplus_{i\geq 0} A_i$ has a free, graded A-resolution of the trivial A-module $\mathbb{Q}$,

$$\cdots \xrightarrow{\varphi_3} A^{b_2} \xrightarrow{\varphi_2} A^{b_1} \xrightarrow{\varphi_1} A \longrightarrow \mathbb{Q}\,. \tag{2.20}$$

Such a resolution is said to be *minimal* if all the nonzero entries of the matrices φ_i have positive degrees.

A *Koszul algebra* is a graded algebra for which the minimal graded resolution of $\mathbb{Q}$ is linear, or, equivalently, $\operatorname{Ext}_A(\mathbb{Q},\mathbb{Q}) = \operatorname{Ext}_A^1(\mathbb{Q},\mathbb{Q})$. Such an algebra is always quadratic, but the converse is far from true. If A is a Koszul algebra, then the quadratic dual $A^!$ is also a Koszul algebra, and the following 'Koszul duality' formula holds:

$$\operatorname{Hilb}(A,t)\cdot\operatorname{Hilb}(A^!,-t) = 1. \tag{2.21}$$

Furthermore, if A is a graded algebra of the form $A = T(V)/I$, where I is an ideal admitting a (noncommutative) quadratic Gröbner basis, then A is a Koszul algebra (see [58] by Fröberg).

Corollary 2.2.8. *Let A be a connected, commutative graded algebra. If $\bar{A}$ is a Koszul algebra, then* $\operatorname{Hilb}(\bar{A}, -t) \cdot \operatorname{Hilb}(U(\mathfrak{h}(A)), t) = 1$.

Example 2.2.9. Consider the quadratic algebra $A = \bigwedge(u_1, u_2, u_3, u_4)/(u_1u_2 - u_3u_4)$. Clearly, $\operatorname{Hilb}(A, t) = 1 + 4t + 5t^2$. If A were Koszul, then formula (2.21) would give $\operatorname{Hilb}(A^!, t) = 1 + 4t + 11t^2 + 24t^3 + 41t^4 + 44t^5 - 29t^6 + \cdots$, which is impossible.

Example 2.2.10. The quasitriangular Lie algebra $\mathfrak{qtr}_n$ defined in [11] is generated by x_{ij}, $1 \leq i \neq j \leq n$ with relations $[x_{ij}, x_{ik}] + [x_{ij}, x_{jk}] + [x_{ik}, x_{jk}] = 0$ for distinct i, j, k and $[x_{ij}, x_{kl}] = 0$ for distinct i, j, k, l. The Lie algebra $\mathfrak{tr}_n$ is the quotient Lie algebra of $\mathfrak{qtr}_n$ by the ideal generated by $x_{ij} + x_{ji}$ for distinct $i \neq j$. In [11], Bartholdi et al. show that the quadratic dual algebras $U(\mathfrak{qtr}_n)^!$ and $U(\mathfrak{tr}_n)^!$ are Koszul, and compute their Hilbert series. They also state that neither $\mathfrak{qtr}_n$ nor $\mathfrak{tr}_n$ is filtered-formal for $n \geq 4$, and sketch a proof of this fact. We will provide a detailed proof in Chapter 7.

2.3 Minimal models and (partial) formality

In this section, we discuss two basic notions in non-simply-connected rational homotopy theory: the minimal model and the (partial) formality properties of a differential graded algebra.

2.3.1 Minimal models of cdgas

We follow the approach of Sullivan [148], Deligne et al. [39], and Morgan [113], as further developed by Félix et al. [51, 52, 53], Griffiths and Morgan [63], Halperin and Stasheff [67], Kohno [81], and Măcinic [99]. We start with some basic algebraic notions.

Definition 2.3.1. A *differential graded algebra* (for short, a CDGA) over a field $\mathbb{Q}$ of characteristic 0 is a graded $\mathbb{Q}$-algebra $A^* = \bigoplus_{n \geq 0} A^n$ equipped with a differential $d\colon A \to A$ of

degree 1 satisfying $ab = (-1)^{mn}ba$ and $d(ab) = d(a) \cdot b + (-1)^{|a|}a \cdot d(b)$ for any $a \in A^m$ and $b \in A^n$. We denote the CDGA by (A^*, d) or simply by A^* if there is no confusion.

A morphism $f\colon A^* \to B^*$ between two CDGA's is a degree zero algebra map which commutes with the differentials. A *Hirsch extension* (of degree i) is a CDGA inclusion $\alpha\colon (A^*, d_A) \hookrightarrow (A^* \otimes \bigwedge(V), d)$, where V is a $\mathbb{Q}$-vector space concentrated in degree i, while $\bigwedge(V)$ is the free graded-commutative algebra generated by V, and d sends V into A^{i+1}. We say this is a *finite* Hirsch extension if $\dim V < \infty$.

We now come to a crucial definition in rational homotopy theory, due to Sullivan [148].

Definition 2.3.2. A CDGA (A^*, d) is called *minimal* if $A^0 = \mathbb{Q}$, and the following two conditions are satisfied:

1. $A^* = \bigcup_{j\geq 0} A^*_j$, where $A_0 = \mathbb{Q}$, and A_j is a Hirsch extension of A_{j-1}, for all $j \geq 0$.

2. The differential is *decomposable*, i.e., $dA^* \subset A^+ \wedge A^+$, where $A^+ = \bigoplus_{i\geq 1} A^i$.

The first condition implies that A^* has an increasing, exhausting filtration by the sub-CDGA's A^*_j; equivalently, A^* is free as a graded-commutative algebra on generators of degree ≥ 1. (Note that we use the lower-index for the filtration, and the upper-index for the grading.) The second condition is automatically satisfied if A is generated in degree 1.

Two CDGAs A^* and B^* are said to be *quasi-isomorphic* if there is a morphism $f\colon A \to B$ inducing isomorphisms in cohomology. The two CDGAs are called *weakly equivalent* (written $A \simeq B$) if there is a sequence of quasi-isomorphisms (in either direction) connecting them. Likewise, for an integer $i \geq 0$, we say that a morphism $f\colon A \to B$ is an *i-quasi-isomorphism* if $f^*\colon H^j(A) \to H^j(B)$ is an isomorphism for each $j \leq i$ and $f^{i+1}\colon H^{i+1}(A) \to H^{i+1}(B)$ is injective. Furthermore, we say that A and B are *i-weakly equivalent* ($A \simeq_i B$) if there is a zig-zag of i-quasi-isomorphisms connecting A to B.

The next two lemmas follow directly from the definitions.

Lemma 2.3.3. *Any* CDGA *morphism* $\phi\colon (A, d_A) \to (B, d_B)$ *extends to a* CDGA *morphism of Hirsch extensions,* $\bar{\phi}\colon (A, d_A) \otimes \bigwedge(x) \to (B, d_B) \otimes \bigwedge(y)$, *provided that* $d(y) = \phi(d(x))$. *Moreover, if* ϕ *is a (quasi-) isomorphism, then so is* $\bar{\phi}$.

Lemma 2.3.4. *Let* $\alpha\colon A \to B$ *be the inclusion map of Hirsch extension of degree* $i+1$. *Then* α *is an* i*-quasi-isomorphism.*

Given a CDGA A, we say that another CDGA B is a *minimal model* for A if B is a minimal CDGA and there exists a quasi-isomorphism $f\colon B \to A$. Likewise, we say that a minimal CDGA B is an *i-minimal model* for A if B is generated by elements of degree at most i, and there exists an i-quasi-isomorphism $f\colon B \to A$. A basic result in rational homotopy theory is the following existence and uniqueness theorem, first proved for (full) minimal models by Sullivan [148], and in full generality by Morgan in [113, Theorem 5.6].

Theorem 2.3.5 ([113, 148])**.** *Each connected* CDGA (A, d) *has a minimal model* $\mathcal{M}(A)$, *unique up to isomorphism. Likewise, for each* $i \geq 0$, *there is an* i*-minimal model* $\mathcal{M}(A, i)$, *unique up to isomorphism.*

It follows from the proof of Theorem 2.3.5 that the minimal model $\mathcal{M}(A)$ is isomorphic to a minimal model built from the i-minimal model $\mathcal{M}(A, i)$ by means of Hirsch extensions in degrees $i+1$ and higher. Thus, in view of Lemma 2.3.4, $\mathcal{M}(A) \simeq_i \mathcal{M}(A, i)$.

2.3.2 Minimal models and holonomy Lie algebras

Let $\mathcal{M} = (\mathcal{M}^*, d)$ be a minimal CDGA over $\mathbb{Q}$, generated in degree 1. Following [113, 81], let us consider the filtration

$$\mathbb{Q} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M} = \bigcup_i \mathcal{M}_i, \tag{2.22}$$

where $\mathcal{M}_1$ is the subalgebra of $\mathcal{M}$ generated by $x \in \mathcal{M}^1$ such that $dx = 0$, and $\mathcal{M}_i$ is the subalgebra of $\mathcal{M}$ generated by $x \in \mathcal{M}^1$ such that $dx \in \mathcal{M}_{i-1}$ for $i > 1$. Each

inclusion $\mathcal{M}_{i-1} \subset \mathcal{M}_i$ is a Hirsch extension of the form $\mathcal{M}_i = \mathcal{M}_{i-1} \otimes \bigwedge(V_i)$, where $V_i := \ker(H^2(\mathcal{M}_{i-1}) \to H^2(\mathcal{M}))$. Taking the degree 1 part of the filtration (2.22), we obtain the filtration

$$\mathbb{Q} = \mathcal{M}_0^1 \subset \mathcal{M}_1^1 \subset \mathcal{M}_2^1 \subset \cdots \subset \mathcal{M}^1. \tag{2.23}$$

Now assume each of the above Hirsch extensions is finite, i.e., $\dim(V_i) < \infty$ for all i. Using the fact that $d(V_i) \subset \mathcal{M}_{i-1}$, we see that each dual vector space $\mathfrak{L}_i = (\mathcal{M}_i^1)^*$ acquires the structure of a $\mathbb{Q}$-Lie algebra by setting

$$\langle [u^*, v^*], w \rangle = \langle u^* \wedge v^*, dw \rangle \tag{2.24}$$

for $v, v, w \in \mathcal{M}_i^1$. Clearly, $d(V_1) = 0$, and thus $\mathfrak{L}_1 = (V_1)^*$ is an abelian Lie algebra. Using the vector space decompositions $\mathcal{M}_i^1 = \mathcal{M}_{i-1}^1 \oplus V_i$ and $\mathcal{M}_i^2 = \mathcal{M}_{i-1}^2 \oplus (\mathcal{M}_{i-1}^1 \otimes V_i) \oplus \bigwedge^2(V_i)$ we easily see that the canonical projection $\mathfrak{L}_i \twoheadrightarrow \mathfrak{L}_{i-1}$ (i.e., the dual of the inclusion map $\mathcal{M}_{i-1} \hookrightarrow \mathcal{M}_i$) has kernel V_i^*, and this kernel is central inside $\mathfrak{L}_i$. Therefore, we obtain a tower of finite-dimensional nilpotent $\mathbb{Q}$-Lie algebras,

$$0 \twoheadleftarrow \mathfrak{L}_1 \twoheadleftarrow \mathfrak{L}_2 \twoheadleftarrow \cdots \twoheadleftarrow \mathfrak{L}_i \twoheadleftarrow \cdots . \tag{2.25}$$

The inverse limit of this tower, $\mathfrak{L} = \mathfrak{L}(\mathcal{M})$, endowed with the inverse limit filtration, is a complete, filtered Lie algebra with the property that $\mathfrak{L}/\widehat{\Gamma}_{i+1}\mathfrak{L} = \mathfrak{L}_i$, for each $i \geq 1$. Conversely, from a tower of the form (2.25), we can construct a sequence of finite Hirsch extensions $\mathcal{M}_i$ as in (2.22). It is readily seen that the CDGA $\mathcal{M}_i$, with differential defined by (2.24), coincides with the Chevalley–Eilenberg complex $(\bigwedge(\mathfrak{L}_i^*), d)$ associated to the finite-dimensional Lie algebra $\mathfrak{L}_i = \mathfrak{L}(\mathcal{M}_i)$, as in [70, Section VII]. In particular,

$$H^*(\mathcal{M}_i) \cong H^*(\mathfrak{L}_i; \mathbb{Q}) . \tag{2.26}$$

The direct limit of the above sequence of Hirsch extensions, $\mathcal{M} = \bigcup_i \mathcal{M}_i$, is a minimal $\mathbb{Q}$-CDGA generated in degree 1, which we denote by $\mathcal{M}(\mathfrak{L})$. We obtain in this fashion an adjoint correspondence that sends $\mathcal{M}$ to the pronilpotent Lie algebra $\mathfrak{L}(\mathcal{M})$ and conversely,

sends a pronilpotent Lie algebra $\mathfrak{L}$ to the minimal algebra $\mathcal{M}(\mathfrak{L})$. Under this correspondence, filtration-preserving CDGA morphisms $\mathcal{M} \to \mathcal{N}$ get sent to filtration-preserving Lie morphisms $\mathfrak{L}(\mathcal{N}) \to \mathfrak{L}(\mathcal{M})$, and vice-versa.

2.3.3 Positive weights

Following Body et al. [19], Morgan [113], and Sullivan [148], we say that a CGA A^* has *positive weights* if each graded piece has a vector space decomposition $A^i = \bigoplus_{\alpha\in\mathbb{Z}} A^{i,\alpha}$ with $A^{1,\alpha} = 0$ for $\alpha \leq 0$, such that $xy \in A^{i+j,\alpha+\beta}$ for $x \in A^{i,\alpha}$ and $y \in A^{j,\beta}$. Furthermore, we say that a CDGA (A^*, d) has *positive weights* if the underlying CGA A^* has positive weights, and the differential is homogeneous with respect to those weights, i.e., $d(x) \in A^{i+1,\alpha}$ for $x \in A^{i,\alpha}$.

Now let $(\mathcal{M}^*, d)$ be a minimal CDGA generated in degree one, endowed with the canonical filtration $\{\mathcal{M}_i\}_{i\geq 0}$ constructed in (2.22), where each sub-CDGA $\mathcal{M}_i$ given by a Hirsch extension of the form $\mathcal{M}_{i-1} \otimes \bigwedge(V_i)$. The underlying CGA $\mathcal{M}^*$ possesses a natural set of positive weights, which we will refer to as the *Hirsch weights*: simply declare V_i to have weight i, and extend those weights to $\mathcal{M}^*$ multiplicatively. We say that the CDGA $(\mathcal{M}^*, d)$ has *positive Hirsch weights* if the differential d is homogeneous with respect to those weights. If this is the case, each sub-CDGA $\mathcal{M}_i$ also has positive Hirsch weights.

Lemma 2.3.6. *Let $\mathcal{M} = (\mathcal{M}^*, d)$ be a minimal* CDGA *generated in degree one, with dual Lie algebra $\mathfrak{L}$. Then $\mathcal{M}$ has positive Hirsch weights if and only if $\mathfrak{L} = \widehat{\mathrm{gr}}(\mathfrak{L})$.*

Proof. As usual, write $\mathcal{M} = \bigcup \mathcal{M}_i$, with $\mathcal{M}_i = \mathcal{M}_{i-1} \otimes \bigwedge(V_i)$. Since $\mathcal{M}$ is generated in degree one, the differential is homogeneous with respect to the Hirsch weights if and only if $d(V_s) \subset \bigoplus_{i+j=s} V_i \wedge V_j$, for all $s \geq 1$. Passing now to the dual Lie algebra $\mathfrak{L} = \mathfrak{L}(\mathcal{M})$ and using formula (2.24), we see that this condition is equivalent to having $[V_i^*, V_j^*] \subset V_{i+j}^*$, for all $i, j \geq 1$. In turn, this is equivalent to saying that each Lie algebra $\mathfrak{L}_i$ is a graded Lie algebra with $\mathrm{gr}_k(\mathfrak{L}_i) = V_k^*$, for each $k \leq i$, which means that the filtered Lie algebra $\mathfrak{L} = \varprojlim_i \mathfrak{L}_i$ coincides with the completion of its associated graded Lie algebra, $\widehat{\mathrm{gr}}(\mathfrak{L})$. □

Remark 2.3.7. The property that the differential of $\mathcal{M}$ be homogeneous with respect to the Hirsch weights is stronger than saying that the Lie algebra $\mathfrak{L} = \mathfrak{L}(\mathcal{M})$ is filtered-formal. The fact that this can happen is illustrated in Example 9.1.2.

Remark 2.3.8. If a minimal CDGA is generated in degree 1 and has positive weights, but these weights do not coincide with the Hirsch weights, then the dual Lie algebra need not be filtered-formal. This phenomenon is illustrated in Example 9.1.4: there is a finitely generated nilpotent Lie algebra $\mathfrak{m}$ for which the Chevalley–Eilenberg complex $\mathcal{M}(\mathfrak{m}) = \bigwedge(\mathfrak{m}^*)$ has positive weights, but those weights are not the Hirsch weights; moreover, $\mathfrak{m}$ is not filtered-formal.

2.3.4 Dual Lie algebra and holonomy Lie algebra

Let (B^*, d) be a CDGA, and let $A = H^*(B)$ be its cohomology algebra. Assume A is connected and $\dim A^1 < \infty$, and let $\mu\colon A^1 \wedge A^1 \to A^2$ be the multiplication map. By the discussion from §2.3.1, there is a 1-minimal model $\mathcal{M}(B, 1)$ for (B^*, d), unique up to isomorphism.

A concrete way to build such a model can be found in [39, 63, 113]. The first two steps of this construction are easy to describe. Set $V_1 = A^1$ and define $\mathcal{M}(B, 1)_1 = \bigwedge(V_1)$, with differential $d = 0$. Next, set $V_2 = \ker(\mu)$ and define $\mathcal{M}(B, 1)_2 = \bigwedge(V_1 \oplus V_2)$, with $d|_{V_2}$ equal to the inclusion map $V_2 \hookrightarrow A^1 \wedge A^1$.

Let $\mathfrak{L}(B) = \mathfrak{L}(\mathcal{M}(B, 1))$ be the Lie algebra corresponding to the 1-minimal model of B. The next proposition, which generalizes a result of Kohno ([81, Lemma 4.9]), relates this Lie algebra to the holonomy Lie algebra $\mathfrak{h}(A)$ from Definition 2.2.3.

Proposition 2.3.9. *Let $\phi\colon \mathbf{L} \to \mathfrak{L}(B)$ be the morphism defined by extending the identity map of V_1^* to the free Lie algebra $\mathbf{L} = \mathfrak{lie}(V_1^*)$, and let $J = \ker(\phi)$. There exists then an isomorphism of graded Lie algebras, $\mathfrak{h}(A) \cong \mathbf{L}/\langle J \cap \mathbf{L}_2 \rangle$, where $\mathfrak{h}(A)$ is the holonomy Lie algebra of $A = H^*(B)$.*

Proof. Let $\mathrm{gr}(\phi)\colon \mathbf{L} \to \mathrm{gr}^{\widehat{\Gamma}}(\mathfrak{L}(B))$ be the associated graded morphism of ϕ. Then the first graded piece $\mathrm{gr}_1(\phi)\colon V_1^* \to V_1^*$ is the identity, while the second graded piece $\mathrm{gr}_2(\phi)$ can be identified with the Lie bracket map $V_1^* \wedge V_1^* \to V_2^*$, which is the dual of the differential $d\colon V_2 \to V_1 \wedge V_1$. From the construction of $\mathcal{M}(B,1)_2$, there is an isomorphism $\ker d^* \cong \operatorname{im} \mu^*$. Since $J \cap \mathbf{L}_2 = \ker(\mathrm{gr}_2(\phi))$, we have that $\operatorname{im} \mu^* = J \cap \mathbf{L}_2$, and the claim follows. □

2.3.5 The completion of the holonomy Lie algebra

Let A^* be a commutative graded $\mathbb{Q}$-algebra with $A^0 = \mathbb{Q}$. Proceeding as above, by taking $B = A$ and $d = 0$ so that $H^*(B) = A$, we can construct a 1-minimal model $\mathcal{M} = \mathcal{M}(A,1)$ for the algebra A in a 'formal' way, following the approach outlined by Carlson and Toledo in [24]. (A construction of the full, bigraded minimal model of a CGA can be found in [67, §3].)

As before, set $\mathcal{M}_1 = (\bigwedge(V_1), d = 0)$ where $V_1 = A^1$, and $\mathcal{M}_2 = (\bigwedge(V_1 \oplus V_2), d)$, where $V_2 = \ker(\mu\colon A^1 \wedge A^1 \to A^2)$ and $d\colon V_2 \hookrightarrow V_1 \wedge V_1$ is the inclusion map. After that, define inductively $\mathcal{M}_i$ as $\mathcal{M}_{i-1} \otimes \bigwedge(V_i)$, where the vector space V_i fits into the short exact sequence

$$0 \longrightarrow V_i \longrightarrow H^2(\mathcal{M}_{i-1}) \longrightarrow \operatorname{im}(\mu) \longrightarrow 0\,, \tag{2.27}$$

while the differential d includes V_i into $V_1 \wedge V_{i-1} \subset \mathcal{M}_{i-1}$. In particular, the subalgebras $\mathcal{M}_i$ constitute the canonical filtration (2.22) of $\mathcal{M}$, and the differential d preserves the Hirsch weights on $\mathcal{M}$. For these reasons, we call $\mathcal{M} = \mathcal{M}(A,1)$ the *canonical* 1-minimal model of A.

The next theorem relates the Lie algebra dual to the 1-minimal model of a CGA as above to its holonomy Lie algebra. A similar result was obtained by Markl and Papadima in [105]; see also Morgan [113, Theorem 9.4] and Remark 3.3.3.

Theorem 2.3.10. *Let A^* be a connected* CGA *with $\dim A^1 < \infty$. Let $\mathfrak{L}(A) := \mathfrak{L}(\mathcal{M}(A,1))$ be the Lie algebra corresponding to the 1-minimal model of A, and let $\mathfrak{h}(A)$ be the holonomy*

Lie algebra of A. There exists then an isomorphism of complete, filtered Lie algebras between $\mathfrak{L}(A)$ and the degree completion $\widehat{\mathfrak{h}}(A)$.

Proof. By Definition 2.2.3, the holonomy Lie algebra of A has presentation $\mathfrak{h}(A) = \mathbf{L}/\mathfrak{r}$, where $\mathbf{L} = \mathfrak{lie}(V_1^*)$ and $\mathfrak{r}$ is the ideal generated by $\operatorname{im}(\mu^*) \subset \mathbf{L}_2$. It follows that, for each $i \geq 1$, the nilpotent quotient $\mathfrak{h}_i(A) := \mathfrak{h}(A)/\Gamma_{i+1}\mathfrak{h}(A)$ has presentation $\mathbf{L}/(\mathfrak{r} + \Gamma_{i+1}\mathbf{L})$.

Consider now the dual Lie algebra $\mathfrak{L}_i(A) = \mathfrak{L}(\mathcal{M}_i)$. By construction, we have a vector space decomposition, $\mathfrak{L}_i(A) = \bigoplus_{s \leq i} V_s^*$. The fact that $d(V_s) \subset V_1 \wedge V_{s-1}$ implies that the Lie bracket maps $V_1^* \wedge V_{s-1}^*$ onto V_s^*, for every $1 < s \leq i$. In turn, this implies that $\mathfrak{L}_i(A)$ is an i-step nilpotent, graded Lie algebra generated in degree 1, with $\operatorname{gr}_s(\mathfrak{L}_i(A)) = V_s^*$ for $s \leq i$.

Let $\mathfrak{r}_i$ be the kernel of the canonical projection $\pi_i \colon \mathbf{L} \twoheadrightarrow \mathfrak{L}_i(A)$. By the Hopf formula, there is an isomorphism of graded vector spaces between $H_2(\mathfrak{L}_i(A); \mathbb{Q})$ and $\mathfrak{r}_i/[\mathbf{L}, \mathfrak{r}_i]$, the space of (minimal) generators for the homogeneous ideal $\mathfrak{r}_i$. On the other hand, $H^2(\mathcal{M}_i) \cong H^2(\mathfrak{L}_i; \mathbb{Q})$, by (2.26). Taking the dual of the exact sequence (2.27), we find that $H_2(\mathfrak{L}_i(A); \mathbb{Q}) \cong \operatorname{im}(\mu^*) \oplus V_{i+1}^*$. We conclude that the ideal $\mathfrak{r}_i$ is generated by $\operatorname{im}(\mu^*)$ in degree 2 and a copy of V_{i+1}^* in degree $i+1$.

Since $\operatorname{gr}_2(\mathfrak{r}) = \operatorname{im}(\mu^*)$, we infer that $\bigoplus_{s \leq i} \operatorname{gr}_s(\mathfrak{r}_i) = \bigoplus_{s \leq i} \operatorname{gr}_s(\mathfrak{r})$. Since $\mathfrak{L}_i(A)$ is an i-step nilpotent Lie algebra, $\bigoplus_{s > i} \operatorname{gr}_s(\mathfrak{r}_i) = \Gamma_{i+1}\mathbf{L}$. Therefore, $\Gamma_{i+1}\mathbf{L} + \mathfrak{r} = \mathfrak{r}_i$. It follows that the identity map of $\mathbf{L}$ induces an isomorphism $\mathfrak{L}_i(A) \cong \mathfrak{h}_i(A)$, for each $i \geq 1$. Hence, $\mathfrak{L}(A) \cong \widehat{\mathfrak{h}}(A)$, as filtered Lie algebras. □

Corollary 2.3.11. *The graded ranks of the holonomy Lie algebra of a connected, graded algebra A are given by $\dim \mathfrak{h}_i(A) = \dim V_i$, where $\mathcal{M} = \bigwedge \left(\bigoplus_{i \geq 1} V_i \right)$ is the 1-minimal model of $(A, d = 0)$.*

2.3.6 Partial formality and field extensions

The following notion, introduced by Sullivan in [148], and further developed in [39, 63, 99, 113], will play a central role in our study.

Definition 2.3.12. A CDGA (A^*, d) over $\mathbb{Q}$ is said to be *formal* if there exists a quasi-isomorphism $\mathcal{M}(A) \to (H^*(A), d = 0)$. Likewise, (A^*, d) is said to be *i-formal* if there exists an i-quasi-isomorphism $\mathcal{M}(A, i) \to (H^*(A), d = 0)$.

In [99], Măcinic studies in detail these concepts. Evidently, if A is formal, then it is i-formal, for all $i \geq 0$, and, if A is i-formal, then it is j-formal for every $j \leq i$. Moreover, A is 0-formal if and only if $H^0(A) = \mathbb{Q}$.

Lemma 2.3.13 ([99])**.** *A* CDGA (A^*, d) *is i-formal if and only if* (A^*, d) *is i-weakly equivalent to* $(H^*(A), d = 0)$.

As a corollary, we deduce that i-formality is invariant under i-weakly equivalence.

Corollary 2.3.14. *Suppose $A \simeq_i B$. Then A is i-formal if and only if B is i-formal.*

Given a CDGA (A, d) over a field $\mathbb{Q}$ of characteristic 0, and a field extension $\mathbb{Q} \subset \mathbb{K}$, let $(A \otimes \mathbb{K}, d \otimes \mathrm{id}_{\mathbb{K}})$ be the corresponding CDGA over $\mathbb{K}$. (If the underlying field $\mathbb{Q}$ is understood, we will usually omit it from the tensor product $A \otimes_{\mathbb{Q}} \mathbb{K}$.) The following result will be crucial to us in the sequel.

Theorem 2.3.15 (Theorem 6.8 in [67])**.** *Let (A^*, d_A) and (B^*, d_B) be two* CDGA*s over $\mathbb{Q}$ whose cohomology algebras are connected and of finite type. Suppose there is an isomorphism of graded algebras, $f\colon H^*(A) \to H^*(B)$, and suppose $f \otimes \mathrm{id}_{\mathbb{K}}\colon H^*(A) \otimes \mathbb{K} \to H^*(B) \otimes \mathbb{K}$ can be realized by a weak equivalence between $(A^* \otimes \mathbb{K}, d_A \otimes \mathrm{id}_{\mathbb{K}})$ and $(B^* \otimes \mathbb{K}, d_B \otimes \mathrm{id}_{\mathbb{K}})$. Then f can be realized by a weak equivalence between (A^*, d_A) and (B^*, d_B).*

This theorem has an important corollary, based on the following lemma. For completeness, we provide proofs for these statements, which are omitted in [67] by Halperin and Stasheff.

Lemma 2.3.16 ([67])**.** *A* CDGA (A^*, d_A) *with $H^*(A)$ of finite-type is formal if and only if the identity map of $H^*(A)$ can be realized by a weak equivalence between (A^*, d_A) and $(H^*(A), d = 0)$.*

Proof. The backwards implication is obvious. So assume (A^*, d_A) is formal, that is, there is a zig-zag of quasi-isomorphisms between (A^*, d_A) and $(H^*(A), d = 0)$. This yields an isomorphism in cohomology, $\phi\colon H^*(A) \to H^*(A)$. The inverse of ϕ defines a quasi-isomorphism between $(H^*(A), d = 0)$ and $(H^*(A), d = 0)$. Composing this quasi-isomorphism with the given zig-zag of quasi-isomorphisms defines a new weak equivalence between (A^*, d_A) and $(H^*(A), d = 0)$, which induces the identity map in cohomology. □

Corollary 2.3.17 ([67])**.** *A* $\mathbb{Q}$-CDGA (A^*, d_A) *with* $H^*(A)$ *of finite-type is formal if and only if the* $\mathbb{K}$-CDGA $(A^* \otimes \mathbb{K}, d_A \otimes \mathrm{id}_{\mathbb{K}})$ *is formal.*

Proof. As the forward implication is obvious, we only prove the converse. Suppose our $\mathbb{K}$-CDGA is formal. By Lemma 2.3.16, there exists a weak equivalence between $(A^* \otimes \mathbb{K}, d_A \otimes \mathrm{id}_{\mathbb{K}})$ and $(H^*(A) \otimes \mathbb{K}, d = 0)$ inducing the identity on $H^*(A) \otimes \mathbb{K}$. By Theorem 2.3.15, the map $\mathrm{id}\colon H^*(A) \to H^*(A)$ can be realized by a weak equivalence between (A^*, d_A) and $(H^*(A), d = 0)$. That is, (A^*, d_A) is formal (over $\mathbb{Q}$). □

2.3.7 Field extensions and i-formality

We now use the aforementioned result of Halperin and Stasheff on full formality to establish an analogous result for partial formality. First we need an auxiliary construction, and a lemma.

Let $\mathcal{M}(A, i)$ be the i-minimal model of a CDGA (A^*, d_A). The degree $i + 1$ piece, $\mathcal{M}(A, i)^{i+1}$, is isomorphic to $(\ker d^{i+1}) \oplus \mathcal{C}_{i+1}$, where $d^{i+1}\colon \mathcal{M}(A, i)^{i+1} \to \mathcal{M}(A, i)^{i+2}$ is the differential, and $\mathcal{C}_{i+1}$ is a complement to its kernel. It is readily checked that the vector subspace

$$\mathcal{I}_i := \mathcal{C}_{i+1} \oplus \bigoplus_{s \geq i+2} \mathcal{M}(A, i)^s \tag{2.28}$$

is an ideal of $\mathcal{M}(A,i)$, left invariant by the differential. Consider the quotient CDGA,

$$\begin{aligned}\mathcal{M}[A,i] :&= \mathcal{M}(A,i)/\mathcal{I}_i \\ &= \mathbb{Q} \oplus \mathcal{M}(A,i)^1 \oplus \cdots \oplus \mathcal{M}(A,i)^i \oplus \ker d^{i+1}.\end{aligned} \tag{2.29}$$

Lemma 2.3.18. *The following statements are equivalent:*

1. (A^*, d_A) *is* i*-formal.*
2. $\mathcal{M}(A,i)$ *is* i*-formal.*
3. $\mathcal{M}[A,i]$ *is* i*-formal.*
4. $\mathcal{M}[A,i]$ *is formal.*

Proof. Since $\mathcal{M}(A,i)$ is an i-minimal model for (A^*, d_A), the two CDGAs are i-quasi-isomorphic. The equivalence (1) $\Leftrightarrow$ (2) follows from Corollary 2.3.14.

Now let $\psi\colon \mathcal{M}(A,i) \to \mathcal{M}[A,i]$ be the canonical projection. It is readily checked that the induced homomorphism, $\psi^*\colon H^*(\mathcal{M}(A,i)) \to H^*(\mathcal{M}[A,i])$, is an isomorphism in degrees up to and including $i+1$. In particular, this shows that $\mathcal{M}(A,i)$ is an i-minimal model for $\mathcal{M}[A,i]$. The equivalence (2) $\Leftrightarrow$ (3) again follows from Corollary 2.3.14.

Implication (4) $\Rightarrow$ (3) is trivial, so it remains to establish (3) $\Rightarrow$ (4). Assume the CDGA $\mathcal{M}[A,i]$ is i-formal. Since $\mathcal{M}(A,i)$ is an i-minimal model for $\mathcal{M}[A,i]$, there is an i-quasi-isomorphism β as in diagram (2.30). In particular, the homomorphism, $\beta^*\colon H^{i+1}(\mathcal{M}(A,i)) \to H^{i+1}(\mathcal{M}[A,i])$, is injective. On the other hand, we know from the previous paragraph that $H^{i+1}(\mathcal{M}[A,i])$ and $H^{i+1}(\mathcal{M}(A,i))$ have the same dimension; thus, β^* is an isomorphism in degree $i+1$, too.

$$\begin{array}{ccc} \mathcal{M}(A,i) & \xrightarrow{\beta} & (H^*(\mathcal{M}[A,i]),0) \\ {\scriptstyle\psi}\downarrow & \overset{\alpha}{\searrow} & \uparrow{\scriptstyle\gamma}\ {\scriptstyle\simeq} \\ \mathcal{M}[A,i] & \underset{\simeq}{\overset{\phi}{\leftarrow}} & \mathcal{M}\,. \end{array} \tag{2.30}$$

Let $\mathcal{M} = \mathcal{M}(\mathcal{M}[A,i])$ be the full minimal model of $\mathcal{M}[A,i]$. As mentioned right after Theorem 2.3.5, this model can be constructed by Hirsch extensions of degree $k \geq i+1$, starting from the i-minimal model of $\mathcal{M}[A,i]$, which we can take to be $\mathcal{M}(A,i)$. Hence, the inclusion map, $\alpha\colon \mathcal{M}(A,i) \to \mathcal{M}$, induces isomorphisms in cohomology up to degree i, and a monomorphism in degree $i+1$. Now, since $H^{i+1}(\mathcal{M})$ has the same dimension as $H^{i+1}(\mathcal{M}[A,i])$, and thus as $H^{i+1}(\mathcal{M}(A,i))$, the map α^* is also an isomorphism in degree $i+1$.

The CDGA morphism β extends to a CGA map $\gamma\colon \mathcal{M} \to H^*(\mathcal{M}[A,i])$ as in diagram (2.30), by sending the new generators to zero. Since the target of β vanishes in degrees $k \geq i+2$ and has differential $d=0$, the map γ is a CDGA morphism. Furthermore, since $\gamma \circ \alpha = \beta$, we infer that γ induces isomorphisms in cohomology in degrees $k \leq i+1$. Since $H^k(\mathcal{M}) = H^k(\mathcal{M}[A,i]) = 0$ for $k \geq i+2$, we conclude that γ^* is an isomorphism in all degrees, i.e., γ is a quasi-isomorphism.

Finally, let $\phi\colon \mathcal{M} \to \mathcal{M}[A,i]$ be a quasi-isomorphism from the minimal model of $\mathcal{M}[A,i]$ to this CDGA. The maps ϕ and γ define a weak equivalence between $\mathcal{M}[A,i]$ and $H^*(\mathcal{M}[A,i])$, thereby showing that $\mathcal{M}[A,i]$ is formal. □

Since $H^{\geq i+2}(\mathcal{M}[A,i]) = 0$, the equivalence of conditions (3) and (4) in the above lemma also follows from the (quite different) proof of Proposition 3.4 from [99]; see Remark 2.3.21 for more on this. We are now ready to prove descent for partial formality of CDGAs.

Theorem 2.3.19. *Let (A^*, d_A) be a CDGA over $\mathbb{Q}$, and let $\mathbb{Q} \subset \mathbb{K}$ be a field extension. Suppose $H^{\leq i+1}(A)$ is finite-dimensional. Then (A^*, d_A) is i-formal if and only if $(A^* \otimes \mathbb{K}, d_A \otimes \mathrm{id}_{\mathbb{K}})$ is i-formal.*

Proof. By Lemma 2.3.18, (A^*, d_A) is i-formal if and only if $\mathcal{M}[A,i]$ is formal. By construction, $H^q(\mathcal{M}[A,i])$ equals $H^q(A)$ for $q \leq i$, injects into $H^q(A)$ for $q = i+1$, and vanishes for $q > i+1$; hence, in view of our hypothesis, $H^*(\mathcal{M}[A,i])$ is of finite-type. By Corollary

2.3.17, $\mathcal{M}[A,i]$ is formal if and only if $\mathcal{M}[A,i]\otimes\mathbb{K}$ is formal. By Lemma 2.3.18 again, this is equivalent to the i-formality of $\mathcal{M}[A,i]\otimes\mathbb{K}$. □

2.3.8 Formality notions for spaces

To every space X, Sullivan [148] associated in a functorial way a CDGA of 'rational polynomial forms', denoted $A^*_{PL}(X)$. As shown in [51, §10], there is a natural identification $H^*(A^*_{PL}(X)) = H^*(X,\mathbb{Q})$ under which the respective induced homomorphisms in cohomology correspond. In particular, the weak isomorphism type of $A^*_{PL}(X)$ depends only on the rational homotopy type of X.

A CDGA (A,d) over $\mathbb{K}$ is called a *model* for the space X if A is weakly equivalent to Sullivan's algebra $A_{PL}(X;\mathbb{K}) := A_{PL}(X)\otimes_{\mathbb{Q}}\mathbb{K}$. In other words, $\mathcal{M}(A)$ is isomorphic to $\mathcal{M}(X;\mathbb{K}) := \mathcal{M}(X)\otimes_{\mathbb{Q}}\mathbb{K}$, where $\mathcal{M}(A)$ is the minimal model of A and $\mathcal{M}(X)$ is the minimal model of $A_{PL}(X)$. In the same vein, A is an *i-model* for X if $(A,d)\simeq_i A_{PL}(X;\mathbb{K})$. For instance, if X is a smooth manifold, then the de Rham algebra $\Omega^*_{dR}(X)$ is a model for X over $\mathbb{R}$.

A space X is said to be *formal* over $\mathbb{K}$ if the model $A_{PL}(X;\mathbb{K})$ is formal, that is, there is a quasi-isomorphism $\mathcal{M}(X;\mathbb{K})\to(H^*(X;\mathbb{K}),d=0)$. Likewise, X is said to be *i-formal*, for some $i\geq 0$, if there is an i-quasi-isomorphism $\mathcal{M}(A_{PL}(X;\mathbb{K}),i)\to(H^*(X;\mathbb{K}),d=0)$. Note that X is 0-formal if and only if X is path-connected. Also, since a homotopy equivalence $X\simeq Y$ induces an isomorphism $H^*(Y;\mathbb{Q})\xrightarrow{\cong}H^*(X;\mathbb{Q})$, it follows from Corollary 2.3.14 that the i-formality property is preserved under homotopy equivalences.

The following theorem of Papadima and Yuzvinsky [126] nicely relates the properties of the minimal model of X to the Koszulness of its cohomology algebra.

Theorem 2.3.20 ([126]). *Let X be a connected space with finite Betti numbers. If $\mathcal{M}(X)\cong\mathcal{M}(X,1)$, then $H^*(X;\mathbb{Q})$ is a Koszul algebra. Moreover, if X is formal, then the converse holds.*

Remark 2.3.21. In [99, Proposition 3.4], Măcinic shows that every i-formal space X for which $H^{\geq i+2}(X;\mathbb{Q})$ vanishes is formal. In particular, the notions of formality and i-formality coincide for $(i+1)$-dimensional CW-complexes. In general, though, full formality is a much stronger condition than partial formality.

Remark 2.3.22. There is a competing notion of i-formality, due to Fernández and Muñoz [56]. As explained in [99], the two notions differ significantly, even for $i = 1$. In the sequel, we will use exclusively the classical notion of i-formality given above.

As is well-known, the (full) formality property behaves well with respect to field extensions of the form $\mathbb{Q} \subset \mathbb{K}$. Indeed, it follows from Halperin and Stasheff's Corollary 2.3.17 that a connected space X with finite Betti numbers is formal over $\mathbb{Q}$ if and only if X is formal over $\mathbb{K}$. This result was first stated and proved by Sullivan [148], using different techniques. An independent proof was given by Neisendorfer and Miller [115] in the simply-connected case.

These classical results on descent of formality may be strengthened to a result on descent of partial formality. More precisely, using Theorem 2.3.19, we obtain the following immediate corollary.

Corollary 2.3.23. *Let X be a connected space with finite Betti numbers $b_1(X), \dots, b_{i+1}(X)$. Then X is i-formal over $\mathbb{Q}$ if and only if X is i-formal over $\mathbb{K}$.*

Chapter 3

Formality of finitely generated groups

We now turn to finitely generated groups, and study the associated graded Lie algebras, the holonomy Lie algebra, and the Malcev Lie algebras attached to such groups, and the ranks of these Lie algebras. We specially emphasize on the relationship between these Lie algebras, relating to the notion of 1-formality and leading to the notions of graded-formality and filtered-formality. We investigate the propagation properties for these formality properties, with respect to split injections, products and coproducts. The most intricate of these Lie algebras, and in many ways, the most important, is the Malcev Lie algebra, for which we describe several equivalent definitions. The study of filtered formality of the Malcev Lie algebra relates to many research directions in different fields. This chapter is based on the work in my papers [143, 145, 146] with Alex Suciu.

3.1 Groups, Lie algebras, and graded formality

In this section, we study two graded Lie algebras associated to a finitely generated group and the graded formality property.

3.1.1 Central filtrations on groups

We start with some general background on lower central series and the associated graded Lie algebra of a group. For more details on this classical topic, we refer to Lazard [89] and Magnus et al. [102].

Let G be a group. For elements $x, y \in G$, let $[x, y] = xyx^{-1}y^{-1}$ be their group commutator. Likewise, for subgroups $H, K < G$, let $[H, K]$ be the subgroup of G generated by all commutators $[x, y]$ with $x \in H$, $y \in K$.

A *(central) filtration* on the group G is a decreasing sequence of subgroups, $G = \mathcal{F}_1 G > \mathcal{F}_2 G > \mathcal{F}_3 G > \cdots$, such that $[\mathcal{F}_r G, \mathcal{F}_s G] \subset \mathcal{F}_{r+s} G$. It is readily verified that, for each $k > 1$, the group $\mathcal{F}_{k+1} G$ is a normal subgroup of $\mathcal{F}_k G$, and the quotient group $\operatorname{gr}_k^{\mathcal{F}}(G) = \mathcal{F}_k G / \mathcal{F}_{k+1} G$ is abelian. As before, let $\mathbb{Q}$ be a field of characteristic 0. The direct sum

$$\operatorname{gr}^{\mathcal{F}}(G; \mathbb{Q}) = \bigoplus_{k \geq 1} \operatorname{gr}_k^{\mathcal{F}}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{3.1}$$

is a graded Lie algebra over $\mathbb{Q}$, with Lie bracket induced from the group commutator: If $x \in \mathcal{F}_r G$ and $y \in \mathcal{F}_s G$, then $[x + \mathcal{F}_{r+1} G, y + \mathcal{F}_{s+1} G] = xyx^{-1}y^{-1} + \mathcal{F}_{r+s+1} G$. We can view $\operatorname{gr}^{\mathcal{F}}(-; \mathbb{Q})$ as a functor from groups to graded $\mathbb{Q}$-Lie algebras. Moreover, $\operatorname{gr}^{\mathcal{F}}(G; \mathbb{K}) = \operatorname{gr}^{\mathcal{F}}(G; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{K}$, for any field extension $\mathbb{Q} \subset \mathbb{K}$. (Once again, if the underlying ring in a tensor product is understood, we will write $\otimes$ for short.)

Let H be a normal subgroup of G, and let $Q = G/H$ be the quotient group. Define filtrations on H and Q by $\widetilde{\mathcal{F}}_k H = \mathcal{F}_k G \cap H$ and $\overline{\mathcal{F}}_k Q = \mathcal{F}_k G / \widetilde{\mathcal{F}}_k H$, respectively. We then have the following classical result of Lazard.

Proposition 3.1.1 (Theorem 2.4 in [89]). *The canonical projection $G \twoheadrightarrow G/H$ induces a natural isomorphism of graded Lie algebras,*

$$\operatorname{gr}^{\mathcal{F}}(G) / \operatorname{gr}^{\widetilde{\mathcal{F}}}(H) \xrightarrow{\simeq} \operatorname{gr}^{\overline{\mathcal{F}}}(G/H) \,.$$

3.1.2 The associated graded Lie algebra

Any group G comes endowed with the lower central series (LCS) filtration $\{\Gamma_k G\}_{k\geq 1}$, defined inductively by $\Gamma_1 G = G$ and

$$\Gamma_{k+1} G = [\Gamma_k G, G]. \tag{3.2}$$

If $\Gamma_k G \neq 1$ but $\Gamma_{k+1} G = 1$, then G is said to be a *k-step nilpotent group*. In general, though, the LCS filtration does not terminate.

The Lie algebra $\operatorname{gr}(G;\mathbb{Q}) = \operatorname{gr}^{\Gamma}(G;\mathbb{Q})$ is called the *associated graded Lie algebra* (over $\mathbb{Q}$) of the group G. For instance, if $F = F_n$ is a free group of rank n, then $\operatorname{gr}(F;\mathbb{Q})$ is the free graded Lie algebra $\mathfrak{lie}(\mathbb{Q}^n)$. A group homomorphism $f\colon G_1 \to G_2$ induces a morphism of graded Lie algebras, $\operatorname{gr}(f;\mathbb{Q})\colon \operatorname{gr}(G_1;\mathbb{Q}) \to \operatorname{gr}(G_2;\mathbb{Q})$; moreover, if f is surjective, then $\operatorname{gr}(f;\mathbb{Q})$ is also surjective.

For each $k \geq 2$, the factor group $G/\Gamma_k(G)$ is the maximal $(k-1)$-step nilpotent quotient of G. The canonical projection $G \to G/\Gamma_k(G)$ induces an epimorphism $\operatorname{gr}(G;\mathbb{Q}) \to \operatorname{gr}(G/\Gamma_k(G);\mathbb{Q})$, which is an isomorphism in degrees $s < k$.

From now on, unless otherwise specified, we will assume that the group G is finitely generated. That is, there is a free group F of finite rank, and an epimorphism $\varphi\colon F \twoheadrightarrow G$. Let $R = \ker(\varphi)$; then $G = F/R$ is called a presentation for G. Note that the induced morphism $\operatorname{gr}(\varphi;\mathbb{Q})\colon \operatorname{gr}(F;\mathbb{Q}) \to \operatorname{gr}(G;\mathbb{Q})$ is surjective. Thus, $\operatorname{gr}(G;\mathbb{Q})$ is a finitely generated Lie algebra, with generators in degree 1.

Let $H \lhd G$ be a normal subgroup, and let $Q = G/H$. If $\widetilde{\Gamma}_r H = \Gamma_r G \cap H$ is the induced filtration on H, it is readily seen that the filtration $\overline{\Gamma}_r Q = \Gamma_r G/\widetilde{\Gamma}_r H$ coincides with the LCS filtration on Q. Hence, by Proposition 3.1.1,

$$\operatorname{gr}(Q) \cong \operatorname{gr}(G)/\operatorname{gr}^{\widetilde{\Gamma}}(H). \tag{3.3}$$

Now suppose $G = H \rtimes Q$ is a semi-direct product of groups. In general, there is not much of a relation between the respective associated graded Lie algebras. Nevertheless, we have the

following well-known result of Falk and Randell [48], which shows that $\mathrm{gr}(G) = \mathrm{gr}(H) \rtimes \mathrm{gr}(Q)$ for 'almost-direct' products of groups.

Theorem 3.1.2 (Theorem 3.1 in [48]). *Let $G = H \rtimes Q$ be a semi-direct product of groups, and suppose Q acts trivially on H_{ab}. Then the filtrations $\{\widetilde{\Gamma}_r H\}_{r\geq 1}$ and $\{\Gamma_r H\}_{r\geq 1}$ coincide, and there is a split exact sequence of graded Lie algebras,*

$$0 \longrightarrow \mathrm{gr}(H) \longrightarrow \mathrm{gr}(G) \longrightarrow \mathrm{gr}(Q) \longrightarrow 0\,.$$

3.1.3 LCS ranks

The next two lemmas give explicit ways to compute the LCS ranks of a group G, under a common rationality hypothesis for the Hilbert series of the graded Lie algebra U.

Lemma 3.1.3. *Suppose there is a polynomial $f(t) = 1 + \sum_{i=1}^{n} b_i t^i \in \mathbb{Z}[t]$ such that*

$$\mathrm{Hilb}(U(\mathrm{gr}(G)), -t) \cdot f(t) = 1. \tag{3.4}$$

Then the LCS ranks of G are given by

$$\phi_k(G) = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) \left[\sum_{m_1+2m_2+\cdots+nm_n=d} (-1)^{s_n} d(m!) \prod_{j=1}^{n} \frac{(b_j)^{m_j}}{(m_j)!} \right], \tag{3.5}$$

where $0 \leq m_j \in \mathbb{Z}$, $s_n = \sum_{i=1}^{[n/2]} m_{2i}$, $m = \sum_{i=1}^{n} m_i - 1$ and μ is the Möbius function.

Proof. From formula (2.12) and assumption (3.4), we have that

$$\prod_{k=1}^{\infty} (1-t^k)^{\phi_k(G)} = 1 + \sum_{i=1}^{n} b_i (-t)^i. \tag{3.6}$$

Taking logarithms on both sides, we find that

$$\sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \phi_s(G) \frac{t^{sj}}{j} = \sum_{w=1}^{\infty} \frac{1}{w} \left(-\sum_{i=1}^{n} b_i (-t)^i \right)^w. \tag{3.7}$$

Comparing the coefficients of t^k on each side gives

$$\sum_{d|k} \phi_d(G) \frac{d}{k} = \sum_{m_1+2m_2+\cdots+nm_n=k} (-1)^{s_n} (m!) \prod_{j=1}^{n} \frac{(b_j)^{m_j}}{(m_j)!}, \tag{3.8}$$

where $s_n = \sum_{i=1}^{[n/2]} m_{2i}$ and $m = \sum_{i=1}^{n} m_i - 1$. Finally, multiplying both sides by k and using the Möbius inversion formula yields the desired formula. □

The advantage of Lemma 3.1.3 is that it is easy to use it to compute low-index LCS ranks. For instance, we obtain the following formulas for a group G satisfying (3.4):

$$\begin{aligned}
\phi_2(G) &= \frac{1}{2}(-b_1 + b_1^2) - b_2,\\
\phi_3(G) &= \frac{1}{3}(-b_1 + b_1^3) - b_1 b_2 + b_3,\\
\phi_4(G) &= \frac{1}{4}(-b_1^2 + 2b_2 + b_1^4 + 2b_2^2) - b_1^2 b_2 + b_1 b_3 - b_4,\\
\phi_5(G) &= \frac{1}{5}(-b_1 + b_1^5) + b_1 b_2^2 - b_1 b_4 + b_1^2 b_3 - b_1^3 b_2 - b_2 b_3 + b_5.
\end{aligned}$$

An alternative way of computing the LCS ranks of a group G satisfying the assumptions from Lemma 3.1.3 was given by Weigel in [150].

Lemma 3.1.4 ([150])**.** *Suppose there is a polynomial $f(t) = 1 + \sum_{i=1}^{n} b_i t^i \in \mathbb{Z}[t]$ such that $\operatorname{Hilb}(U(\operatorname{gr}(G), -t)) \cdot f(t) = 1$. Let $z_1, \dots, z_n$ be the (complex) roots of $f(-t)$. Then the LCS ranks of G are given by*

$$\phi_k(G) = \frac{1}{k} \sum_{1 \le i \le n} \sum_{d \mid k} \mu\left(\frac{k}{d}\right) \frac{1}{z_i^d}. \tag{3.9}$$

Proposition 3.1.5. *Suppose the group G is graded-formal, and its cohomology algebra, $A = H^*(G; \mathbb{C})$, is Koszul. Then $\operatorname{Hilb}(U(\operatorname{gr}(G)), -t) \cdot \operatorname{Hilb}(A, t) = 1$.*

Proof. Let $U = U(\operatorname{gr}(G))$. and let $U^!$ be its quadratic dual. By assumption, $\operatorname{gr}(G) = \mathfrak{h}(A)$ is a quadratic Lie algebra. Thus, U is a quadratic algebra. Furthermore, since A is also quadratic, $U = U(\mathfrak{h}(A))$ is isomorphic to $A^!$, the quadratic dual of A, see [126].

On the other hand, since A is Koszul, the Koszul duality formula gives $\operatorname{Hilb}(A^!, -t) \cdot \operatorname{Hilb}(A, t) = 1$. The conclusion follows. □

Corollary 3.1.6. *Suppose the group G is graded-formal, and its cohomology algebra is Koszul and finite-dimensional. Then the LCS ranks $\phi_k(G)$ are given by formula (3.5), where $b_i = b_i(G)$.*

3.1.4 The holonomy Lie algebra

The holonomy Lie algebra of a finitely generated group was introduced by Kohno [81] following the work of K.-T. Chen [28], and further studied in [105, 117].

Definition 3.1.7. Let G be a finitely generated group. The *holonomy Lie algebra* of G is the holonomy Lie algebra of the cohomology ring $A = H^*(G;\mathbb{Q})$, that is,

$$\mathfrak{h}(G;\mathbb{Q}) = \mathfrak{lie}(H_1(G;\mathbb{Q}))/\langle \operatorname{im} \partial_G \rangle, \tag{3.10}$$

where ∂_G is the dual to the cup-product map $\cup_G \colon H^1(G;\mathbb{Q}) \wedge H^1(G;\mathbb{Q}) \to H^2(G;\mathbb{Q})$.

By construction, $\mathfrak{h}(G;\mathbb{Q})$ is a quadratic Lie algebra. If $f \colon G_1 \to G_2$ is a group homomorphism, then the induced homomorphism in cohomology, $f^* \colon H^1(G_2;\mathbb{Q}) \to H^1(G_1;\mathbb{Q})$ yields a morphism of graded Lie algebras, $\mathfrak{h}(f;\mathbb{Q}) \colon \mathfrak{h}(G_1;\mathbb{Q}) \to \mathfrak{h}(G_2;\mathbb{Q})$. Moreover, if f is surjective, then $\mathfrak{h}(f;\mathbb{Q})$ is also surjective. Finally, $\mathfrak{h}(G;\mathbb{K}) = \mathfrak{h}(G;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{K}$, for any field extension $\mathbb{Q} \subset \mathbb{K}$

In the definition of the holonomy Lie algebra of G, we used the cohomology ring of a classifying space $K(G,1)$. As the next lemma shows, we may replace this space by any other connected CW-complex with the same fundamental group.

Lemma 3.1.8. *Let X be a connected CW-complex with $\pi_1(X) = G$. Then $\mathfrak{h}(H^*(X;\mathbb{Q})) \cong \mathfrak{h}(G;\mathbb{Q})$.*

Proof. We may construct a classifying space for G by adding cells of dimension 3 and higher to X in a suitable way. The inclusion map, $j \colon X \to K(G,1)$, induces a map on cohomology rings, $j^* \colon H^*(K(G,1);\mathbb{Q}) \to H^*(X;\mathbb{Q})$, which is an isomorphism in degree 1 and an injection in degree 2. Consequently, j^2 restricts to an isomorphism from $\operatorname{im}(\cup_G)$ to $\operatorname{im}(\cup_X)$. Taking duals, we find that $\operatorname{im}(\partial_X) = \operatorname{im}(\partial_G)$. The conclusion follows. □

In particular, if K_G is the 2-complex associated to a presentation of G, then $\mathfrak{h}(G;\mathbb{Q}) \cong \mathfrak{h}(H^*(K_G;\mathbb{Q}))$. Let $\bar{\phi}_n(G) := \dim \mathfrak{h}_n(G;\mathbb{Q})$ be the dimensions of the graded pieces of the

holonomy Lie algebra of G. The next corollary is an algebraic version of the LCS formula from Papadima and Yuzvinsky [126], but with no formality assumption.

Corollary 3.1.9. *Let X be a connected CW-complex with $\pi_1(X) = G$, let $A = H^*(X;\mathbb{Q})$ be its cohomology algebra, and let $\bar{A}$ be the quadratic closure of A. Then $\prod_{n\geq 1}(1-t^n)^{\bar{\phi}_n(G)} = \sum_{i\geq 0} b_{ii}t^i$, where $b_{ii} = \dim \operatorname{Ext}^i_A(\mathbb{Q},\mathbb{Q})_i$. Moreover, if $\bar{A}$ is a Koszul algebra, then*

$$\prod_{n\geq 1}(1-t^n)^{\bar{\phi}_n} = \operatorname{Hilb}(\bar{A}, -t).$$

Proof. The first claim follows from Lemma 3.1.8, the Poincaré–Birkhoff–Witt formula (2.12), and Löfwall's formula from Proposition 2.2.6. The second claim follows from the Koszul duality formula stated in Corollary 2.2.8. □

3.1.5 A comparison map

Once again, let G be a finitely generated group. Although the next lemma is known, we provide a proof, both for the sake of completeness, and for later use.

Lemma 3.1.10 ([105, 117])**.** *There exists a natural epimorphism of graded $\mathbb{Q}$-Lie algebras,*

$$\Phi_G \colon \mathfrak{h}(G;\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G;\mathbb{Q}), \tag{3.11}$$

inducing isomorphisms in degrees 1 and 2. Furthermore, this epimorphism is natural with respect to field extensions $\mathbb{Q} \subset \mathbb{K}$.

Proof. As first noted by Sullivan [147] in a particular case, and proved by Lambe [87] in general, there is a natural exact sequence

$$0 \longrightarrow (\Gamma_2 G/\Gamma_3 G \otimes \mathbb{Q})^* \xrightarrow{\beta} H^1(G;\mathbb{Q}) \wedge H^1(G;\mathbb{Q}) \xrightarrow{\cup} H^2(G;\mathbb{Q}), \tag{3.12}$$

where β is the dual of Lie bracket product. In particular, $\operatorname{im}(\partial_G) = \ker(\beta^*)$.

Recall that the associated graded Lie algebra $\operatorname{gr}(G;\mathbb{Q})$ is generated by its degree 1 piece, $H_1(G;\mathbb{Q}) \cong \operatorname{gr}_1(G) \otimes \mathbb{Q}$. Hence, there is a natural epimorphism of graded $\mathbb{Q}$-Lie algebras,

$$\varphi_G \colon \mathfrak{lie}(H_1(G;\mathbb{Q})) \twoheadrightarrow \operatorname{gr}(G;\mathbb{Q}), \tag{3.13}$$

restricting to the identity in degree 1, and to the Lie bracket map $[\,,\,]\colon \bigwedge^2 \mathrm{gr}_1(G;\mathbb{Q}) \to \mathrm{gr}_2(G;\mathbb{Q})$ in degree 2. In the exact sequence (3.12), the image of ∂_G coincides with the kernel of the Lie bracket map. Thus, the morphism φ_G factors through the desired morphism Φ_G. The fact that Φ_G commutes with the morphisms $\mathfrak{h}(G;\mathbb{Q}) \to \mathfrak{h}(G;\mathbb{K})$ and $\mathrm{gr}(G;\mathbb{Q}) \to \mathrm{gr}(G;\mathbb{K})$ readily follows. □

Corollary 3.1.11. *Let $V = H_1(G;\mathbb{Q})$. Suppose the associated graded Lie algebra $\mathfrak{g} = \mathrm{gr}(G;\mathbb{Q})$ has presentation $\mathfrak{lie}(V)/\mathfrak{r}$. Then the holonomy Lie algebra $\mathfrak{h}(G;\mathbb{Q})$ has presentation $\mathfrak{lie}(V)/\langle\mathfrak{r}_2\rangle$, where $\mathfrak{r}_2 = \mathfrak{r} \cap \mathfrak{lie}_2(V)$. Furthermore, if $A = U(\mathfrak{g})$, then $\mathfrak{h}(G;\mathbb{Q}) = \mathfrak{h}\big(\bar{A}^!\big)$.*

Proof. Taking the dual of the exact sequence (3.13), we find that $\mathrm{im}(\partial_G) = \ker(\beta^*)$, where $\beta\colon V \wedge V \to \mathfrak{lie}_2(V)$ is the Lie bracket in $\mathfrak{lie}(V)$. Hence, $\langle\mathfrak{r}_2\rangle = \langle\mathrm{im}(\partial_G)\rangle$ as ideals of $\mathfrak{lie}(V)$; thus, $\mathfrak{h}(G;\mathbb{Q}) = \mathfrak{lie}(V)/\langle\mathfrak{r}_2\rangle$. The last claim follows from Corollary 2.2.5. □

Recall we denote by $\phi_n(G)$ and $\bar{\phi}_n(G)$ the dimensions on the n-th graded pieces of $\mathrm{gr}(G;\mathbb{Q})$ and $\mathfrak{h}(G;\mathbb{Q})$, respectively. By Lemma 3.1.10, $\bar{\phi}_n(G) \geq \phi_n(G)$, for all $n \geq 1$, and equality always holds for $n \leq 2$. Nevertheless, these inequalities can be strict for $n \geq 3$.

As a quick application, let us compare the holonomy Lie algebras of G and its nilpotent quotients.

Proposition 3.1.12. *Let G be a finitely generated group. Then,*

$$\mathfrak{h}(G/\Gamma_k G;\mathbb{Q}) = \begin{cases} \mathfrak{h}(G;\mathbb{Q})/\mathfrak{h}(G;\mathbb{Q})' & \text{for } k = 2, \\ \mathfrak{h}(G;\mathbb{Q}) & \text{for } k \geq 3. \end{cases} \tag{3.14}$$

In particular, the holonomy Lie algebra of G depends only on the second nilpotent quotient, $G/\Gamma_3 G$.

Proof. The case $k = 2$ is trivial, so let us assume $k \geq 3$. By a previous remark, the projection $G \to G/\Gamma_k(G)$ induces an isomorphism $\mathrm{gr}_2(G;\mathbb{Q}) \to \mathrm{gr}_2(G/\Gamma_k(G);\mathbb{Q})$. Furthermore, $H_1(G;\mathbb{Q}) \cong H_1(G/\Gamma_k(G);\mathbb{Q})$. Using now the dual of the exact sequence (3.12), we see that $\mathrm{im}(\partial_G) = \mathrm{im}(\partial_{G/\Gamma_k(G)})$. The desired conclusion follows. □

3.1.6 Graded-formality

We continue our discussion of the associated graded and holonomy Lie algebras of a finitely generated group with a formality notion that will be important in the sequel.

Definition 3.1.13. A finitely generated group G is *graded-formal* (over $\mathbb{Q}$) if the canonical projection $\Phi_G\colon \mathfrak{h}(G;\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G;\mathbb{Q})$ is an isomorphism of graded Lie algebras.

This notion was introduced by Lee in [91], where it is called graded 1-formality. Next, we give two alternate definitions, which oftentimes are easier to verify.

Lemma 3.1.14. *A finitely generated group G is graded-formal over $\mathbb{Q}$ if and only if* $\operatorname{gr}(G;\mathbb{Q})$ *is quadratic.*

Proof. The forward implication is immediate. So assume $\operatorname{gr}(G;\mathbb{Q})$ is quadratic, that is, it admits a presentation of the form $\mathfrak{lie}(V)/\langle U\rangle$, where V is a $\mathbb{Q}$-vector space in degree 1 and U is a $\mathbb{Q}$-vector subspace of $\mathfrak{lie}_2(V)$. In particular, $V = \operatorname{gr}_1(G;\mathbb{Q}) = H_1(G;\mathbb{Q})$.

From the exact sequence (3.12), we see that the image of ∂_G coincides with the kernel of the Lie bracket map $[\,,\,]\colon \bigwedge^2 \operatorname{gr}_1(G;\mathbb{Q}) \to \operatorname{gr}_2(G;\mathbb{Q})$, which can be identified with U. Hence the surjection $\varphi_G\colon \mathfrak{lie}(G_{\mathbb{Q}}) \twoheadrightarrow \operatorname{gr}(G;\mathbb{Q})$ induces an isomorphism $\Phi_G\colon \mathfrak{h}(G;\mathbb{Q}) \xrightarrow{\cong} \operatorname{gr}(G;\mathbb{Q})$. □

Lemma 3.1.15. *A finitely generated group G is graded-formal over $\mathbb{Q}$ if and only if*

$$\dim_{\mathbb{Q}} \mathfrak{h}_n(G;\mathbb{Q}) = \dim_{\mathbb{Q}} \operatorname{gr}_n(G;\mathbb{Q}), \text{ for all } n \geq 1.$$

Proof. The homomorphisms $(\Phi_G)_n\colon \mathfrak{h}_n(G;\mathbb{Q}) \to \operatorname{gr}_n(G;\mathbb{Q})$ are always isomorphisms for $n \leq 2$ and epimorphisms $n \geq 3$. Our assumption, together with the fact that each $\mathbb{Q}$-vector space $\mathfrak{h}_n(G;\mathbb{Q})$ is finite-dimensional implies that all homomorphisms $(\Phi_G)_n$ are isomorphisms. Therefore, the map $\Phi_G\colon \mathfrak{h}(G;\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G;\mathbb{Q})$ is an isomorphism of graded Lie algebras. □

The lemma implies that the definition of graded formality is independent of the choice of coefficient field $\mathbb{K}$ of characteristic 0. More precisely, we have the following corollary.

Corollary 3.1.16. *A finitely generated group G is graded-formal over $\mathbb{K}$ if and only if is graded-formal over $\mathbb{Q}$.*

Proof. The dimension of a finite-dimensional vector space does not change upon the extensions of scalars $\mathbb{Q} \subset \mathbb{K}$. The conclusion follows at once from Lemma 3.1.15. □

3.1.7 Split injections

We are now in a position to state and prove the main result of this section, which proves the first part of Theorem 1.2.2 from the Introduction.

Theorem 3.1.17. *Let G be a finitely generated group. Suppose there is a split monomorphism $\iota\colon K \to G$. If G is a graded-formal group, then K is also graded-formal.*

Proof. In view of our hypothesis, we have an epimorphism $\sigma\colon G \twoheadrightarrow K$ such that $\sigma \circ \iota = \mathrm{id}$. In particular, K is also finitely generated. Furthermore, the induced maps $\mathfrak{h}(\iota)$ and $\mathrm{gr}(\iota)$ are also injective.

Let $\pi\colon F \twoheadrightarrow G$ be a presentation for G. There is then an induced presentation for K, given by the composition $\sigma\pi\colon F \twoheadrightarrow K$. By Lemma 3.1.10, there exist epimorphisms Φ_K and Φ making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{h}(K;\mathbb{Q}) & \xrightarrow{\Phi_K} & \mathrm{gr}(K;\mathbb{Q}) \\ \big\downarrow{\scriptstyle \mathfrak{h}(\iota)} & & \big\downarrow{\scriptstyle \mathrm{gr}(\iota)} \\ \mathfrak{h}(G;\mathbb{Q}) & \xrightarrow{\Phi} & \mathrm{gr}(G;\mathbb{Q})\,. \end{array} \tag{3.15}$$

If the group G is graded-formal, then Φ is an isomorphism of graded Lie algebras. Hence, the epimorphism Φ_K is also injective, and so K is a graded-formal. □

Theorem 3.1.18. *Let $G = K \rtimes Q$ be a semi-direct product of finitely generated groups, and suppose G is graded-formal. Then:*

1. *The group Q is graded-formal.*

2. If, moreover, Q acts trivially on K_{ab}, then K is also graded-formal.

Proof. The first assertion follows at once from Theorem 3.1.17. So assume Q acts trivially on K_{ab}. By Theorem 3.1.2, there exists a split exact sequence of graded Lie algebras, which we record in the top row of the next diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{gr}(K;\mathbb{Q}) & \rightleftarrows & \mathrm{gr}(G;\mathbb{Q}) & \rightleftarrows & \mathrm{gr}(Q;\mathbb{Q}) & \longrightarrow & 0 \\ & & \uparrow & & \cong\uparrow & & \cong\uparrow & & \\ & & \mathfrak{h}(K;\mathbb{Q}) & \rightleftarrows & \mathfrak{h}(G;\mathbb{Q}) & \rightleftarrows & \mathfrak{h}(Q;\mathbb{Q}) & \longrightarrow & 0\,. \end{array} \tag{3.16}$$

Let $\iota\colon K \to G$ be the inclusion map. By the above, we have an epimorphism σ from $\mathrm{gr}(G;\mathbb{Q})$ to $\mathrm{gr}(K;\mathbb{Q})$ such that $\sigma \circ \mathrm{gr}(\iota) = \mathrm{id}$. Consequently, $\mathrm{gr}(K;\mathbb{Q})$ is finitely generated.

By Corollary 3.1.11, the map σ induces a morphism $\bar{\sigma}\colon \mathfrak{h}(G;\mathbb{Q}) \to \mathfrak{h}(G;\mathbb{Q})$ such that $\bar{\sigma} \circ \mathfrak{h}(\iota) = \mathrm{id}$. Consequently, $\mathfrak{h}(\iota)$ is injective. Therefore, the morphism $\mathfrak{h}(K;\mathbb{Q}) \to \mathrm{gr}(K;\mathbb{Q})$ is also injective. Hence, K is graded-formal. □

If the hypothesis of Theorem 3.1.18, part (2) does not hold, the subgroup K may not be graded-formal, even when the group G is 1-formal. We illustrate this phenomenon with an example adapted from [120].

Example 3.1.19. Let $K = \langle x, y \mid [x,[x,y]],[y,[x,y]]\rangle$ be the discrete Heisenberg group. Consider the semidirect product $G = K \rtimes_{\phi} \mathbb{Z}$, defined by the automorphism $\phi\colon K \to K$ given by $x \to y$, $y \to xy$. We have that $b_1(G) = 1$, and so G is 1-formal, yet K is not graded-formal.

3.1.8 Products and coproducts

We conclude this section with a discussion of the functors gr and $\mathfrak{h}$ and how the notion of graded formality behaves with respect to products and coproducts.

Lemma 3.1.20 ([95, 125]). *The functors* gr *and* $\mathfrak{h}$ *preserve products and coproducts, that is, we have the following natural isomorphisms of graded Lie algebras,*

$$\begin{cases} \mathrm{gr}(G_1 \times G_2; \mathbb{Q}) \cong \mathrm{gr}(G_1; \mathbb{Q}) \times \mathrm{gr}(G_2; \mathbb{Q}) \\ \mathrm{gr}(G_1 * G_2; \mathbb{Q}) \cong \mathrm{gr}(G_1; \mathbb{Q}) * \mathrm{gr}(G_2; \mathbb{Q}), \end{cases} \quad \textit{and} \quad \begin{cases} \mathfrak{h}(G_1 \times G_2; \mathbb{Q}) \cong \mathfrak{h}(G_1; \mathbb{Q}) \times \mathfrak{h}(G_2; \mathbb{Q}) \\ \mathfrak{h}(G_1 * G_2; \mathbb{Q}) \cong \mathfrak{h}(G_1; \mathbb{Q}) * \mathfrak{h}(G_2; \mathbb{Q}). \end{cases}$$

Proof. The first statement on the $\mathrm{gr}(-)$ functor is well-known, while the second statement is the main theorem from [95]. The statements regarding the $\mathfrak{h}(-)$ functor can be found in [125]. □

Regarding graded-formality, we have the following result, which sharpens and generalizes Lemma 4.5 from Plantiko [127], and proves the first part of Theorem 1.2.3 from the Introduction.

Proposition 3.1.21. *Let G_1 and G_2 be two finitely generated groups. Then, the following conditions are equivalent.*

1. *G_1 and G_2 are graded-formal.*
2. *$G_1 * G_2$ is graded-formal.*
3. *$G_1 \times G_2$ is graded-formal.*

Proof. Since there exist split injections from G_1 and G_2 to the product $G_1 \times G_2$ and the coproduct $G_1 * G_2$, Theorem 3.1.17 shows that implications (2)⇒(1) and (3)⇒(1) hold. Implications (1)⇒(2) and (1)⇒(3) follow from Lemma 3.1.20 and the naturality of the map Φ from (3.11). □

3.2 Malcev Lie algebras and filtered formality

In this section we consider the Malcev Lie algebra of a finitely generated group, and study the ensuing notions of filtered formality and 1-formality.

3.2.1 Prounipotent completions and Malcev Lie algebras

Once again, let G be a finitely generated group, and let $\{\Gamma_k G\}_{k\geq 1}$ be its LCS filtration. The successive quotients of G by these normal subgroups form a tower of finitely generated, nilpotent groups,

$$\cdots \longrightarrow G/\Gamma_4 G \longrightarrow G/\Gamma_3 G \longrightarrow G/\Gamma_2 G = G_{\mathrm{ab}}\,. \tag{3.17}$$

Let $\mathbb{Q}$ be a field of characteristic 0. It is possible to replace each nilpotent quotient $N_k = G/\Gamma_k G$ by $N_k \otimes \mathbb{Q}$, the (rationally defined) nilpotent Lie group associated to the discrete, torsion-free nilpotent group $N_k/\mathrm{tors}(N_k)$ via a procedure which will be discussed in more detail in §9.1.1. The corresponding inverse limit,

$$\mathfrak{M}(G;\mathbb{Q}) = \varprojlim_k ((G/\Gamma_k G)\otimes\mathbb{Q}), \tag{3.18}$$

is a prounipotent $\mathbb{Q}$-Lie group over $\mathbb{Q}$, which is called the *prounipotent completion*, or *Malcev completion* of G over $\mathbb{Q}$. Let $\mathfrak{Lie}((G/\Gamma_k G)\otimes\mathbb{Q})$ be the Lie algebra of the nilpotent Lie group $(G/\Gamma_k G)\otimes\mathbb{Q}$. The pronilpotent Lie algebra

$$\mathfrak{m}(G;\mathbb{Q}) := \varprojlim_k \mathfrak{Lie}((G/\Gamma_k G)\otimes\mathbb{Q}), \tag{3.19}$$

with the inverse limit filtration, is called the *Malcev Lie algebra* of G (over $\mathbb{Q}$). By construction, $\mathfrak{m}(-;\mathbb{Q})$ is a functor from the category of finitely generated groups to the category of complete, separated, filtered $\mathbb{Q}$-Lie algebras.

In [131], Quillen gave a different construction of this Lie algebra, as follows. The group-algebra $\mathbb{Q}G$ has a natural Hopf algebra structure, with comultiplication $\Delta\colon \mathbb{Q}G \to \mathbb{Q}G\otimes\mathbb{Q}G$ given by $\Delta(g) = g\otimes g$ for $g\in G$, and counit the augmentation map $\varepsilon\colon \mathbb{Q}G\to\mathbb{Q}$. The powers of the augmentation ideal $I = \ker\varepsilon$ form a descending filtration of $\mathbb{Q}G$ by two-sided ideals; let $\widehat{\mathbb{Q}G} = \varprojlim_k \mathbb{Q}G/I^k$ be the completion of the group-algebra with respect to this filtration. The comultiplication map Δ extends to a map $\widehat{\Delta}\colon \widehat{\mathbb{Q}G} \to \widehat{\mathbb{Q}G}\,\hat{\otimes}\,\widehat{\mathbb{Q}G}$, making $\widehat{\mathbb{Q}G}$ into a complete Hopf algebra. An element $x\in\widehat{\mathbb{Q}G}$ is called 'primitive' if $\widehat{\Delta}x = x\hat{\otimes}1 + 1\hat{\otimes}x$. The

set of all primitive elements in $\widehat{\mathbb{Q}G}$, with bracket $[x,y] = xy - yx$, and endowed with the induced filtration, is a Lie algebra, isomorphic to the Malcev Lie algebra of G,

$$\mathfrak{m}(G;\mathbb{Q}) \cong \operatorname{Prim}\big(\widehat{\mathbb{Q}G}\big). \tag{3.20}$$

The filtration topology on $\widehat{\mathbb{Q}G}$ is a metric topology; hence, the filtration topology on $\mathfrak{m}(G;\mathbb{Q})$ is also metrizable, and thus separated. We shall denote by $\operatorname{gr}(\mathfrak{m}(G;\mathbb{Q}))$ the associated graded Lie algebra of $\mathfrak{m}(G;\mathbb{Q})$ with respect to the induced inverse limit filtration.

A non-zero element $x \in \widehat{\mathbb{Q}G}$ is called 'group-like' if $\widehat{\Delta}x = x\hat{\otimes}x$. The set of all such elements, with multiplication inherited from $\widehat{\mathbb{Q}G}$, forms a group, which is isomorphic to $\mathfrak{M}(G;\mathbb{Q})$. The group G naturally embeds as a subgroup of $\mathfrak{M}(G;\mathbb{Q})$. Composing this inclusion with the logarithmic map $\log\colon \mathfrak{M}(G;\mathbb{Q}) \to \mathfrak{m}(G;\mathbb{Q})$, we obtain a map $\rho\colon G \to \mathfrak{m}(G;\mathbb{Q})$; see Massuyeau [109] for details. As shown by Quillen in [132], the map ρ induces an isomorphism of graded Lie algebras,

$$\operatorname{gr}(\rho)\colon\ \operatorname{gr}(G;\mathbb{Q}) \xrightarrow{\cong} \operatorname{gr}(\mathfrak{m}(G;\mathbb{Q}))\ . \tag{3.21}$$

In particular, $\operatorname{gr}(\mathfrak{m}(G;\mathbb{Q}))$ is generated in degree 1. If G admits a finite presentation, one can use this approach to find a presentation for the Malcev Lie algebra $\mathfrak{m}(G;\mathbb{Q})$, see Massuyeau [109] and Papadima [116].

3.2.2 Minimal models and Malcev Lie algebras

Every group G has a classifying space $K(G,1)$, which can be chosen to be a connected CW-complex. Such a CW-complex is unique up to homotopy, and thus, up to rational homotopy equivalence. Hence, by the discussion from §2.3.8 the weak equivalence type of the Sullivan algebra $A = A_{PL}(K(G,1))$ depends only on the isomorphism type of G. By Theorem 2.3.5, the CDGA $A \otimes_{\mathbb{Q}} \mathbb{Q}$ has a 1-minimal model, $\mathcal{M}(A \otimes_{\mathbb{Q}} \mathbb{Q}, 1)$, unique up to isomorphism. Moreover, the assignment $G \rightsquigarrow \mathcal{M}(A \otimes_{\mathbb{Q}} \mathbb{Q}, 1)$ is functorial.

Assume now that the group G is finitely generated. Let $\mathcal{M} = \mathcal{M}(G;\mathbb{Q})$ be a 1-minimal model of G, with the canonical filtration constructed in (2.22). The starting point is the finite-dimensional vector space $\mathcal{M}_1^1 = V_1 := H^1(G;\mathbb{Q})$. Each sub-CDGA $\mathcal{M}_i$ is a Hirsch extension of $\mathcal{M}_{i-1}$ by the finite-dimensional vector space $V_i := \ker(H^2(\mathcal{M}_{i-1}) \to H^2(A))$.

Define $\mathfrak{L}(G;\mathbb{Q}) = \varprojlim_i \mathfrak{L}_i(G;\mathbb{Q})$ as the pronilpotent Lie algebra associated to the 1-minimal model $\mathcal{M}(G;\mathbb{Q})$ in the manner described in §2.3.2, and note that the assignment $G \rightsquigarrow \mathfrak{L}(G;\mathbb{Q})$ is also functorial.

Theorem 3.2.1 ([25, 63, 148])**.** *There exist natural isomorphisms of towers of nilpotent Lie algebras,*

$$\begin{array}{ccccccc}
\cdots & \longleftarrow & \mathfrak{L}_{i-1}(G;\mathbb{Q}) & \longleftarrow & \mathfrak{L}_i(G;\mathbb{Q}) & \longleftarrow & \cdots \\
 & & \downarrow\cong & & \downarrow\cong & & \\
\cdots & \longleftarrow & \mathfrak{m}(G/\Gamma_i G;\mathbb{Q}) & \longleftarrow & \mathfrak{m}(G/\Gamma_{i+1} G;\mathbb{Q}) & \longleftarrow & \cdots .
\end{array}$$

Hence, there is a functorial isomorphism $\mathfrak{L}(G;\mathbb{Q}) \cong \mathfrak{m}(G;\mathbb{Q})$ of complete, filtered Lie algebras.

This functorial isomorphism $\mathfrak{m}(G;\mathbb{Q}) \cong \mathfrak{L}(G;\mathbb{Q})$, together with the dualization correspondence $\mathfrak{L}(G;\mathbb{Q}) \leftrightsquigarrow \mathcal{M}(G;\mathbb{Q})$ define adjoint functors between the category of Malcev Lie algebras of finitely generated groups and the category of 1-minimal models of finitely generated groups.

3.2.3 Filtered formality of groups

We now define the notion of filtered formality for groups (also known as weak formality by Lee [91]), based on the notion of filtered formality for Lie algebras from Definition 2.1.4.

Definition 3.2.2. A finitely generated group G is said to be *filtered-formal* (over $\mathbb{Q}$) if its Malcev Lie algebra $\mathfrak{m}(G;\mathbb{Q})$ is filtered-formal, with respect to the inverse limit filtration.

Here are some more direct ways to think of this notion.

Proposition 3.2.3. *A finitely generated group G is filtered-formal over $\mathbb{Q}$ if and only if one of the following conditions is satisfied.*

1. $\mathfrak{m}(G;\mathbb{Q}) \cong \widehat{\mathrm{gr}}(G;\mathbb{Q})$ *as filtered Lie algebras.*

2. $\mathfrak{m}(G;\mathbb{Q})$ *admits a homogeneous presentation.*

Proof. (1) We know from Quillen's isomorphism (3.21) that $\mathrm{gr}(\mathfrak{m}(G;\mathbb{Q})) \cong \mathrm{gr}(G;\mathbb{Q})$. The forward implication follows straight from the definitions, while the backward implication follows from Lemma 2.1.5.

(2) Choose a presentation $\mathrm{gr}(G;\mathbb{Q}) = \mathfrak{lie}(H_1(G;\mathbb{Q}))/\mathfrak{r}$, where r is a homogeneous ideal. By Lemma 2.1.3, we have

$$\mathfrak{m}(G;\mathbb{Q}) = \widehat{\mathfrak{lie}}(H_1(G;\mathbb{Q}))/\overline{\mathfrak{r}}, \tag{3.22}$$

which is a homogeneous presentation for $\mathfrak{m}(G;\mathbb{Q})$. Conversely, if (3.22) holds, then $\mathfrak{m}(G;\mathbb{Q}) \cong \widehat{\mathfrak{g}}$, where $\mathfrak{g} = \mathfrak{lie}(H_1(G;\mathbb{Q}))/\mathfrak{r}$. □

The notion of filtered formality can also be interpreted in terms of minimal models. Let $\mathcal{M}(G;\mathbb{Q})$ be the 1-minimal model of G, endowed with the canonical filtration, which is the minimal CDGA dual to the Malcev Lie algebra $\mathfrak{m}(G;\mathbb{Q})$ under the correspondence described in §2.3.2. Likewise, let $\mathcal{N}(G;\mathbb{Q})$ be the minimal CDGA (generated in degree 1) corresponding to the prounipotent Lie algebra $\widehat{\mathrm{gr}}(G;\mathbb{Q})$. Recall that both $\mathcal{M}(G;\mathbb{Q})$ and $\mathcal{N}(G;\mathbb{Q})$ come equipped with increasing filtrations as in (2.22), which correspond to the inverse limit filtrations on $\mathfrak{m}(G;\mathbb{Q})$ and $\widehat{\mathrm{gr}}(G;\mathbb{Q})$, respectively.

Proposition 3.2.4. *A finitely generated group G is filtered-formal over $\mathbb{Q}$ if and only if one of the following conditions is satisfied.*

1. *there is a filtration-preserving* CDGA *isomorphism between $\mathcal{M}(G;\mathbb{Q})$ and $\mathcal{N}(G;\mathbb{Q})$.*

2. *there is a* CDGA *isomorphism between $\mathcal{M}(G;\mathbb{Q})$ and $\mathcal{N}(G;\mathbb{Q})$ inducing the identity on first cohomology.*

Proof. (1) Recall Proposition 3.2.3 that G is filtered-formal if and only if $\mathfrak{m}(G;\mathbb{Q}) \cong \widehat{\mathrm{gr}}(G;\mathbb{Q})$, as filtered Lie algebras. Dualizing, this condition becomes equivalent to $\mathcal{M}(G;\mathbb{Q}) \cong \mathcal{N}(G;\mathbb{Q})$, as filtered CDGA's.

(2) Recall that G is filtered-formal if and only if $\mathfrak{m}(G;\mathbb{Q}) \cong \widehat{\mathrm{gr}}(\mathfrak{m}(G;\mathbb{Q}))$ inducing identity on their associated graded Lie algebras.

Likewise, both $\mathcal{M}_1^1$ and $\mathcal{N}_1^1$ can be canonically identified with $\mathrm{gr}_1(G;\mathbb{Q})^* = H^1(G;\mathbb{Q})$. The desired conclusion follows. □

Here is another description of filtered formality, suggested to us by R. Porter.

Theorem 3.2.5. *A finitely generated group G is filtered-formal over $\mathbb{Q}$ if and only if the canonical 1-minimal model $\mathcal{M}(G;\mathbb{Q})$ is filtered-isomorphic to a 1-minimal model $\mathcal{M}$ with positive Hirsch weights.*

Proof. First suppose G is filtered-formal, and let $\mathcal{N} = \mathcal{N}(G;\mathbb{Q})$ be the minimal CDGA dual to $\mathfrak{L} = \widehat{\mathrm{gr}}(G,\mathbb{Q})$. By Proposition 3.2.4, this CDGA is a 1-minimal model for G. Since by construction $\mathfrak{L} = \widehat{\mathrm{gr}}(\mathfrak{L})$, Lemma 2.3.6 shows that the differential on $\mathcal{N}$ is homogeneous with respect to the Hirsch weights.

Now suppose $\mathcal{M}$ is a 1-minimal model for G over $\mathbb{Q}$, with homogeneous differential on Hirsch weights. By Lemma 2.3.6 again, the dual Lie algebra $\mathfrak{L}(\mathcal{M})$ is filtered-formal. On the other hand, the assumption that $\mathcal{M} \cong \mathcal{M}(G;\mathbb{Q})$ and Theorem 3.2.1 together imply that $\mathfrak{L}(\mathcal{M}) \cong \mathfrak{m}(G;\mathbb{Q})$. Hence, the group G is filtered-formal by Definition 3.2.2. □

We would like to thank Y. Cornulier for asking whether the next result holds, and for pointing out the connection it would have with [37, Theorem 3.14].

Proposition 3.2.6. *Let G be a finitely generated group, and let $\mathbb{Q} \subset \mathbb{K}$ be a field extension. Then G is filtered-formal over $\mathbb{Q}$ if and only if G is filtered-formal over $\mathbb{K}$.*

3.3 Filtered-formality and 1-formality

In this section, we consider the 1-formality property of finitely generated groups, and the way it relates to Massey products, graded-formality, and filtered-formality. We also study the way various formality properties behave under free and direct products, as well as retracts.

3.3.1 1-formality of groups

We start with a basic definition. As usual, let $\mathbb{Q}$ be a field of characteristic 0.

Definition 3.3.1. A finitely generated group G is called 1-*formal* (over $\mathbb{Q}$) if a classifying space $K(G,1)$ is 1-formal over $\mathbb{Q}$.

Since any two classifying spaces for G are homotopy equivalent, the discussion from §2.3.8 shows that this notion is well-defined. A similar argument shows that the 1-formality property of a path-connected space X depends only on its fundamental group, $G = \pi_1(X)$.

The next, well-known theorem provides an equivalent, purely group-theoretic definition of 1-formality. Although proofs can be found in the literature (see for instance Markl–Papadima [105], Carlson–Toledo [24], and Remark 3.3.3 below), we provide here an alternative proof, based on Theorem 2.3.10 and the discussion from §3.2.2.

Theorem 3.3.2. *A finitely generated group G is 1-formal over $\mathbb{Q}$ if and only if the Malcev Lie algebra of G is isomorphic to the degree completion of the holonomy Lie algebra $\mathfrak{h}(G;\mathbb{Q})$.*

Proof. Let $\mathcal{M}(G;\mathbb{Q}) = \mathcal{M}(A_{PL}(K(G,1)),1) \otimes_{\mathbb{Q}} \mathbb{Q}$ be the 1-minimal model of G. The group G is 1-formal if and only if there exists a CDGA morphism $\mathcal{M}(G;\mathbb{Q}) \to (H^*(G;\mathbb{Q}), d=0)$ inducing an isomorphism in first cohomology and a monomorphism in second cohomology, i.e., $\mathcal{M}(G;\mathbb{Q})$ is a 1-minimal model for $(H^*(G;\mathbb{Q}), d=0)$.

Let $\mathfrak{L}(G;\mathbb{Q})$ be the dual Lie algebra of $\mathcal{M}(G;\mathbb{Q})$. By Theorem 3.2.1, the Malcev Lie algebra of G is isomorphic to $\mathfrak{L}(G;\mathbb{Q})$. By Theorem 2.3.10, the degree completion of the holonomy Lie algebra of G is isomorphic to $\mathfrak{L}(G;\mathbb{Q})$. This completes the proof. □

Remark 3.3.3. Theorem 3.3.2 admits the following generalization: if G is a finitely generated group, and if (A, d) is a connected CDGA with $\dim A^1 < \infty$ whose 1-minimal model is isomorphic to $\mathcal{M}(G; \mathbb{Q})$, then the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{Q})$ is isomorphic to the completion with respect to the degree filtration of the Lie algebra $\mathfrak{h}(A, d) := \mathfrak{lie}((A^1)^*)/\langle \mathrm{im}((d^1)^* + \mu_A^*)\rangle$. A proof of this result is given by Berceanu et al. in [13]; related results can be found in work of Bezrukavnikov [16], Bibby–Hilburn [17], and Polishchuk–Positselski [128].

An equivalent formulation of Theorem 3.3.2 is given by Papadima and Suciu in [120]: A finitely generated group G is 1-formal over $\mathbb{Q}$ if and only if its Malcev Lie algebra $\mathfrak{m}(G; \mathbb{Q})$ is isomorphic to the degree completion of a quadratic Lie algebra, as filtered Lie algebras. For instance, if $b_1(G)$ equals 0 or 1, then G is 1-formal.

Clearly, finitely generated free groups are 1-formal; indeed, if F is such a group, then $\mathfrak{m}(F; \mathbb{Q}) \cong \widehat{\mathfrak{lie}}(H_1(F; \mathbb{Q}))$. Other well-known examples of 1-formal groups include fundamental groups of compact Kähler manifolds, cf. Deligne et al. [39], fundamental groups of complements of complex algebraic hypersurfaces, cf. Kohno [81], and the pure braid groups of surfaces of genus different from 1, cf. Bezrukavnikov [16] and Hain [65].

3.3.2 Massey products

A well-known obstruction to 1-formality is provided by the higher-order Massey products (introduced in [107]). For our purposes, we will discuss here only triple Massey products of degree 1 cohomology classes.

Let γ_1, γ_2 and γ_3 be cocycles of degrees 1 in the (singular) chain complex $C^*(G; \mathbb{Q})$, with cohomology classes $u_i = [\gamma_i]$ satisfying $u_1 \cup u_2 = 0$ and $u_2 \cup u_3 = 0$. That is, we assume there are 1-cochains γ_{12} and γ_{23} such that $d\gamma_{12} = \gamma_1 \cup \gamma_2$ and $d\gamma_{23} = \gamma_2 \cup \gamma_3$. It is readily seen that the 2-cochain $\omega = \gamma_{12} \cup \gamma_3 + \gamma_1 \cup \gamma_{23}$ is, in fact, a cocycle. The set of all cohomology classes $[\omega]$ obtained in this way is the *Massey triple product* $\langle u_1, u_2, u_3\rangle$ of the classes u_1, u_2 and u_3. Due to the ambiguity in the choice of γ_{12} and γ_{23}, the Massey triple product $\langle u_1, u_2, u_3\rangle$ is a

representative of the coset

$$H^2(G;\mathbb{Q})/(u_1 \cup H^1(G;\mathbb{Q}) + H^1(G;\mathbb{Q}) \cup u_3). \tag{3.23}$$

In [129], Porter gave a topological method for computing cup products products and Massey products in $H^2(G;\mathbb{Q})$. Building on work of Dwyer [44], Fenn and Sjerve gave in [55] another method for computing these products in the second cohomology of a commutator-relators group, directly from a presentation of the group. We will briefly review the latter method in Remark 4.2.8, and use it in the computations from Examples 3.3.6, 9.2.12, and 9.2.15.

If a group G is 1-formal, then all triple Massey products vanish in the quotient $\mathbb{Q}$-vector space (3.23). However, if G is only graded-formal, these Massey products need not vanish. As we shall see in Example 9.2.12, even a one-relator group G may be graded-formal, yet not 1-formal.

3.3.3 Filtered formality, graded formality and 1-formality

The next result pulls together the various formality notions for groups, and establishes the basic relationship among them.

Proposition 3.3.4. *A finitely generated group G is 1-formal if and only if G is graded-formal and filtered-formal.*

Proof. First suppose G is 1-formal. Then, by Theorem 3.3.2, $\mathfrak{m}(G;\mathbb{Q}) \cong \widehat{\mathfrak{h}}(G;\mathbb{Q})$, and thus, $\operatorname{gr}(G;\mathbb{Q}) \cong \mathfrak{h}(G;\mathbb{Q})$, by (3.21). Hence, G is graded-formal, by Lemma 3.1.14. It follows that $\mathfrak{m}(G;\mathbb{Q}) \cong \widehat{\operatorname{gr}}(G;\mathbb{Q})$, and hence G is filtered-formal, by Proposition 3.2.3.

Now suppose G filtered-formal. Then, by Proposition 3.2.3, we have that $\mathfrak{m}(G;\mathbb{Q}) \cong \widehat{\operatorname{gr}}(G;\mathbb{Q})$. Thus, if G is also graded-formal, $\mathfrak{m}(G;\mathbb{Q}) \cong \widehat{\mathfrak{h}}(G;\mathbb{Q})$. Hence, G is 1-formal. □

Using this proposition, together with Proposition 3.2.6 and Corollary 3.1.16, we obtain the following corollary.

Corollary 3.3.5. *A finitely generated group G is 1-formal over $\mathbb{Q}$ if and only if G is 1-formal over $\mathbb{K}$.*

In other words, the 1-formality property of a finitely generated group is independent of the choice of coefficient field of characteristic 0.

In general, a filtered-formal group need not be 1-formal. Examples include some of the free nilpotent groups from Example 9.1.1 or the unipotent groups from Example 9.1.7. In fact, the triple Massey products in the cohomology of a filtered-formal group need not vanish (modulo indeterminacy).

Example 3.3.6. Let $G = F_2/\Gamma_3 F_2 = \langle x_1, x_2 \mid [x_1,[x_1,x_2]] = [x_2,[x_1,x_2]] = 1\rangle$ be the Heisenberg group. Then G is filtered-formal, yet has non-trivial triple Massey products $\langle u_1, u_1, u_2\rangle$ and $\langle u_2, u_1, u_2\rangle$ in $H^2(G;\mathbb{Q})$. Hence, G is not graded-formal.

As shown by in Hain in [65, 66] the Torelli groups in genus 4 or higher are 1-formal, but the Torelli group in genus 3 is filtered-formal, yet not graded-formal.

Example 3.3.7. In [11], Bartholdi et al. consider two infinite families of groups. The first are the quasitriangular groups QTr_n, which have presentations with generators x_{ij} $(1 \leq i \neq j \leq n)$, and relations $x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}$ and $x_{ij}x_{kl} = x_{kl}x_{ij}$ for distinct i,j,k,l. The second are the triangular groups Tr_n, each of which is the quotient of QTr_n by the relations of the form $x_{ij} = x_{ji}$ for $i \neq j$. As shown by Lee in [91], the groups QTr_n and Tr_n are all graded-formal. On the other hand, as indicated in [11], these groups are non-1-formal (and hence, not filtered-formal) for all $n \geq 4$. A detailed proof of this fact will be given in Chapter 7.

3.3.4 Propagation of filtered formality

The next theorem shows that filtered formality is inherited upon taking nilpotent quotients.

Theorem 3.3.8. *Let G be a finitely generated group, and suppose G is filtered-formal. Then all the nilpotent quotients $G/\Gamma_i(G)$ are filtered-formal.*

Proof. Set $\mathfrak{g} = \operatorname{gr}(G;\mathbb{Q})$ and $\mathfrak{m} = \mathfrak{m}(G;\mathbb{Q})$, and write $\mathfrak{g} = \bigoplus_{k\geq 1}\mathfrak{g}_k$. Then, for each $i \geq 1$, the canonical projection $\phi_i\colon G \twoheadrightarrow G/\Gamma_i G$ induces an epimorphism of complete, filtered Lie algebras, $\mathfrak{m}(\phi_i)\colon \mathfrak{m} \twoheadrightarrow \mathfrak{m}(G/\Gamma_i G;\mathbb{Q})$. In each degree $k \geq i$, we have that $\widehat{\Gamma}_k\mathfrak{m}(G/\Gamma_i G;\mathbb{Q}) = 0$, and so $\mathfrak{m}(\phi_i)(\widehat{\Gamma}_k\mathfrak{m}) = 0$. Therefore, there exists an induced epimorphism

$$\Phi_{k,i}\colon \mathfrak{m}/\widehat{\Gamma}_k\mathfrak{m} \twoheadrightarrow \mathfrak{m}(G/\Gamma_i G;\mathbb{Q})\,. \tag{3.24}$$

Passing to associated graded Lie algebras, we obtain an epimorphism $\operatorname{gr}(\Phi_{k,i})$ from $\operatorname{gr}(\mathfrak{m}/\widehat{\Gamma}_k\mathfrak{m})$ to $\operatorname{gr}(\mathfrak{m}(G/\Gamma_i G;\mathbb{Q}))$, which is readily seen to be an isomorphism for $k = i$. Using now Lemma 2.1.2, we conclude that the map $\Phi_{i,i}$ is an isomorphism of completed, filtered Lie algebras.

On the other hand, our filtered-formality assumption on G allows us to identify $\mathfrak{m} \cong \widehat{\mathfrak{g}} = \prod_{k\geq 1}\mathfrak{g}_k$. Using now formula (2.7), we find that $\mathfrak{m}/\widehat{\Gamma}_k\mathfrak{m} = \widehat{\mathfrak{g}}/\widehat{\Gamma}_k\widehat{\mathfrak{g}} = \mathfrak{g}/\Gamma_k\mathfrak{g}$, for all $k \geq 1$. Using these identifications for $k = i$, together with the isomorphism $\Phi_{i,i}$ from above, we obtain isomorphisms

$$\mathfrak{m}(G/\Gamma_i G;\mathbb{Q}) \cong \mathfrak{g}/\Gamma_i\mathfrak{g} \cong \operatorname{gr}(G/\Gamma_i G;\mathbb{Q}). \tag{3.25}$$

This shows that the nilpotent quotient $G/\Gamma_i G$ is filtered-formal, and we are done. □

Proposition 3.3.9. *Suppose $\phi\colon G_1 \to G_2$ is a homomorphism between two finitely generated groups, inducing an isomorphism $H_1(G_1;\mathbb{Q}) \to H_1(G_2;\mathbb{Q})$ and an epimorphism $H_2(G_1;\mathbb{Q}) \to H_2(G_2;\mathbb{Q})$. Then we have the following statements.*

1. *If G_2 is 1-formal, then G_1 is also 1-formal.*
2. *If G_2 is filtered-formal, then G_1 is also filtered-formal.*
3. *If G_2 is graded-formal, then G_1 is also graded-formal.*

Proof. A celebrated theorem of Stallings [138] (see also Dwyer [44] and Freedman et al. [57]) insures that the homomorphism ϕ induces isomorphisms $\phi_k\colon (G_1/\Gamma_k G_1)\otimes\mathbb{Q} \to (G_2/\Gamma_k G_2)\otimes \mathbb{Q}$, for all $k \geq 1$. Hence, ϕ induces an isomorphism $\mathfrak{m}(\phi)\colon \mathfrak{m}(G_1;\mathbb{Q}) \to \mathfrak{m}(G_2;\mathbb{Q})$ between the respective Malcev completions, thereby proving claim (1). The other two claims follow directly from (3.21). □

3.3.5 Split injections

We are now ready to state and prove the main result of this section, which completes the proof of Theorem 1.2.2 from the Introduction.

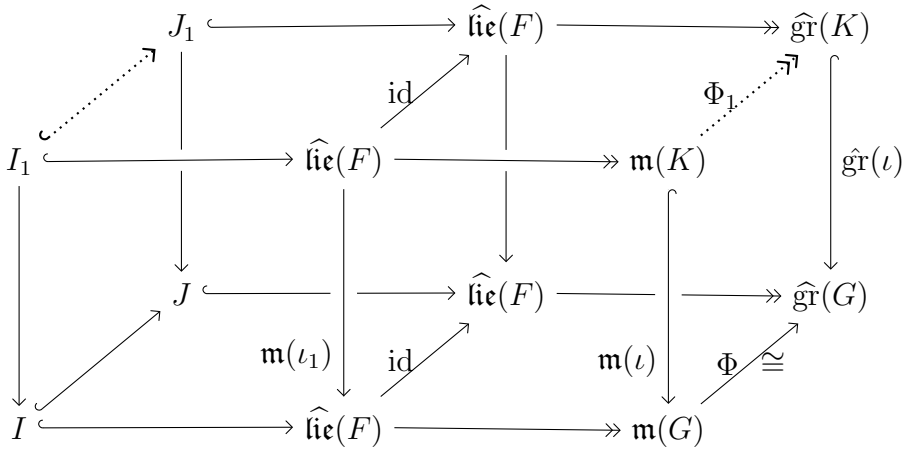
Theorem 3.3.10. *Let G be a finitely generated group. Suppose there is a split monomorphism $\iota\colon K \to G$. The following statements then hold.*

1. *If G is filtered-formal, then K is also filtered-formal.*

2. *If G is 1-formal, then K is also 1-formal.*

Proof. By hypothesis, we have an epimorphism $\sigma\colon G \twoheadrightarrow K$ such that $\sigma \circ \iota = \mathrm{id}$. It follows that the induced maps $\mathfrak{m}(\iota)$ and $\hat{\mathrm{gr}}(\iota)$ are also split injections.

Let $\pi\colon F \twoheadrightarrow G$ be a presentation for G. We then have an induced presentation for K, given by the composition $\pi_1 := \sigma\pi\colon F \twoheadrightarrow K$. There is also a map $\iota_1\colon F \to F$ which is a lift of ι, that is, $\iota\pi_1 = \pi\iota_1$. Consider the following diagram (for simplicity, we will suppress the coefficient field $\mathbb{Q}$ from the notation).



(3.26)

We have $\mathfrak{m}(\iota_1) = \hat{\mathrm{gr}}(\iota_1)$. By assumption, G is filtered-formal; hence, there exists a filtered Lie algebra isomorphism $\Phi\colon \mathfrak{m}(G) \to \widehat{\mathrm{gr}}(G)$ as in diagram (3.26), which induces the identity on associated graded algebras. It follows that Φ is induced from the identity map of $\widehat{\mathfrak{lie}}(F)$ upon projecting onto source and target, i.e., the bottom right square in the diagram commutes.

First, we show that the identity map $\mathrm{id}\colon \widehat{\mathfrak{lie}}(F) \to \widehat{\mathfrak{lie}}(F)$ in the above diagram induces an inclusion map $I_1 \to J_1$. Suppose there is an element $c \in \widehat{\mathfrak{lie}}(F)$ such that $c \in I_1$ and $c \notin J_1$, i.e., $[c] = 0$ in $\mathfrak{m}(K)$ and $[c] \neq 0$ in $\widehat{\mathrm{gr}}(G)$. Since $\hat{\mathrm{gr}}(\iota)$ is injective, we have that $\hat{\mathrm{gr}}(\iota)([c]) \neq 0$, i.e., $\hat{\mathrm{gr}}(\iota_1)(c) \notin I$. We also have $\mathfrak{m}(\iota)([c]) = 0 \in \mathfrak{m}(G)$, i.e., $\mathfrak{m}(\iota_1)(c) \in J$. This contradicts the fact that the inclusion $I \hookrightarrow J$ is induced by the identity map. Thus, $I_1 \subset J_1$.

In view of the above, we may define a Lie algebra morphism $\Phi_1\colon \mathfrak{m}(K) \to \widehat{\mathrm{gr}}(K)$ as the quotient of the identity on $\widehat{\mathfrak{lie}}(F)$. By construction, Φ_1 is an epimorphism. We also have $\hat{\mathrm{gr}}(\iota) \circ \Phi_1 = \Phi \circ \mathfrak{m}(\iota)$. Since the maps $\mathfrak{m}(\iota)$, $\hat{\mathrm{gr}}(\iota)$ and Φ are all injective, the map Φ_1 is also injective. Therefore, Φ_1 is an isomorphism, and so the group K is filtered-formal.

Finally, part (2) follows at once from part (1) and Theorem 3.1.17. □

This completes the proof of Theorem 1.2.2 from the Introduction. As we shall see in Example 3.3.7, this theorem is useful for deciding whether certain infinite families of groups are 1-formal.

We now proceed with the proof of Theorem 1.2.3. First, we need a lemma.

Lemma 3.3.11 ([42]). *Let G_1 and G_2 be two finitely generated groups. Then $\mathfrak{m}(G_1 \times G_2; \mathbb{Q}) \cong \mathfrak{m}(G_1; \mathbb{Q}) \times \mathfrak{m}(G_2; \mathbb{Q})$ and $\mathfrak{m}(G_1 * G_2; \mathbb{Q}) \cong \mathfrak{m}(G_1; \mathbb{Q}) \hat{*} \mathfrak{m}(G_2; \mathbb{Q})$.*

Proposition 3.3.12. *For any two finitely generated groups G_1 and G_2, the following conditions are equivalent.*

1. *G_1 and G_2 are filtered-formal.*
2. *$G_1 * G_2$ is filtered-formal.*
3. *$G_1 \times G_2$ is filtered-formal.*

Proof. Since there exist split injections from G_1 and G_2 to the product $G_1 \times G_2$ and coproduct $G_1 * G_2$, we may apply Theorem 3.3.10 to conclude that implications (2)⇒(1) and (3)⇒(1) hold. Implications (1)⇒(2) and (1)⇒(3) follow from Lemmas 2.1.8, 2.1.9, and 3.3.11. □

Remark 3.3.13. As we shall see in Example 9.2.17, the implication (1)⇒(3) from Proposition 3.3.12 cannot be strengthened from direct products to arbitrary semi-direct products. More precisely, there exist split extensions of the form $G = F_n \rtimes_\alpha \mathbb{Z}$, for certain automorphisms $\alpha \in \mathrm{Aut}(F_n)$, such that the group G is not filtered-formal, although of course both F_n and $\mathbb{Z}$ are 1-formal.

Corollary 3.3.14. *Suppose G_1 and G_2 are finitely generated groups such that G_1 is not graded-formal and G_2 is not filtered-formal. Then the product $G_1 \times G_2$ and the free product $G_1 * G_2$ are neither graded-formal, nor filtered-formal.*

Proof. Follows at once from Propositions 3.1.21 and 3.3.12. □

As mentioned in the Introduction, concrete examples of groups which do not possess either formality property can be obtained by taking direct products of groups which enjoy one property but not the other.

Chapter 4

Magnus expansions and the holonomy Lie algebra

The free differential calculus on free groups, defined by R. Fox in the 1950s, is an important tool in the study of Alexander invariants and Alexander polynomials in knot theory. Closely related to the Fox derivatives, the Magnus expansion is a ring homomorphism from the group ring of a free group to a non-commutative power series ring. The Magnus expansion is important in studying the group cohomology in low degrees and holonomy Lie algebras. In this chapter, we construct a similar Magnus expansion for a finitely presented group. We use this Magnus-type expansion to compute cup products, and find an explicit presentation for the holonomy Lie algebra. This chapter is based on the work in my paper [143] with Alex Suciu.

4.1 Magnus expansions for finitely generated groups

In this section, we introduce and study a Magnus-type expansion for an arbitrary finitely generated group.

4.1.1 The Magnus expansion for a free group

We start by reviewing some standard material on Fox calculus and Magnus expansions, following the exposition from Magnus et al. [102], Fenn–Sjerve [55], and Matei–Suciu [110].

As before, let $\mathbb{Q}$ denote a field of characteristic 0. Let F be the free group generated by $\mathbf{x} = \{x_1, \dots, x_n\}$, and set $F_{\mathbb{Q}} = F_{\mathrm{ab}} \otimes \mathbb{Q}$. The completed tensor algebra $\widehat{T}(F_{\mathbb{Q}})$ can be identified with $\mathbb{Q}\langle\langle \mathbf{x} \rangle\rangle$, the power series ring over $\mathbb{Q}$ in n non-commuting variables.

Let $\mathbb{Q}F$ be the group ring of F, with augmentation map $\epsilon\colon \mathbb{Q}F \to \mathbb{Q}$ given by $\epsilon(x_i) = 1$. There is a well-defined ring morphism $M\colon \mathbb{Q}F \to \mathbb{Q}\langle\langle \mathbf{x} \rangle\rangle$, called the *Magnus expansion*, given by

$$M(x_i) = 1 + x_i \quad \text{and} \quad M(x_i^{-1}) = 1 - x_i + x_i^2 - x_i^3 + \cdots . \tag{4.1}$$

The *Fox derivatives* are the ring morphisms $\partial_i\colon \mathbb{Z}F \to \mathbb{Z}F$ defined by the rules $\partial_i(1) = 0$, $\partial_i(x_j) = \delta_{ij}$, and $\partial_i(uv) = \partial_i(u)\epsilon(v) + u\partial_i(v)$ for $u, v \in \mathbb{Z}F$. The higher Fox derivatives $\partial_{i_1,\dots,i_k}$ are then defined inductively.

The Magnus expansion can be computed in terms of Fox derivatives, as follows. Given $y \in F$, if we write $M(y) = 1 + \sum a_I x_I$, then $a_I = \epsilon_I(y)$, where $I = (i_1, \dots, i_s)$, and $\epsilon_I = \epsilon \circ \partial_I$ is the composition of $\epsilon\colon \mathbb{Q}F \to \mathbb{Q}$ with $\partial_I\colon \mathbb{Q}F \to \mathbb{Q}F$. Let M_k be the composite

$$\mathbb{Q}F \xrightarrow{M} \widehat{T}(F_{\mathbb{Q}}) \xrightarrow{\mathrm{gr}_k} \mathrm{gr}_k(\widehat{T}(F_{\mathbb{Q}})) , \qquad \text{with } M_k \text{ the composite} \tag{4.2}$$

In particular, for each $y \in F$, we have $M_1(y) = \sum_{i=1}^n \epsilon_i(y) x_i$, while for each $y \in [F, F]$ we have

$$M_2(y) = \sum_{i<j} \epsilon_{i,j}(y)(x_i x_j - x_j x_i). \tag{4.3}$$

Notice that $M_2(y)$ is a primitive element in the Hopf algebra $\widehat{T}(F_{\mathbb{Q}})$, which corresponds to the element $\sum_{i<j} \epsilon_{i,j}(y)[x_i, x_j]$ in the free Lie algebra $\mathfrak{lie}(F_{\mathbb{Q}})$.

Remark 4.1.1. The map M extends to a map $\hat{M}\colon \widehat{\mathbb{Q}F} \to \widehat{T}(F_{\mathbb{Q}})$ which is an isomorphism of complete, filtered algebras, but $\hat{M}$ is not compatible with the respective comultiplications

if $\operatorname{rank} F > 1$. On the other hand, X. Lin constructed in [96] an exponential expansion, $\exp\colon \widehat{\mathbb{Q}F} \to \widehat{T}(F_{\mathbb{Q}})$, while Massuyeau showed in [109] that the map exp is an isomorphism of complete Hopf algebras. Restricting this map to the Lie algebras of primitive elements gives an isomorphism $\mathfrak{m}(F;\mathbb{Q}) \xrightarrow{\simeq} \widehat{\mathfrak{lie}}(F_{\mathbb{Q}})$.

4.1.2 The Magnus expansion for finitely generated groups

Given a finitely generated group G, there exists an epimorphism $\varphi\colon F \twoheadrightarrow G$ from a free group F of finite rank. Let π be the induced epimorphism in homology from $F_{\mathbb{Q}} := H_1(F;\mathbb{Q})$ to $G_{\mathbb{Q}} := H_1(G;\mathbb{Q})$.

Definition 4.1.2. The *Magnus expansion* for a finitely generated group G, denoted by κ, is the composite

$$\mathbb{Q}F \xrightarrow{M} \widehat{T}(F_{\mathbb{Q}}) \xrightarrow{\widehat{T}(\pi)} \widehat{T}(G_{\mathbb{Q}})\,, \qquad \overset{\kappa}{\frown} \tag{4.4}$$

where M is the classical Magnus expansion for the free group F, and the epimorphism $\widehat{T}(\pi)$ from $\widehat{T}(F_{\mathbb{Q}})$ to $\widehat{T}(G_{\mathbb{Q}})$ is induced by the projection $\pi\colon F_{\mathbb{Q}} \twoheadrightarrow G_{\mathbb{Q}}$.

In particular, if the group G is a commutator-relators group, then π identifies $G_{\mathbb{Q}}$ with $F_{\mathbb{Q}}$, and the Magnus expansion κ coincides with the classical Magnus expansion M.

More generally, let G be a group generated by $\mathbf{x} = \{x_1, \dots, x_n\}$, and let F be the free group generated by the same set. Pick a basis $\mathbf{y} = \{y_1, \dots, y_b\}$ for $G_{\mathbb{Q}}$, and identify $\widehat{T}(G_{\mathbb{Q}})$ with $\mathbb{Q}\langle\langle \mathbf{y} \rangle\rangle$. Let $\kappa(r)_I$ be the coefficient of $y_I := y_{i_1} \cdots y_{i_s}$ in $\kappa(r)$, for $I = (i_1, \dots, i_s)$. Then we can write

$$\kappa(r) = 1 + \sum_I \kappa(r)_I \cdot y_I. \tag{4.5}$$

Lemma 4.1.3. *If $r \in \Gamma_k F$, then $\kappa(r)_I = 0$, for $|I| < k$. Furthermore, if $r \in \Gamma_2 F$, then $\kappa(r)_{i,j} = -\kappa(r)_{j,i}$.*

Proof. Since $M(r)_I = \epsilon_I(r) = 0$ for $|I| < k$ (see for instance [110]), we have that $\kappa(r)_I = 0$ for $|I| < k$. To prove the second assertion, identify the completed symmetric algebras $\widehat{\operatorname{Sym}}(F_{\mathbb{Q}})$

and $\widehat{\mathrm{Sym}}(G_{\mathbb{Q}})$ with the power series rings $\mathbb{Q}[\![\mathbf{x}]\!]$ and $\mathbb{Q}[\![\mathbf{y}]\!]$ in the following commutative diagram of linear maps.

$$\begin{array}{ccccc} \mathbb{Q}F & \xrightarrow{M} & \widehat{T}(F_{\mathbb{Q}}) & \xrightarrow{\alpha_1} & \widehat{\mathrm{Sym}}(F_{\mathbb{Q}}) \\ & \searrow^{\kappa} & \downarrow{\scriptstyle \widehat{T}(\pi)} & & \downarrow{\scriptstyle \widehat{\mathrm{Sym}}(\pi)} \\ & & \widehat{T}(G_{\mathbb{Q}}) & \xrightarrow{\alpha_2} & \widehat{\mathrm{Sym}}(G_{\mathbb{Q}})\,. \end{array} \tag{4.6}$$

When $r \in [F,F]$, we have that $\alpha_2 \circ \kappa(r) = \widehat{\mathrm{Sym}}(\pi) \circ \alpha_1 \circ M(r) = 1$. Thus, $\kappa_i(r) = 0$ and $\kappa(r)_{i,j} + \kappa(r)_{j,i} = 0$. □

Lemma 4.1.4. *If $u, v \in F$ satisfy $\kappa(u)_J = \kappa(v)_J = 0$ for all $|J| < s$, for some $s \geq 2$, then*

$$\kappa(uv)_I = \kappa(u)_I + \kappa(v)_I, \quad \text{for } |I| = s.$$

Moreover, the above formula is always true for $s = 1$.

Proof. We have that $\kappa(uv) = \kappa(u)\kappa(v)$ for $u, v \in F$. If $\kappa(u)_J = \kappa(v)_J = 0$ for all $|J| < s$, then $\kappa(u) = 1 + \sum_{|I|=s} \kappa(u)_I y_I$ up to higher-order terms, and similarly for $\kappa(v)$. Then

$$\kappa(uv) = \kappa(u)\kappa(v) = 1 + \sum_{|I|=s} (\kappa(u)_I + \kappa(u)_I) y_I + \text{higher-order terms.} \tag{4.7}$$

Therefore, $\kappa(uv)_i = \kappa(u)_i + \kappa(v)_i$, and so $\kappa(uv)_I = \kappa(u)_I + \kappa(v)_I$. □

4.1.3 Truncating the Magnus expansions

Recall from (4.2) that we defined truncations M_k of the Magnus expansion M of a free group F. In a similar manner, we can also define the truncations of the Magnus expansion κ for any finitely generated group G.

Lemma 4.1.5. *The following diagram commutes.*

$$\begin{array}{ccccc} \mathbb{Q}F & \xrightarrow{M} & \widehat{T}(F_{\mathbb{Q}}) & \xrightarrow{\mathrm{gr}_k} & \mathrm{gr}_k(\widehat{T}(F_{\mathbb{Q}})) = \bigotimes^k \mathbb{Q}^n \\ & \searrow^{\kappa} & \downarrow{\scriptstyle \widehat{T}(\pi)} & & \downarrow{\scriptstyle \otimes^k \pi} \\ & & \widehat{T}(G_{\mathbb{Q}}) & \xrightarrow{\mathrm{gr}_k} & \mathrm{gr}_k(\widehat{T}(G_{\mathbb{Q}})) = \bigotimes^k \mathbb{Q}^b. \end{array} \tag{4.8}$$

Proof. The triangle on the left of diagram (4.8) commutes, since it consists of ring homomorphisms by the definition of the Magnus expansion for a group.

The morphisms in the square on the right side of (4.8) are homomorphisms between $\mathbb{Q}$-vector spaces. The square commutes, since π is a linear map. □

In diagram (4.8), denote the composition of κ and gr_k by κ_k.

$$\mathbb{Q}F \xrightarrow{\kappa} \widehat{T}(G_{\mathbb{Q}}) \xrightarrow{\mathrm{gr}_k} \mathrm{gr}_k(\widehat{T}(G_{\mathbb{Q}})) \,, \qquad \kappa_k \text{ over both} \tag{4.9}$$

In particular, $\kappa_1(r) = \sum_{i=1}^{b} \kappa(r)_i y_i$ for $r \in F$. By Lemma 4.1.3, if $r \in [F, F]$, then

$$\kappa_2(r) = \sum_{1 \le i < j \le b} \kappa(r)_{i,j}(y_i y_j - y_j y_i). \tag{4.10}$$

The next lemma provides a close connection between the Magnus expansion κ and the classical Magnus expansion M.

Lemma 4.1.6. *Let $(a_{i,s})$ be the $b \times n$ matrix for the linear map $\pi\colon F_{\mathbb{Q}} \to G_{\mathbb{Q}}$, and let $r \in F$ be an arbitrary element. Then, for each $1 \le i, j \le b$, we have*

$$\kappa(r)_i = \sum_{s=1}^{n} a_{i,s}\epsilon_s(r), \tag{4.11}$$

$$\kappa(r)_{i,j} = \sum_{s,t=1}^{n} \left(a_{i,s}a_{j,t}\epsilon_{s,t}(r)\right). \tag{4.12}$$

Proof. By assumption, we have $\pi(x_s) = \sum_{i=1}^{b} a_{i,s}y_i$. By Lemma 4.1.5 (for $k = 1$), we have

$$\kappa_1(r) = \pi \circ M_1(r) = \pi(\sum_{s=1}^{n} \epsilon_s(r)x_s) = \sum_{s=1}^{n}\sum_{i=1}^{b} a_{i,s}\epsilon_s(r)y_i,$$

which gives formula (4.11). By Lemma 4.1.5 (for $k = 2$), we have

$$\kappa_2(r) = \pi \otimes \pi \circ M_2(r) = \pi \otimes \pi \left(\sum_{s,t=1}^{n} \epsilon_{s,t}(r)x_s \otimes x_t\right) = \sum_{s,t=1}^{n}\sum_{i,j=1}^{b} \epsilon_{s,t}(r)a_{i,s}a_{j,t}y_i \otimes y_j,$$

which gives formula (4.12). □

4.1.4 Echelon presentations

Let G be a group with finite presentation $P = F/R = \langle \mathbf{x} \mid \mathbf{w} \rangle$ where $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\mathbf{w} = \{w_1, \dots, w_m\}$. Then R is a free subgroup of F generated by the set $\mathbf{w}$, and R_{ab} is a free abelian group with the same generating set.

Let K_P be the 2-complex associated to this presentation of G. We may view $\mathbf{x}$ as a basis for $C_1(K_P; \mathbb{Q})$ and $\mathbf{w}$ as a basis for $C_2(K_P; \mathbb{Q}) = \mathbb{Q}^m$. With this choice of bases, the matrix of the boundary map $d_2^P \colon C_2(K_P; \mathbb{Q}) \to C_1(K_P; \mathbb{Q})$ is the $m \times n$ Jacobian matrix $J_P = (\epsilon_i(w_k))$.

Definition 4.1.7. A group G has an *echelon presentation* $P = \langle \mathbf{x} \mid \mathbf{w} \rangle$ if the matrix $(\epsilon_i(w_k))$ is in row-echelon form.

Example 4.1.8. Let G be the group generated by $x_1, \dots, x_6$, with relations $w_1 = x_1^2 x_2^1 x_3^3 x_4^5$, $w_2 = x_3^2 x_4^{-2} x_6^4$, $w_3 = x_4^3 x_5^{-2} x_6^3$, and $w_4 = [x_1, x_2]$. The given presentation is already an echelon presentation, since the matrix

$$d_2^G = \begin{pmatrix} \boxed{2} & 1 & 3 & 5 & 0 & 0 \\ 0 & 0 & \boxed{2} & -2 & 0 & 4 \\ 0 & 0 & 0 & \boxed{3} & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has the required form.

The next proposition shows that for any finitely generated group, we can construct a group with an echelon presentation such that the two groups have the same holonomy Lie algebra.

Proposition 4.1.9. *Let G be a group with a finite presentation P. There exists then a group $\widetilde{G}$ with an echelon presentation $\widetilde{P}$ and a surjective homomorphism $\rho \colon \widetilde{G} \twoheadrightarrow G$ inducing the following isomorphisms:*

(i) $\rho_* \colon H_i(K_{\widetilde{P}}; \mathbb{Q}) \xrightarrow{\simeq} H_i(K_P; \mathbb{Q})$ *for* $i = 1, 2$*;*

(ii) $\rho^*\colon H^i(K_P;\mathbb{Q}) \xrightarrow{\cong} H^i(K_{\widetilde{P}};\mathbb{Q})$ *for* $i = 1, 2$*;*

(iii) $\mathfrak{h}(\rho)\colon \mathfrak{h}(\widetilde{G};\mathbb{Q}) \xrightarrow{\cong} \mathfrak{h}(G;\mathbb{Q})$.

Proof. Suppose G has presentation $P = \langle \mathbf{x} \mid \mathbf{r} \rangle$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\mathbf{r} = \{r_1, \dots, r_m\}$. Consider the diagram

$$\begin{array}{ccccccc} H_2(K_G;\mathbb{Q}) & \longrightarrow & C_2(K_G;\mathbb{Q}) & \xrightarrow{d_2^G} & C_1(K_G;\mathbb{Q}) & \xrightarrow{\pi} & G_{\mathbb{Q}} \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ H^2(K_G;\mathbb{Q}) & \longleftarrow & C^2(K_G;\mathbb{Q}) & \xleftarrow{(d_2^G)^*} & C^1(K_G;\mathbb{Q}) & \xleftarrow{\pi^*} & H^1(K_G;\mathbb{Q}), \end{array} \tag{4.13}$$

where the vertical arrows indicate duality isomorphisms. By Gaussian elimination over $\mathbb{Z}$, there exists a matrix $C = (c_{l,k}) \in \mathrm{GL}(m;\mathbb{Z})$ such that $C \cdot (d_2^G)^*$ is in row-echelon form. We define a new group,

$$\widetilde{G} = \langle x_1, \dots, x_n \mid w_1, \dots, w_m \rangle, \tag{4.14}$$

where $w_k = r_1^{c_{1,k}} r_2^{c_{2,k}} \cdots r_m^{c_{m,k}}$ for $1 \leq k \leq m$. The surjection $\rho\colon \widetilde{G} \twoheadrightarrow G$, defined by sending a generator $x_i \in \widetilde{G}$ to the same generator $x_i \in G$ for $1 \leq i \leq n$, induces a chain map from the cellular chain complex $C_*(K_{\widetilde{G}};\mathbb{Q})$ to $C_*(K_G;\mathbb{Q})$, as follows:

$$\begin{array}{ccccccccc} C_0(K_G;\mathbb{Q}) & \xleftarrow{d_1^G=0} & C_1(K_G;\mathbb{Q}) & \xleftarrow{d_2^G} & C_2(K_G;\mathbb{Q}) & \longleftarrow & 0 & \longleftarrow & \cdots \\ \uparrow{\scriptstyle \rho_0=\mathrm{id}} & & \uparrow{\scriptstyle \rho_1=\mathrm{id}} & & \uparrow{\scriptstyle \rho_2} & & \uparrow & & \\ C_0(K_{\widetilde{G}};\mathbb{Q}) & \xleftarrow{d_1^{\widetilde{G}}=0} & C_1(K_{\widetilde{G}};\mathbb{Q}) & \xleftarrow{d_2^{\widetilde{G}}} & C_2(K_{\widetilde{G}};\mathbb{Q}) & \longleftarrow & 0 & \longleftarrow & \cdots . \end{array} \tag{4.15}$$

The map ρ_2 is given by the matrix C, while $d_2^{\widetilde{G}}$ is given by the composition $d_2^G \circ \rho_2$. The homomorphism ρ induces isomorphisms on homology groups. In particular, $\rho_*\colon H_1(K_{\widetilde{G}};\mathbb{Q}) \to H_1(K_G;\mathbb{Q})$ is the identity. Then, we see that $\pi_G = \pi_{\widetilde{G}}$.

The last statement follows from the functoriality of the cup-product and Lemma 3.1.8. □

4.2 Group presentations and (co)homology

We compute in this section the cup products of degree 1 classes in the cohomology of a finitely presented group in terms of the Magnus expansion associated to the group.

4.2.1 A chain transformation

We start by reviewing the classical bar construction. Let G be a discrete group, and let $B_*(G)$ be the normalized bar resolution (see e.g. Brown [22], and Fenn and Sjerve [55]), where $B_p(G)$ is the free left $\mathbb{Z}G$-module on generators $[g_1|\dots|g_p]$, with $g_i \in G$ and $g_i \neq 1$, and $B_0(G) = \mathbb{Z}G$ is free on one generator, $[\ ]$. The boundary operators are G-module homomorphisms, $d_p\colon B_p(G) \to B_{p-1}(G)$, defined by

$$d_p[g_1|\dots|g_p] = g_1[g_2|\dots|g_p] + \sum_{i=1}^{p-1}(-1)^i[g_1|\dots|g_ig_{i+1}|\dots|g_p] + (-1)^p[g_1|\dots|g_{p-1}]. \quad (4.16)$$

In particular, $d_1[g] = (g-1)[\ ]$ and $d_2[g_1|g_2] = g_1[g_2] - [g_1g_2] + [g_1]$. Let $\epsilon\colon B_0(G) \to \mathbb{Q}$ be the augmentation map. We then have a free resolution of the trivial G-module $\mathbb{Q}$,

$$\cdots \longrightarrow B_2(G) \xrightarrow{d_2} B_1(G) \xrightarrow{d_1} B_0(G) \xrightarrow{\epsilon} \mathbb{Q} \longrightarrow 0\,. \quad (4.17)$$

As before, $\mathbb{Q}$ will denote a field of characteristic 0. We shall view $\mathbb{Q}$ as a right $\mathbb{Q}G$-module, with action induced by the augmentation map. An element of the cochain group $B^p(G;\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}G}(B_p(G),\mathbb{Q})$ can be viewed as a set function $u\colon G^p \to \mathbb{Q}$ satisfying the normalization condition $u(g_1,\dots,g_p) = 0$ if some $g_i = 1$. The cup-product of two 1-dimensional classes $u, u' \in H^1(G;\mathbb{Q}) \cong B^1(G;\mathbb{Q}) \cong \mathrm{Hom}(G,\mathbb{Q})$ is given by

$$u \cup u'[g_1|g_2] = u(g_1)u'(g_2). \quad (4.18)$$

Now suppose the group $G = \langle x_1,\dots x_n \mid r_1,\dots,r_m\rangle$ is finitely presented. Let $\varphi\colon F \twoheadrightarrow G$ be the presenting homomorphism, and let K_G be the 2-complex associated to this presentation of G. Denote the cellular chain complex (over $\mathbb{Q}$) of the universal cover of this 2-complex by $C_*(\widetilde{K_G})$. The differentials in this chain complex are given by

$$\delta_1(\lambda_1,\dots,\lambda_n) = \sum_{i=1}^{n}\lambda_i(x_i - 1)\,, \quad (4.19)$$

$$\delta_2(\mu_1,\dots,\mu_m) = \Big(\sum_{j=1}^{m}\mu_j\varphi(\partial_1 w_j),\dots,\sum_{j=1}^{m}\mu_j\varphi(\partial_n w_j)\Big)\,,$$

for $\lambda_i, \mu_j \in \mathbb{Z}G$.

Lemma 4.2.1 ([55]). *There exists a chain transformation $T\colon C_*(\widetilde{K_G}) \to B_*(G)$ commuting with the augmentation map,*

$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & \mathbb{Z} & \xleftarrow{\epsilon} & C_0(\widetilde{K_G}) & \xleftarrow{\delta_1} & C_1(\widetilde{K_G}) & \xleftarrow{\delta_2} & C_2(\widetilde{K_G}) & \longleftarrow & 0 & \longleftarrow & \cdots \\
 & & \| & & \downarrow{\scriptstyle T_0} & & \downarrow{\scriptstyle T_1} & & \downarrow{\scriptstyle T_2} & & \downarrow & & \\
0 & \longleftarrow & \mathbb{Z} & \xleftarrow{\epsilon} & B_0(G) & \xleftarrow{d_1} & B_1(G) & \xleftarrow{d_2} & B_2(G) & \longleftarrow & B_3(G) & \longleftarrow & \cdots .
\end{array}$$

Here

$$T_0(\lambda) := \lambda[\,], \quad T_1(\lambda_1, \ldots, \lambda_n) = \sum_i \lambda_i [x_i], \quad T_2(\mu_1, \ldots, \mu_m) = \sum_{j=1}^{m} \mu_j \tau_1 T_1 \delta_2(e_j), \tag{4.20}$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in (\mathbb{Z}G)^m$ has a 1 only in position j, and $\tau_0\colon B_0(G) \to B_1(G)$ and $\tau_1\colon B_1(G) \to B_2(G)$ are the homomorphisms defined by

$$\tau_0(g[\,]) = [g] \text{ and } \tau_1(g[g_1]) = [g|g_1], \tag{4.21}$$

for all $g, g_1 \in G$.

4.2.2 Cup products for echelon presentations

Now let G be a group with echelon presentation $G = \langle \mathbf{x} \mid \mathbf{w} \rangle$, where $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $\mathbf{w} = \{w_1, \ldots, w_m\}$, as in Definition 4.1.7. Suppose the pivot elements of the $m \times n$ matrix $(\epsilon_i(w_k))$ are in position $\{i_1, \ldots, i_d\}$, and let $b = n - d$.

Lemma 4.2.2. *Let K_G be the 2-complex associated to the above presentation for G. Then*

(i) The vector space $H_1(K_G; \mathbb{Q}) = \mathbb{Q}^b$ has basis $\mathbf{y} = \{y_1, \ldots, y_b\}$, where $y_j = x_{i_{d+j}}$ for $1 \le j \le b$.

(ii) The vector space $H_2(K_G; \mathbb{Q}) = \mathbb{Q}^{m-d}$ has basis $\{w_{d+1}, \ldots, w_m\}$ which coincide with the augmentation of the basis $\{e_{d+1}, \ldots, e_m\}$ in Lemma 4.2.1.

Proof. The lemma follows from the fact that the matrix $(\epsilon_i(w_k))$ is in row-echelon form. □

We will choose as basis for $H^1(K_G;\mathbb{Q})$ the set $\{u_1,\dots,u_b\}$, where u_i is the Kronecker dual to y_i.

Lemma 4.2.3. *For each basis element $u_i \in H^1(K_G;\mathbb{Q}) \cong H^1(G;\mathbb{Q})$ as above, and each $r \in F$, we have that*

$$u_i([\varphi(r)]) = \sum_{s=1}^{n} \epsilon_s(r) a_{i,s} = \kappa_i(r),$$

where $(a_{i,s})_{b\times n}$ is the matrix for the projection map $\pi\colon F_{\mathbb{Q}} \to G_{\mathbb{Q}}$.

Proof. If $r \in F$, then $\varphi(r) \in G$ and $[\varphi(r)] \in B_1(G)$. Hence,

$$u_i([\varphi(r)]) = \sum_{s=1}^{n} \epsilon_s(r) u_i([x_s]) = \sum_{s=1}^{n} \epsilon_s(r) a_{i,s} = \kappa_i(r). \tag{4.22}$$

Since $H^1(G;\mathbb{Q}) \cong B^1(G;\mathbb{Q}) \cong \mathrm{Hom}(G,\mathbb{Q})$, we may view u_i as a group homomorphism. This yields the first equality. Since $\pi(x_s) = \sum_{j=1}^{b} a_{i,s} y_i$ and $u_i = y_i^*$, the second equality follows. The last equality follows from Lemma 4.1.6. □

Theorem 4.2.4. *The cup-product map $H^1(K_G;\mathbb{Q}) \wedge H^1(K_G;\mathbb{Q}) \to H^2(K_G;\mathbb{Q})$ is given by*

$$(u_i \cup u_j, w_k) = \kappa(w_k)_{i,j}\,,$$

for $1 \le i,j \le b$ and $d+1 \le k \le m$, where κ is the Magnus expansion of G.

Proof. Let us write the Fox derivative $\partial_t(w_k)$ as a finite sum, $\sum_{x\in F} p_{tk}^x x$, for $1 \le t \le n$, and $1 \le k \le m$. We then have

$$\begin{aligned}
T_2(e_k) &= \tau_1 T_1(\delta_2(e_k)) && \text{by (4.20)} \\
&= \tau_1 T_1\left(\varphi(\partial_1(w_k)),\dots,\varphi(\partial_n(w_k))\right) && \text{by (4.19)} \\
&= \tau_1\Big(\sum_{t=1}^{n} \varphi\big(\partial_t(w_k)\big)[x_t]\Big) && \text{by (4.20)} \\
&= \sum_{t=1}^{n}\sum_{x\in F} p_{tk}^x [\varphi(x)|x_t]. && \text{by (4.21)}
\end{aligned} \tag{4.23}$$

The chain transformation $T\colon C_*(\widetilde{K_G}) \to B_*(G)$ induces an isomorphism on first cohomology, $T^*\colon H^1(G;\mathbb{Q}) \to H^1(K_G;\mathbb{Q})$. Let us view u_i and u_j as elements in $H^1(G;\mathbb{Q})$. We then have

$$\begin{aligned}
(u_i \cup u_j, 1 \otimes_{\mathbb{Z}G} e_k) &= (u_i \cup u_j, 1 \otimes_{\mathbb{Z}G} T_2(e_k)) \\
&= (u_i \cup u_j, \sum_{t=1}^{n} \sum_{x \in F} p_{tk}^x [\varphi(x)|x_t]) && \text{by (4.23)} \\
&= \sum_{t=1}^{n} \sum_{x \in F} p_{tk}^x u_i(\varphi(x)) u_j(x_t) && \text{by (4.18)} \\
&= \sum_{t=1}^{n} \sum_{x \in F} p_{tk}^x u_i(\varphi(x)) a_{j,t} && \text{by Lemma 4.2.3} \\
&= \sum_{t=1}^{n} \sum_{x \in F} p_{tk}^x \sum_{s=1}^{n} a_{i,s} \epsilon_s(x) a_{j,t} && \text{by Lemma 4.2.3} \\
&= \sum_{t=1}^{n} \sum_{s=1}^{n} (a_{j,t} a_{i,s} \epsilon_{s,t}(w_k)) && \text{Fox derivative} \\
&= \kappa(w_k)_{i,j}, && \text{by Lemma 4.1.6}
\end{aligned}$$

and this completes the proof. □

Example 4.2.5. Let K_G be the presentation 2-complex for the group G in Example 4.1.8, with the homology basis as in Lemma 4.2.2. A basis of $H_1(K_G;\mathbb{Q}) = \mathbb{Q}^3$ is $\{x_2, x_5, x_6\}$ while a basis of $H_2(K_G;\mathbb{Q}) = \mathbb{Q}$ is $\{w_4\}$, and a basis of $H^1(K_G;\mathbb{Q})$ is $\{u_1, u_2, u_3\}$. With these choices, we have that $(u_1 \cup u_2, w_4) = 8/3$, $(u_1 \cup u_3, w_4) = -7$, and $(u_2 \cup u_3, w_4) = 0$.

4.2.3 Cup products for finite presentations

Let G be a group with a finite presentation $\langle \mathbf{x} \mid \mathbf{r} \rangle$. By Proposition 4.1.9, there exists a group $\widetilde{G}$ with an echelon presentation $\langle \mathbf{x} \mid \mathbf{w} \rangle$. Let us choose a basis $\mathbf{y} = \{y_1, \ldots, y_b\}$ for $H_1(K_G;\mathbb{Q}) \cong H_1(K_{\widetilde{G}};\mathbb{Q})$ and the dual basis $\{u_1, \ldots, u_b\}$ for $H^1(K_G;\mathbb{Q}) \cong H^1(K_{\widetilde{G}};\mathbb{Q})$. Choose also a basis $\{r_1, \ldots, r_m\}$ for $C_2(K_G;\mathbb{Q})$ and a basis $\{w_1, \ldots, w_m\}$ for $C_2(K_{\widetilde{G}};\mathbb{Q})$. Set

$$\gamma_k := \rho_*(w_k) = \sum_{l=1}^{m} c_{l,k} r_l \,. \tag{4.24}$$

Then $\{\gamma_k \mid 1 \leq k \leq m\}$ is another basis for $C_2(K_G;\mathbb{Q})$. Furthermore, $\{w_{d+1},\ldots,w_m\}$ is a basis for $H_2(K_{\widetilde{G}};\mathbb{Q})$ and $\{\gamma_{d+1},\ldots,\gamma_m\}$ is a basis for $H_2(K_G;\mathbb{Q})$. Thus, $H^2(K_G;\mathbb{Q})$ has dual basis $\{\beta_{d+1},\ldots,\beta_m\}$.

Theorem 4.2.6. *The cup-product map* $\cup\colon H^1(K_G;\mathbb{Q}) \wedge H^1(K_G;\mathbb{Q}) \to H^2(K_G;\mathbb{Q})$ *is given by the formula*

$$u_i \cup u_j = \sum_{k=d+1}^{m} \kappa(w_k)_{i,j}\beta_k,$$

That is, $(u_i \cup u_j, \gamma_k) = \kappa(w_k)_{i,j}$ *for all* $1 \leq i,j \leq b$.

Proof. By Proposition 4.1.9, we have that $\gamma_k := \rho_*(w_k) = \rho_*(1 \otimes_{\mathbb{Z}\widetilde{G}} e_k)$, for all $d+1 \leq k \leq m$. Hence,

$$\begin{aligned}
(u_i \cup u_j, \gamma_k) &= \big(u_i \cup u_j, \rho_*(1 \otimes_{\mathbb{Z}\widetilde{G}} e_k)\big) \\
&= \big(\rho^*(u_i \cup u_j), 1 \otimes_{\mathbb{Z}\widetilde{G}} e_k\big) \\
&= \big(u_i \cup u_j, 1 \otimes_{\mathbb{Z}\widetilde{G}} e_k\big) && \text{since } \rho^*(u_i) = u_i \\
&= \kappa(w_k)_{i,j} && \text{by Theorem 4.2.4.}
\end{aligned}$$

The claim follows. □

Corollary 4.2.7 ([55])**.** *For a commutator-relators group* $G = \langle \mathbf{x} \mid \mathbf{r}\rangle$*, the cup-product map* $H^1(K_G;\mathbb{Q}) \wedge H^1(K_G;\mathbb{Q}) \to H^2(K_G;\mathbb{Q})$ *is given by*

$$(u_i \cup u_j, r_k) = M(r_k)_{i,j}\,,$$

for $1 \leq i,j \leq n$ *and* $1 \leq k \leq m$.

Remark 4.2.8. In [55], Fenn and Sjerve also gave formulas for the higher-order Massey products in a commutator-relator group, using the classical Magnus expansion. For instance, suppose $G = \langle \mathbf{x} \mid r\rangle$, where the single relator r belongs to $[F,F]$ and is not a proper power. Let $I = (i_1,\ldots,i_k)$, and suppose $\epsilon_{i_s,\ldots,i_{t-1}}(r) = 0$ for all $1 \leq s < t \leq k+1$, $(s,t) \neq (1,k+1)$. Then the evaluation of the Massey product $\langle -u_{i_1},\ldots,-u_{i_k}\rangle$ on the

homology class $[r] \in H_2(G;\mathbb{Z})$ equals $\epsilon_I(r)$. For an alternative approach, in a more general context, see [129, Theorem 2].

4.3 A presentation for the holonomy Lie algebra

In this section, we give a presentation for the holonomy Lie algebra and the Chen holonomy Lie algebra of a finitely presented group. In the process, we complete the proof of the first two parts of Theorem 1.2.4 from the Introduction.

4.3.1 Magnus expansion and holonomy

In view of Theorem 4.1.9, for any group with finite presentation $\langle \mathbf{x} \mid \mathbf{r} \rangle$, there exists a group with echelon presentation $P = \langle \mathbf{x} \mid \mathbf{w} \rangle$ such that the two groups have the same holonomy Lie algebras.

Let $G = F/R$ be a group admitting an echelon presentation P as above, with $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\mathbf{w} = \{w_1, \dots, w_m\}$. We now give a more explicit presentation for the holonomy Lie algebra $\mathfrak{h}(G;\mathbb{Q})$ over $\mathbb{Q}$.

Let $\partial_i(w_k) \in \mathbb{Z}F$ be the Fox derivatives of the relations, and let $\epsilon_i(w_k) \in \mathbb{Z}$ be their augmentations. Recall from Lemma 4.2.2 that we can choose a basis $\mathbf{y} = \{y_1, \dots, y_b\}$ for $H_1(K_P;\mathbb{Q})$ and a basis $\{w_{d+1}, \dots, w_m\}$ for $H_2(K_P;\mathbb{Q})$, where d is the rank of Jacobian matrix $J_P = (\epsilon_i(w_k))$, viewed as an $m \times n$ matrix over $\mathbb{Q}$. Let $\mathfrak{lie}(\mathbf{y})$ be the free Lie algebra over $\mathbb{Q}$ generated by $\mathbf{y}$ in degree 1. Recall that κ_2 is the degree 2 part of the Magnus expansion of G given explicitly in (4.10). Thus, we can identify $\kappa_2(w_k)$ with $\sum_{i<j} \kappa(w_k)_{i,j}[y_i, y_j]$ in $\mathfrak{lie}(\mathbf{y})$ for $d+1 \leq k \leq m$.

Theorem 4.3.1. *Let G be a group admitting an echelon presentation $G = \langle \mathbf{x} \mid \mathbf{w} \rangle$. Then there exists an isomorphism of graded Lie algebras*

$$\mathfrak{h}(G;\mathbb{Q}) \xrightarrow{\cong} \mathfrak{lie}(\mathbf{y})/\mathrm{ideal}(\kappa_2(w_{d+1}), \dots, \kappa_2(w_m)) \,.$$

Proof. Combining Theorem 4.2.4 with the fact that $(u_i \wedge u_j, \cup^*(w_k)) = (\cup(u_i \wedge u_j), w_k)$, we see that the dual cup-product map, $\cup^* \colon H_2(K_P;\mathbb{Q}) \to H_1(K_P;\mathbb{Q}) \wedge H_1(K_P;\mathbb{Q})$, is given by

$$\cup^*(w_k) = \sum_{1 \le i < j \le b} \kappa(w_k)_{i,j}(y_i \wedge y_j). \tag{4.25}$$

Hence, the following diagram commutes.

$$\begin{array}{ccc} H_2(K_P;\mathbb{Q}) & \xrightarrow{\cup^*} & H_1(K_P;\mathbb{Q}) \wedge H_1(K_P;\mathbb{Q}) \\ \downarrow & & \downarrow \\ C_2(K_P;\mathbb{Q}) & \xrightarrow{\kappa_2} & H_1(K_P;\mathbb{Q}) \otimes H_1(K_P;\mathbb{Q}) \end{array} \tag{4.26}$$

Using now the identification of $\kappa_2(w_k)$ and $\sum_{i<j} \kappa(w_k)_{i,j}[y_i, y_j]$ as elements of $\mathfrak{lie}(\mathbf{y})$, the definition of the holonomy Lie algebra, and the fact that $\mathfrak{h}(G;\mathbb{Q}) \cong \mathfrak{h}(K_P;\mathbb{Q})$, we arrive at the desired conclusion. □

Corollary 4.3.2. *The universal enveloping algebra $U(\mathfrak{h})$ of $\mathfrak{h}(G;\mathbb{Q})$ has presentation*

$$U(\mathfrak{h}) = \mathbb{Q}\langle \mathbf{y} \rangle / \operatorname{ideal}(\kappa_2(w_{n-b+1}), \dots, \kappa_2(w_m)).$$

Recall that, for $r \in [F, F]$, the primitive element $M_2(r)$ in $\widehat{T}_2(F_{\mathbb{Q}})$ corresponds to the element $\sum_{i<j} \epsilon_{i,j}(r)[x_i, x_j]$ in $\mathfrak{lie}_2(\mathbf{x})$.

Corollary 4.3.3 ([117])**.** *If $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$ is a commutator-relators group, then*

$$\mathfrak{h}(G;\mathbb{Q}) = \mathfrak{lie}(\mathbf{x}) / \operatorname{ideal}\Big\{ \sum_{i<j} \epsilon_{i,j}(r)[x_i, x_j] \mid r \in \mathbf{r} \Big\}.$$

Corollary 4.3.4. *For every quadratic, rationally defined Lie algebra $\mathfrak{g}$, there exists a commutator relators group G such that $\mathfrak{h}(G;\mathbb{Q}) \cong \mathfrak{g}$.*

Proof. By assumption, we may write $\mathfrak{g} = \mathfrak{lie}(\mathbf{x})/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal generated by elements of the form $\ell_k = \sum c_{ijk}[x_i, x_j]$ for $1 \le k \le m$, and where the coefficients c_{ijk} are in $\mathbb{Q}$. Clearing denominators, we may assume all c_{ijk} are integers. We can then define words $r_k = \prod [x_i, x_j]^{c_{ijk}}$ in the free group generated by $\mathbf{x}$, and set $G = \langle \mathbf{x} \mid r_1, \dots, r_m \rangle$. The desired conclusion follows from Corollary 4.3.3. □

4.3.2 Presentations for the holonomy Chen Lie algebras

The next result (which completes the proof of Theorem 1.2.4 from the Introduction) sharpens and extends the first part of Theorem 7.3 from [117].

Theorem 4.3.5. *Let $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$ be a finitely presented group, and set $\mathfrak{h} = \mathfrak{h}(G; \mathbb{Q})$. Let $\mathbf{y} = \{y_1, \dots, y_b\}$ be a basis of $H_1(G; \mathbb{Q})$. Then, for each $i \geq 2$,*

$$\mathfrak{h}/\mathfrak{h}^{(i)} \cong \mathfrak{lie}(\mathbf{y})/(\mathrm{ideal}(\kappa_2(w_{n-b+1}), \dots, \kappa_2(w_m)) + \mathfrak{lie}^{(i)}(\mathbf{y})),$$

where $b = b_1(G)$ and w_k is defined in (4.14).

Proof. By Theorem 4.3.1, the holonomy Lie algebra $\mathfrak{h}$ is isomorphic to the quotient of the free Lie algebra $\mathfrak{lie}(\mathbf{y})$ by the ideal generated by $\kappa_2(w_{n-b+1}), \dots, \kappa_2(w_m)$. The claim follows from Lemma 2.1.1. □

Using Corollary 4.3.3, we obtain the following corollary.

Corollary 4.3.6. *Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a commutator-relators group, and $\mathfrak{h} = \mathfrak{h}(G; \mathbb{Q})$. Then, for each $i \geq 2$, the Lie algebra $\mathfrak{h}/\mathfrak{h}^{(i)}$ is isomorphic to the quotient of the free Lie algebra $\mathfrak{lie}(\mathbf{x})$ by the sum of the ideals $(M_2(r_1), \dots, M_2(r_m))$ and $\mathfrak{lie}^{(i)}(\mathbf{x})$.*

Now suppose G is 1-formal. Then, in view of Corollary 6.1.7, the Chen Lie algebra $\mathrm{gr}(G/G''; \mathbb{Q})$ is isomorphic to $\mathfrak{h}(G; \mathbb{Q})/\mathfrak{h}(G; \mathbb{Q})^{(i)}$, which has presentation as above.

4.3.3 Koszul properties

We now use our presentation of the holonomy Lie algebra $\mathfrak{h} = \mathfrak{h}(G; \mathbb{Q})$ of a group G with finitely many generators and relators to study the Koszul properties of the corresponding universal enveloping algebra.

Computing the Hilbert Series of $U(\mathfrak{h})$ directly from Corollary 4.3.2 is not easy, since it involves finding a Gröbner basis for a non-commutative algebra. However, if $U(\mathfrak{h})$ is a Koszul

algebra, we can use Proposition 2.2.7 and the Corollary 2.2.8 to reduce the computation to that of the Hilbert series of a graded-commutative algebra, which can be done by a standard Gröbner basis algorithm.

Proposition 4.3.7. *Let G be a group with presentation $\langle \mathbf{x} \mid \mathbf{r} \rangle$, and set $m = |\mathbf{r}|$. If $\operatorname{rank}(\epsilon_i(r_j)) = m$ or $m-1$, then the universal enveloping algebra $U(\mathfrak{h}(G;\mathbb{Q}))$ is Koszul.*

Proof. Let $\mathfrak{h} = \mathfrak{h}(G;\mathbb{Q})$. By Proposition 4.1.9 and Theorem 4.3.1,

$$\mathfrak{h} = \begin{cases} \mathfrak{lie}(\mathbf{x}) & \text{if } \operatorname{rank}(\epsilon_i(r_j)) = m, \\ \mathfrak{lie}(\mathbf{y})/\operatorname{ideal}(\kappa_2(w_m)) & \text{if } \operatorname{rank}(\epsilon_i(r_j)) = m-1. \end{cases} \tag{4.27}$$

In the first case, $U(\mathfrak{h}) = T\langle \mathbf{x} \rangle$, which of course is Koszul. In the second case, Corollary 4.3.2 implies that $U(\mathfrak{h}) = T\langle \mathbf{y} \rangle / I$, where $I = \operatorname{ideal}(\kappa_2(w_m))$. Clearly, I is a principal ideal, generated in degree 2; thus, by [58], $U(\mathfrak{h})$ is again a Koszul algebra. □

Of course, the universal enveloping algebra of the holonomy Lie algebra of a finitely generated group is a quadratic algebra. In general, though, it is not a Koszul algebra (see for instance Example 9.1.10.)

Example 4.3.8. Let $\mathfrak{h}$ be the holonomy Lie algebra of the McCool group wP_n^+. As shown by Conner and Goetz in [36], the algebra $U(\mathfrak{h})$ is not Koszul for $n \geq 4$. For more information on the Lie algebras associated to the McCool groups, see Chapter 8.

Chapter 5

Resonance varieties

One idea from the theory of hyperplane arrangements is to study a family of cochain complexes parametrized by the cohomology ring in degree 1. The resonance varieties are defined from the data of these cochain complexes, capture subtle information as the loci where the cohomology of these cochain complexes jumps. More generally, the resonance varieties of a finite type commutative graded algebra (CGA) A over $\mathbb{C}$, are homogeneous subvarieties of A^1. The resonance varieties of a space X or a group G with appropriate finiteness condition are defined to be the resonance varieties of their cohomology algebras. We study the resonance varieties of product and coproduct of CGAs. Using an explicit presentation for the infinitesimal Alexander invariants, we provide upper and lower bounds for the first resonance varieties, which is important for the computation of the resonance varieties of the pure welded braid groups in a later chapter. This chapter is based on the work in my papers [145, 146] with Alex Suciu.

5.1 Resonance varieties

In this section, we first review the resonance varieties of a locally finite, connected, graded-commutative algebra. Applying the Bernstein–Gelfand–Gelfand correspondence, we give a

description for the first resonance variety.

5.1.1 The Bernstein–Gelfand–Gelfand correspondence

Let V be a complex vector spaces of finite dimension, and let V^* be its dual. We write $E = \bigwedge V$ for the exterior algebra on V, and $S = \mathrm{Sym}(V^*)$ for the symmetric algebra on V^*. Following the approach from [46], the Bernstein–Gelfand–Gelfand (BGG) correspondence is an isomorphism between the category of linear free complexes over E and the category of graded free modules over S.

Let $\{e_1, \dots, e_n\}$ and $\{x_1, \dots, x_n\}$ be dual bases for V and V^*, respectively, and identify the symmetric algebra $\mathrm{Sym}(V^*)$ with the polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$. If we take S to be generated in degree 1, then E is generated in degree -1. Let $\mathbf{L}$ be the functor from the category of graded E-modules to the category of linear free complexes over S which assigns to a graded E-module P the chain complex

$$\mathbf{L}(P): \qquad \cdots \longrightarrow P_i \otimes_{\mathbb{C}} S \xrightarrow{d_i} P_{i-1} \otimes_{\mathbb{C}} S \longrightarrow \cdots, \tag{5.1}$$

with differentials given by $d_i(p \otimes s) = \sum_{j=1}^{n} e_j p \otimes x_j s$, for $p \in P_i$ and $s \in S$.

5.1.2 Resonance varieties

Now let $A = \bigoplus_{i \geq 0} A^i$ be a graded, graded-commutative algebra. We shall assume that A is connected (i.e., $A^0 = \mathbb{C}$), and locally finite (i.e., the Betti numbers $b_i := \dim A^i$ are finite, for each $i \geq 1$). Let $\{e_1, \dots, e_n\}$ be the basis for the complex vector space $V := A^1$, and $\{x_1, \dots, x_n\}$ be the Kronecker dual basis for V^*. We can view A as a module over the exterior algebra $E = \bigwedge(V)$.

Let P be a E-module defined by $P_i := A^{-i}$. The *universal Aomoto complex* of A is the complex $\mathbf{L}(P)$ defined by (5.1), and denoted by $\mathbf{L}(A)$. Notice the change of degree, the

universal Aomoto complex of A is the cochain complex of free S-modules,

$$\mathbf{L}(A): \qquad A^0 \otimes S \xrightarrow{d^0} A^1 \otimes S \xrightarrow{d^1} A^2 \otimes S \xrightarrow{d^2} \cdots, \tag{5.2}$$

with differentials given by

$$d^i(u \otimes s) = \sum_{j=1}^{n} e_j u \otimes x_j s \tag{5.3}$$

for $u \in A^i$ and $s \in S$. According to [121, 141], the evaluation of the universal Aomoto complex at an element $a \in A^1$ coincides with the Aomoto complex, $(A; \delta_a)$, which is the cochain complex of finite-dimensional, complex vector spaces,

$$(A, \delta_a): \ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots, \tag{5.4}$$

with differentials $\delta_a^i(u) = au$ for all $u \in A^i$. By definition, the (degree i, depth d) *resonance varieties* of A are the algebraic sets

$$\mathcal{R}_d^i(A) = \{a \in A^1 \mid b_i(A, a) \geq d\}, \tag{5.5}$$

where (A, a) is the cochain complex (known as the *Aomoto complex*) with differentials $\delta_a^i \colon A^i \to A^{i+1}$ given by $\delta_a^i = a \cdot u$, and $b_i(A, a) := \dim H^i(A, a)$.

Observe that $b_i(A, 0) = b_i(A)$. Thus, $\mathcal{R}_d^i(A)$ is empty if either $d > b_i$ or $d \geq 0$ and $b_i = 0$. Furthermore, $0 \in \mathcal{R}_d^i(A)$ if and only if $d \leq b_i$. In degree zero, we have that $\mathcal{R}_d^0(A) = \{0\}$ for $d = 1$ and $\mathcal{R}_d^0(A) = \emptyset$ for $d \geq 2$. We use the convention that $\mathcal{R}_d^i(A) = A^1$ for $d \leq 0$. The following simple lemma will be useful in computing the resonance varieties of the algebra $A = H^*(vP_3, \mathbb{C})$.

Lemma 5.1.1. *Suppose $A^i \neq 0$ for $i \leq 2$ and $A^i = 0$ for $i \geq 3$. Then $\mathcal{R}_d^2(A) = \mathcal{R}_{d-\chi}^1(A)$ for $d \leq b_2$, where $\chi = 1 - b_1 + b_2$ is the Euler characteristic of A.*

Proof. By the above discussion, $0 \in \mathcal{R}_d^2(A)$ if and only if $d \leq b_2$. But this is equivalent to $0 \in \mathcal{R}_{d-\chi}^1(A)$, since $d - \chi \leq b_2 - \chi \leq b_1 - 1$. Now let $a \in A^1 \backslash \{0\}$. Then $b_2(A, a) = b_1(A, a) + \chi$. Hence, $a \in \mathcal{R}_d^2(A)$ if and only if $a \in \mathcal{R}_{d-\chi}^1(A)$, and we are done. □

We will be mostly interested here in the degree 1 resonance varieties, $\mathcal{R}^1_d(A)$. Equations for these varieties can be obtained as follows (see for instance [141]). Let $\{e_1, \dots, e_n\}$ be a basis for the complex vector space $A^1 = H^1(G; \mathbb{C})$, and let $\{x_1, \dots, x_n\}$ be the dual basis for $A_1 = H_1(G; \mathbb{C})$. Identifying the symmetric algebra $\mathrm{Sym}(A_1)$ with the polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$, we obtain a cochain complex of free S-modules,

$$A^0 \otimes_{\mathbb{C}} S \xrightarrow{\delta^0} A^1 \otimes_{\mathbb{C}} S \xrightarrow{\delta^1} A^2 \otimes_{\mathbb{C}} S \xrightarrow{\delta^2} \cdots, \tag{5.6}$$

with differentials given by $\delta^i(u \otimes 1) = \sum_{j=1}^n e_j u \otimes x_j$ for $u \in A^i$ and extended by S-linearity. The first resonance variety $\mathcal{R}_d(A) := \mathcal{R}^1_d(A)$, then, is the zero locus of the ideal of codimension d minors of the matrix δ^1. An equivalent description for the first resonance varieties can be found in [124].

Now suppose X is a connected, finite-type CW-complex. One defines then the resonance varieties of X to be the sets $\mathcal{R}^i_d(X) := \mathcal{R}^i_d(H^*(X, \mathbb{C}))$. Likewise, the resonance varieties of a group G admitting a finite-type classifying space are defined as $\mathcal{R}^i_d(G) := \mathcal{R}^i_d(H^*(G, \mathbb{C}))$.

5.1.3 Characteristic varieties

Let X be a connected CW-complex with finite k-skeleton ($k \geq 1$). Without lose of generality, suppose X has only one single 0-cell x_0. Let $\mathbb{C}^*$ be the group of units of $\mathbb{C}$. The cellular chain complex of X is denoted by $(C_i(X; \mathbb{C}), \partial_i)$. If the universal cover of X is $\tilde{X} \to X$, then $C_i(\tilde{X}; \mathbb{C})$ is a chain complex of module over $\mathbb{C}G$, where $G = \pi_1(X, x_0)$. The module structure on

$$\mathbb{C}G \times C_i(\tilde{X}; \mathbb{C}) \to C_i(\tilde{X}; \mathbb{C})$$

is given by $g \cdot e_i$ and linear expansion. The group homomorphism $\mathrm{Hom}(G, \mathbb{C}^*)$ is an algebraic group, with multiplication $f_1 \circ f_2(g) = f_1(g) f_2(g)$ and identity $\mathrm{id}(g) = 1$ for $g \in G$ and $f_i \in \mathrm{Hom}(G, \mathbb{C}^*)$. Since $\mathbb{C}^*$ is abelian group, we have

$$\mathrm{Hom}(G, \mathbb{C}^*) = \mathrm{Hom}(G_{\mathrm{ab}}, \mathbb{C}^*) = \mathrm{Hom}(\mathbb{Z}^{b_i} \oplus \bigoplus_i \mathbb{Z}/k_i\mathbb{Z}, \mathbb{C}^*) = (\mathbb{C}^*)^{b_i} \oplus \bigoplus_i \mathbb{C}^*/k_i\mathbb{C}^*.$$

Let $\rho\colon G \to \mathbb{C}^* \in \mathrm{Hom}(G, \mathbb{C}^*)$. The rank 1 local system on X is a 1-dimensional $\mathbb{C}$-vector space $\mathbb{C}_\rho$. There is a is a right $\mathbb{C}G$-module structure $\mathbb{C}_\rho \times G \to \mathbb{C}_\rho$ given by $\rho(g) \circ a$ for $a \in \mathbb{C}_\rho$ and $g \in G$. The homology group of X with coefficient in $\mathbb{C}_\rho$ is defined by

$$H_i(X, \mathbb{C}_\rho) := H_i(C_*(\tilde{X}, \mathbb{C}) \otimes_{\mathbb{C}G} \mathbb{C}_\rho)$$

In particular, for $\rho = \mathrm{id} \in \mathrm{Hom}(G, \mathbb{C}^*)$ gives the homology of X with $\mathbb{C}$ coefficient.

The *characteristic varieties* of X are the jumping loci for cohomology with coefficients in rank 1 local systems,

$$\mathcal{V}^i_k(X) = \{\rho \in \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \mid \dim(H_i(X; \mathbb{C}_\rho)) \geq k\}. \tag{5.7}$$

If a group G admitting a finite-type classifying space $K(G, 1)$, the jump loci of a group G are defined in terms of the jump loci of the corresponding classifying space. Best understood are the degree 1 cohomology jump loci, $\mathcal{R}_k(X) = \mathcal{R}^1_k(X)$ and $\mathcal{V}_k(X) = \mathcal{V}^1_k(X)$, which depend only on the fundamental group $G = \pi_1(X)$.

An important obstruction to 1-formality is provided by the higher-order Massey products, but we will not make use of it in this thesis. Instead, we will use another, better suited obstruction to 1-formality, which is provided by the following theorem.

Theorem 5.1.2 ([42])**.** *Let G be a finitely generated, 1-formal group.*

1. *There exists an isomorphism between the tangent cone variety at the origin, $\mathrm{TC}_1(\mathcal{V}_k(G))$ and the resonance variety $\mathcal{R}_k(G)$.*

2. *Then all irreducible components of $\mathcal{R}_k(G)$ are rationally defined linear subspaces of $H^1(G, \mathbb{C})$, for all $k \geq 0$.*

5.1.4 Resonance varieties of products and coproducts

The next two results are generalizations of Propositions 13.1 and 13.3 from [122]. We will use these results to compute the resonance varieties of the group vP_3.

Proposition 5.1.3. *Let $A = B \otimes C$ be the product of two connected, finite-type graded algebras. Then, for all $i \geq 1$,*

$$\mathcal{R}^1_d(B \otimes C) = \mathcal{R}^1_d(B) \times \{0\} \cup \{0\} \times \mathcal{R}^1_d(C),$$
$$\mathcal{R}^i_1(B \otimes C) = \bigcup_{s+t=i} \mathcal{R}^s_1(B) \times \mathcal{R}^t_1(C).$$

Proof. Let $a = (a_1, a_2)$ be an element in $A^1 = B^1 \oplus C^1$. The cochain complex (A, a) splits as a tensor product of cochain complexes, $(B, a_1) \otimes (C, a_2)$. Therefore,

$$b_i(A, a) = \sum_{s+t=i} b_s(B, a_1) b_t(C, a_2), \tag{5.8}$$

and the second formula follows. In particular, we have $b_1(A, (0, 0)) = b_1(B, 0) + b_1(C, 0)$, $b_1(A, (0, a_2)) = b_1(C, a_2)$ if $a_2 \neq 0$, $b_1(A, (a_1, 0)) = b_1(B, a_1)$ if $a_1 \neq 0$, and $b_1(A, a) = 0$ if $a_1 \neq 0$ and $a_2 \neq 0$. The first formula now easily follows. □

Proposition 5.1.4. *Let $A = B \vee C$ be the coproduct of two connected, finite-type graded algebras. Then, for all $i \geq 1$,*

$$\mathcal{R}^1_d(B \vee C) = \Big(\bigcup_{j+k=d-1} (\mathcal{R}^1_j(B) \backslash \{0\}) \times (\mathcal{R}^1_k(C) \backslash \{0\}) \Big) \cup \big(\{0\} \times \mathcal{R}^1_s(C)\big) \cup \big(\mathcal{R}^1_t(B) \times \{0\}\big),$$
$$\mathcal{R}^i_d(B \vee C) = \bigcup_{j+k=d} \mathcal{R}^i_j(B) \times \mathcal{R}^i_k(C), \qquad \text{if } i \geq 2,$$

where $s = d - \dim B^1$ and $t = d - \dim C^1$.

Proof. Pick an element $a = (a_1, a_2)$ in $A^1 = B^1 \oplus C^1$. The Aomoto complex of A splits (in positive degrees) as a direct sum of chain complexes, $(A^+, a) \cong (B^+, a_1) \oplus (C^+, a_2)$. We then have formulas relating the Betti numbers of the respective Aomoto complexes:

$$b_i(A, a) = \begin{cases} b_i(B, a_1) + b_i(C, a_2) + 1 & \text{if } i = 1, \text{ and } a_1 \neq 0,\ a_2 \neq 0, \\ b_i(B, a_1) + b_i(C, a_2) & \text{otherwise.} \end{cases}$$

The claim follows by a case-by-case analysis of the above formula. □

5.1.5 A description of the first resonance

In this thesis, we focus on the first resonance varieties, $\mathcal{R}_1(A) := \mathcal{R}_1^1(A)$. It is readily seen that this variety depends only on the multiplication map $\mu_A \colon A^1 \wedge A^1 \to A^2$.

More precisely, define the *quadratic closure* of A as $\bar{A} = E/I$, where $E = \bigwedge(A^1)$ is exterior algebra and I is the two-sided ideal of E generated by $K = \ker(\mu_A) \subset A^1 \wedge A^1$. Then we have $\mathcal{R}_1(A) = \mathcal{R}_1(\bar{A})$. Applying the $\mathbf{L}$ functor to the exact sequence $0 \to I \to E \to \bar{A} \to 0$, we obtain a short exact sequence of cochain complexes (see, [133, 134])

$$0 \longrightarrow \mathbf{L}(I) \xrightarrow{\iota} \mathbf{L}(E) \xrightarrow{p} \mathbf{L}(\bar{A}) \longrightarrow 0\,. \tag{5.9}$$

More explicitly in degrees two and three, we have a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^2 \otimes S & \xrightarrow{\iota^2} & E^2 \otimes S & \xrightarrow{\mu_A \otimes \mathrm{id}} & \bar{A}^2 \otimes S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^2} & \searrow^{\Phi} & \downarrow{\scriptstyle d^2} & & \downarrow{\scriptstyle d^2} & & \\ 0 & \longrightarrow & I^3 \otimes S & \xrightarrow[\iota^3]{} & E^3 \otimes S & \longrightarrow & \bar{A}^3 \otimes S & \longrightarrow & 0\,. \end{array} \tag{5.10}$$

Hence, $I^2 = K = \ker \mu_A$. Let Φ be the composition $\iota^3 \circ d^2 \colon I^2 \otimes S \to E^3 \otimes S$. Using this map, we obtain an equivalent description for the first resonance variety.

Lemma 5.1.5. *An element $a \in A^1$ belongs to $\mathcal{R}_1(A)\backslash\{0\}$ if and only if the evaluation of Φ at a is not injective.*

Proof. Recall we fixed a basis $\{e_1, \dots, e_n\}$ for A^1, that $\{x_1, \dots, x_n\}$ is is the dual basis, and that $S = \mathbb{C}[x_1, \dots, x_n]$. Let $a = \sum_{j=1}^n a_j e_j \in A^1$, and let $\mathrm{ev}_a \colon S \to \mathbb{C}$ be the ring morphism given by $g \to g(a_1, \dots, a_n)$. For each $r \in I^2$, by formula (5.3), we have

$$\Phi|_a(r) = (\mathrm{id}_{E^3} \otimes \mathrm{ev}_a) \circ \iota^3 \circ d^2(r \otimes 1) = \sum_{j=1}^n e_j r \otimes \mathrm{ev}_a(x_j) = \sum_{j=1}^n e_j r \cdot a_j = a \cdot r. \tag{5.11}$$

Hence, for each $a \in A^1$, the map $\Phi|_a \colon I^2 \to E^3$ is given by left multiplication of a. If $a \neq 0$, the complex (E, δ_a) is acyclic. Thus, we have

$$H^1(A; \delta_a) \cong H^2(I; \delta_a) \cong \ker(\delta_a \colon I^2 \to I^3) = \ker(\Phi|_a \colon I^2 \to E^3), \tag{5.12}$$

and this finishes the proof. □

5.2 Infinitesimal Alexander invariants

5.2.1 The infinitesimal Alexander invariant of a Lie algebra

We start in a more general context. Let $\mathfrak{g} = \bigoplus_{k\geq 1} \mathfrak{g}_k$ be a finitely generated graded Lie algebra, with graded pieces $\mathfrak{g}_k$, for $k \geq 1$. Then both the derived algebra, $\mathfrak{g}'$, and the second derived algebra $\mathfrak{g}'' = (\mathfrak{g}')'$, are graded sub-Lie algebras. Thus, the maximal metabelian quotient, $\mathfrak{g}/\mathfrak{g}''$, is in a natural way a graded Lie algebra, with derived subalgebra $\mathfrak{g}'/\mathfrak{g}''$. Define the Chen ranks of $\mathfrak{g}$ to be

$$\theta_k(\mathfrak{g}) = \dim(\mathfrak{g}/\mathfrak{g}'')_k. \tag{5.13}$$

Following [117], we associate to $\mathfrak{g}$ a graded module over the symmetric algebra $S = \mathrm{Sym}(\mathfrak{g}_1)$, as follows. The adjoint representation of $\mathfrak{g}_1$ on $\mathfrak{g}/\mathfrak{g}''$ defines an S-action on $\mathfrak{g}'/\mathfrak{g}''$, given by $h \cdot \bar{x} = \overline{[h, x]}$, for $h \in \mathfrak{g}_1$ and $x \in \mathfrak{g}'$. Clearly, this action is compatible with the grading on $\mathfrak{g}'/\mathfrak{g}''$. The *infinitesimal Alexander invariant* of $\mathfrak{g}$ is the graded S-module

$$\mathfrak{B}(\mathfrak{g}) = \mathfrak{g}'/\mathfrak{g}''. \tag{5.14}$$

Here, S is the universal enveloping algebra of $\mathfrak{g}/\mathfrak{g}'$, which can be identified with the symmetric algebra on $\mathfrak{g}_1$, with variables in degree 1. The exact sequence of graded Lie algebras

$$0 \longrightarrow \mathfrak{g}'/\mathfrak{g}'' \longrightarrow \mathfrak{g}/\mathfrak{g}'' \longrightarrow \mathfrak{g}/\mathfrak{g}' \longrightarrow 0$$

gives the graded S-module structure on $\mathfrak{B}(\mathfrak{g})$.

Now suppose $\mathfrak{g}$ admits a finite, quadratic presentation, that is, $\mathfrak{g} = \mathfrak{lie}(H)/\langle \mathfrak{a} \rangle$, where H is a finite-dimensional vector space, and $\mathfrak{a}$ is a finite set of degree two elements in the free Lie algebra $\mathfrak{lie}(H)$. Then, by [117], the S-module $\mathfrak{B}(\mathfrak{g})$ admits a homogeneous, finite presentation of the form

$$\Big(\bigwedge^3 H \oplus \mathfrak{a} \Big) \otimes S \xrightarrow{\delta_3 + (\mathrm{id} \otimes \iota)} \bigwedge^2 H \otimes S \longrightarrow \mathfrak{B}(\mathfrak{g}) \longrightarrow 0\,, \tag{5.15}$$

where ι is the inclusion of $\mathfrak{a}$ into $\mathfrak{lie}(H)_2 \cong H\wedge H$, and $\delta_3(x\wedge y\wedge z) = x\wedge y\otimes z - x\wedge z\otimes y + y\wedge z\otimes x$.

Assume now that the graded Lie algebra $\mathfrak{g} = \bigoplus_{k\geq 1}\mathfrak{g}_k$ is generated in degree 1. We then have $\mathfrak{g}' = \bigoplus_{k\geq 2}\mathfrak{g}_k$. Thus, since the grading for S starts with $S_0 = \mathbb{C}$, we are led to define the grading on $\mathfrak{B}(\mathfrak{g})$ as

$$\mathfrak{B}(\mathfrak{g})_k = (\mathfrak{g}'/\mathfrak{g}'')_{k+2}, \text{ for } k \geq 0. \tag{5.16}$$

We then have the following 'infinitesimal' version of Massey's formula (6.16).

Proposition 5.2.1. *Let $\mathfrak{g}$ be a finitely generated, graded Lie algebra $\mathfrak{g}$ generated in degree 1. Then the Chen ranks of $\mathfrak{g}$ are given by*

$$\sum_{k\geq 2}\theta_k(\mathfrak{g})\cdot t^{k-2} = \operatorname{Hilb}(\mathfrak{B}(\mathfrak{g}), t). \tag{5.17}$$

Proof. Since $\mathfrak{g}$ is generated in degree 1, we have that $\mathfrak{g}/\mathfrak{g}' \cong \mathfrak{g}_1$. Using now the exact sequence of graded Lie algebras $0 \to \mathfrak{g}'/\mathfrak{g}'' \to \mathfrak{g}/\mathfrak{g}'' \to \mathfrak{g}/\mathfrak{g}' \to 0$, we find that $(\mathfrak{g}/\mathfrak{g}'')_k = (\mathfrak{g}'/\mathfrak{g}'')_k$ for all $k \geq 2$. The claim then follows from (5.13) and (5.16). □

5.2.2 The infinitesimal Alexander invariants of a group

Let again G be a finitely generated group. Denote by $H = G_{\mathrm{ab}}$ its abelianization, and identify $\mathfrak{h}_1(G) = \operatorname{gr}_1(G)$ with $H\otimes\mathbb{C}$. Finally, set $S = \operatorname{Sym}(H\otimes\mathbb{C})$. The procedure outlined in §5.2.1 yields two S-modules attached to G.

The first one is $\mathfrak{B}(G) = \mathfrak{B}(\mathfrak{h}(G))$, the infinitesimal Alexander invariant of the holonomy Lie algebra of G. (When G is a finitely presented, commutator-relators group, this S-module coincides with the 'linearized Alexander invariant' from [32, 110], see [117]). The second one is $\mathfrak{B}(\operatorname{gr}(G))$, the infinitesimal Alexander invariant of the associated graded Lie algebra of G. The next result provides a natural comparison map between these S-modules.

Proposition 5.2.2. *The canonical epimorphism $\Psi\colon \mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ from (3.11) induces an epimorphism of S-modules,*

$$\psi\colon \mathfrak{B}(\mathfrak{h}(G)) \twoheadrightarrow \mathfrak{B}(\operatorname{gr}(G)) .$$

Moreover, if G is graded-formal, then ψ is an isomorphism.

Proof. The graded Lie algebra map $\Psi\colon \mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ preserves derived series, and thus induces an epimorphism $\psi\colon \mathfrak{h}(G)'/\mathfrak{h}(G)'' \twoheadrightarrow \operatorname{gr}(G)'/\operatorname{gr}(G)''$. By the discussion from §5.2.1, this map can also be viewed as a map $\psi\colon \mathfrak{B}(\mathfrak{h}(G)) \twoheadrightarrow \mathfrak{B}(\operatorname{gr}(G))$ of graded S-modules.

Finally, if G is graded-formal, i.e., if Ψ is an isomorphism, then clearly ψ is also an isomorphism. □

Remark 5.2.3. For a finitely presented group G, a finite presentation for the S-module $\mathfrak{B}(\mathfrak{h}(G))$ is given in [117]. This presentation may be used to compute the holonomy Chen ranks $\theta_k(\mathfrak{h}(G))$ from the Hilbert series of $\mathfrak{B}(\mathfrak{h}(G))$, using an approach analogous to the one described in Remark 6.2.1. We refer to Chapter 8, for the detailed computations for the (upper) pure welded braid groups.

5.3 Bounds for the first resonance variety

Using the BGG correspondence, we give in this section a presentation for the infinitesimal Alexander invariant. We then obtain upper and lower bounds for the first resonance variety.

Lemma 5.3.1. *The dual of $\Phi\colon I^2 \otimes S \to E^3 \otimes S$ gives a presentation for the infinitesimal Alexander invariant $\mathfrak{B}(A)$.*

Proof. By definition, the ideal I of the exterior algebra $E = \bigwedge V$ is generated by the vector space $K = \ker \mu_A$. Taking duals, we have an isomorphism $K^* \cong \operatorname{coker} \partial_A$. Recall the map Φ from (5.10), defined as the composite

$$K \otimes S \xrightarrow{\iota\otimes\mathrm{id}} \bigwedge^2 V \otimes S \xrightarrow{d^2} \bigwedge^3 V \otimes S\,, \qquad \overset{\Phi}{\frown} \tag{5.18}$$

where $d^2(u \otimes a \wedge b) = u \wedge a \wedge b$. All S-modules in the above diagram are free modules. After taking the dual of the above diagram, we have

$$\operatorname{coker}(\partial_A) \otimes S \xleftarrow{\iota^*} \bigwedge^2 V^* \otimes S \xleftarrow{(d^2)^*} \bigwedge^3 V^* \otimes S \qquad \overset{\Phi^*}{\frown} \tag{5.19}$$

Here, ι^* is the projection map and $(d^2)^*$ is the same as δ_3 in (5.15). Then the infinitesimal Alexander invariant $\mathfrak{B}(G)$ is isomorphic to $\operatorname{coker}\Phi^*$ by presentation (5.15).

□

This result generalizes the formula (2.5) in [34], where they give a presentation for the linearized Alexander invariant for a commutator-relators group.

5.3.1 A decomposition of the resonance varieties

We now briefly review some basic facts about the elementary ideals of a module (for more details, see [45]). Let S be a commutative ring with unit. Assume S is Noetherian and a unique factorization domain. Let M be a S-module with a finite presentation,

$$S^m \xrightarrow{\varphi} S^n \longrightarrow M \longrightarrow 0. \tag{5.20}$$

If we choose basis for S^m and S^n, we shall view φ as a matrix Ω with m rows and n columns. The i-th *elementary ideal* (or, *Fitting ideal*) of M, denoted by $E_i(M) \subseteq S$, is the ideal of S generated by the $(n-i) \times (n-i)$ minors of the $m \times n$ matrix Ω. This ideal is independent of the presentation of M. The ideal of maximal minors $E_0(M)$ is known as the *order ideal*.

The next lemma gives upper and lower bounds for the first resonance variety. The first claim of the lemma is well-known, see e.g.,[41]; we provide a proof for completeness. The two other claims (especially the third one) prove to be useful in the computation of the first resonance variety; we will illustrate their usefulness in Section 8.4.

Lemma 5.3.2. *Let A be a locally finite, connected, graded-commutative algebra. Let Ω be a presentation matrix of the infinitesimal Alexander invariant $\mathfrak{B} = \mathfrak{B}(A)$.*

1. *We have isomorphisms of varieties*

$$\mathbf{V}(E_0(\Omega)) \cong \mathbf{V}(\operatorname{Ann}(\mathfrak{B})) \cong \mathcal{R}_1(A).$$

where Ann *is the annihilator of a module, and* $\mathbf{V}$ *is the variety defined by an ideal.*

2. *Denote the columns of* Ω *by* Ω_i*. Then,*

$$\bigcup_i \mathbf{V}(E_0(\Omega_i)) \subseteq \mathcal{R}_1(A),$$

3. *Suppose* Ω *is a block triangular matrix, with diagonal blocks* Ω_{ii}*. Then,*

$$\mathcal{R}_1(A) \subseteq \bigcup_i \mathbf{V}(E_0(\Omega_{ii}))$$

Proof. (1) Suppose $\mathfrak{B}$ is generated by n elements. By standard commutative algebra, we have that $\mathrm{Ann}(\mathfrak{B})^n \subseteq E_0(\Omega) \subseteq \mathrm{Ann}(\mathfrak{B})$. Thus, $\mathbf{V}(E_0(\Omega)) \cong \mathbf{V}(\mathrm{Ann}(\mathfrak{B}))$. By Lemmas 5.1.5 and 5.3.1, a non-zero element $a \in H^1(A)$ belongs to $\mathcal{R}_1(A)$ if and only if the matrix $\Omega|_a$ does not have full rank, that is, $a \in \mathbf{V}(E_0(\Omega))$.

(2) Suppose $a \in \mathbf{V}(E_0(\Omega_i))$, that is, the matrix $(\Omega_i)|_a$ does not have full rank. Then $\Omega|_a$ does not have full rank, either, that is, $a \in \mathbf{V}(E_0(\Omega)) \cong \mathcal{R}_1(A)$.

(3) Suppose $a \notin \bigcup_i \mathbf{V}(E_0(\Omega_{ii}))$, that is, all matrices $\Omega_{ii}|_a$ have full rank. Then $\Omega|_a$ has full rank, and so $a \notin \mathbf{V}(E_0(\Omega)) \cong \mathcal{R}_1(A)$. □

Chapter 6

Chen Lie algebras

In this chapter, we study the Chen Lie algebra of a finitely generated group G and its relationships with the first resonance varieties and the Alexander invariants of G. K.T. Chen studied the associated graded Lie algebras of the of the maximal metabelian quotient, which are now called the *Chen Lie algebras* of G. The ranks of the Chen Lie algebras are called the *Chen ranks*. Chen computed the Chen ranks of a finitely generated free group F_n, by introducing a path integral technique. The Alexander invariants, which are original from the study of the Alexander polynomials of knots and links, play an important role in investigating resonance varieties, characteristic varieties and the Chen ranks. This chapter is based on the work in my papers [143, 145, 146] with Alex Suciu.

6.1 Derived series and Lie algebras

We now study some of the relationships between the derived series of a group and the derived series of the corresponding Lie algebras.

6.1.1 Derived series

Consider the derived series of a group G, starting at $G^{(0)} = G$, $G^{(1)} = G'$, and $G^{(2)} = G''$, and defined inductively by $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. Note that any homomorphism $\phi\colon G \to H$ takes $G^{(i)}$ to $H^{(i)}$. The quotient groups, $G/G^{(i)}$, are solvable; in particular, $G/G' = G_{\mathrm{ab}}$, while G/G'' is the maximal metabelian quotient of G.

Assume G is a finitely generated group, and fix a coefficient field $\mathbb{Q}$ of characteristic 0.

Proposition 6.1.1. *The holonomy Lie algebras of the derived quotients of G are given by*

$$\mathfrak{h}(G/G^{(i)};\mathbb{Q}) = \begin{cases} \mathfrak{h}(G;\mathbb{Q})/\mathfrak{h}(G;\mathbb{Q})' & \text{for } i = 1, \\ \mathfrak{h}(G;\mathbb{Q}) & \text{for } i \geq 2. \end{cases} \tag{6.1}$$

Proof. For $i = 1$, the statement trivially holds, so we may as well assume $i \geq 2$. It is readily proved by induction that $G^{(i)} \subseteq \Gamma_{2^i}(G)$. Hence, the projections

$$G \twoheadrightarrow G/G^{(i)} \twoheadrightarrow G/\Gamma_{2^i}G \tag{6.2}$$

yield natural projections $\mathfrak{h}(G;\mathbb{Q}) \twoheadrightarrow \mathfrak{h}(G/G^{(i)};\mathbb{Q}) \twoheadrightarrow \mathfrak{h}(G/\Gamma_{2^i}G;\mathbb{Q}) = \mathfrak{h}(G;\mathbb{Q})$. By Proposition 3.1.12, the composition of these projections is an isomorphism of Lie algebras. Therefore, the surjection $\mathfrak{h}(G;\mathbb{Q}) \twoheadrightarrow \mathfrak{h}(G/G^{(i)};\mathbb{Q})$ is an isomorphism. □

The next theorem is the Lie algebra version of Theorem 3.5 from [117].

Theorem 6.1.2 ([117])**.** *For each $i \geq 2$, there is an isomorphism of complete, separated filtered Lie algebras,*

$$\mathfrak{m}(G/G^{(i)}) \cong \mathfrak{m}(G)/\overline{\mathfrak{m}(G)^{(i)}},$$

where $\overline{\mathfrak{m}(G)^{(i)}}$ denotes the closure of $\mathfrak{m}(G)^{(i)}$ with respect to the filtration topology on $\mathfrak{m}(G)$.

6.1.2 Chen Lie algebras

As before, let G be a finitely generated group. For each $i \geq 2$, the *i-th Chen Lie algebra* of G is defined to be the associated graded Lie algebra of the corresponding solvable quotient,

$\operatorname{gr}(G/G^{(i)};\mathbb{Q})$. Clearly, this construction is functorial.

The quotient map, $q_i\colon G \twoheadrightarrow G/G^{(i)}$, induces a surjective morphism between associated graded Lie algebras. Plainly, this morphism is the canonical identification in degree 1. In fact, more is true.

Lemma 6.1.3. *For each $i \geq 2$, the map $\operatorname{gr}(q_i)\colon \operatorname{gr}_k(G;\mathbb{Q}) \twoheadrightarrow \operatorname{gr}_k(G/G^{(i)};\mathbb{Q})$ is an isomorphism for each $k \leq 2^i - 1$.*

Proof. Taking associated graded Lie algebras in sequence (6.2) gives epimorphisms

$$\operatorname{gr}(G;\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G/G^{(i)};\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G/\Gamma_{2^i}G;\mathbb{Q})\,. \tag{6.3}$$

By a previous remark, the composition of these maps is an isomorphism in degrees $k < 2^i$. The conclusion follows. □

We now specialize to the case when $i = 2$, which is the case originally studied by K.-T. Chen in [27]. The *Chen ranks* of G are defined as

$$\theta_k(G) := \dim_{\mathbb{Q}}(\operatorname{gr}_k(G/G'';\mathbb{Q})). \tag{6.4}$$

The projection $\pi\colon G \twoheadrightarrow G/G''$ induces an epimorphism, $\operatorname{gr}(\pi)\colon \operatorname{gr}(G) \twoheadrightarrow \operatorname{gr}(G/G'')$. It is readily seen that $\operatorname{gr}_k(\pi)$ is an isomorphism for $k \leq 3$. For a free group F_n of rank n, Chen showed that

$$\theta_k(F_n) = (k-1)\binom{n+k-2}{k}, \tag{6.5}$$

for all $k \geq 2$. Let us also define the *holonomy Chen ranks* of G as $\bar{\theta}_k(G) = \dim_{\mathbb{Q}}(\mathfrak{h}/\mathfrak{h}'')_k$, where $\mathfrak{h} = \mathfrak{h}(G;\mathbb{Q})$. It is readily seen that $\bar{\theta}_k(G) \geq \theta_k(G)$, with equality for $k \leq 2$.

Lemma 6.1.4. *The Chen Lie algebra of the product of two groups G_1 and G_2 is isomorphic to the direct sum $\operatorname{gr}(G_1/G_1'') \oplus \operatorname{gr}(G_2/G_2'')$, as graded Lie algebras.*

Proof. The canonical projections $G_1 \times G_2 \to G_i$ for $i = 1, 2$ restrict to homomorphisms on the second derived subgroups, $(G_1 \times G_2)'' \to G_i''$. Hence, there is an epimorphism $\phi\colon G_1 \times$

$G_2/(G_1 \times G_2)'' \to G_1/G_1'' \times G_2/G_2''$, inducing an epimorphism

$$\operatorname{gr}(\phi)\colon\ \operatorname{gr}((G_1 \times G_2)/(G_1 \times G_2)'') \twoheadrightarrow \operatorname{gr}(G_1/G_1'') \oplus \operatorname{gr}(G_2/G_2'')\ . \tag{6.6}$$

By [32, Corollary 1.10], we have that

$$\theta_k(G_1 \times G_2) = \theta_k(G_1) + \theta_k(G_2). \tag{6.7}$$

Hence, the homomorphism $\operatorname{gr}(\phi)$ is an isomorphism of graded Lie algebras. □

6.1.3 Chen Lie algebras and formality

We are now ready to state and prove the main result of this section, which (together with the first corollary following it) proves Theorem 1.2.5 from the Introduction.

Theorem 6.1.5. *Let G be a finitely generated group. For each $i \geq 2$, the quotient map $q_i\colon G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded $\mathbb{Q}$-Lie algebras,*

$$\Psi_G^{(i)}\colon\ \operatorname{gr}(G;\mathbb{Q})/\operatorname{gr}(G;\mathbb{Q})^{(i)} \twoheadrightarrow \operatorname{gr}(G/G^{(i)};\mathbb{Q})\ .$$

Moreover, if G is a filtered-formal group, then $\Psi_G^{(i)}$ is an isomorphism and the solvable quotient $G/G^{(i)}$ is filtered-formal.

Proof. The map $q_i\colon G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded $\mathbb{Q}$-Lie algebras, $\operatorname{gr}(q_i)\colon\ \operatorname{gr}(G;\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G/G^{(i)};\mathbb{Q})$. By Proposition 3.1.1, this epimorphism factors through an isomorphism, $\operatorname{gr}(G;\mathbb{Q})/\widetilde{\operatorname{gr}}(G^{(i)};\mathbb{Q}) \xrightarrow{\simeq} \operatorname{gr}(G/G^{(i)};\mathbb{Q})$, where $\widetilde{\operatorname{gr}}$ denotes the graded Lie algebra associated with the filtration $\widetilde{\Gamma}_k G^{(i)} = \Gamma_k G \cap G^{(i)}$.

On the other hand, as shown by Labute in [86, Proposition 10], the Lie ideal $\operatorname{gr}(G;\mathbb{Q})^{(i)}$ is contained in $\widetilde{\operatorname{gr}}(G^{(i)};\mathbb{Q})$. Therefore, the map $\operatorname{gr}(q_i)$ factors through the claimed epimorphism

$\Psi_G^{(i)}$, as indicated in the following commuting diagram,

$$\begin{array}{ccc} \mathrm{gr}(G;\mathbb{Q}) & & \\ \downarrow & \searrow^{\mathrm{gr}(q_i)} & \\ \mathrm{gr}(G;\mathbb{Q})/\,\mathrm{gr}(G;\mathbb{Q})^{(i)} & \xrightarrow{\Psi_G^{(i)}} & \mathrm{gr}(G/G^{(i)};\mathbb{Q})\,. \\ \downarrow & \nearrow_{\simeq} & \\ \mathrm{gr}(G;\mathbb{Q})/\widetilde{\mathrm{gr}}(G^{(i)};\mathbb{Q}) & & \end{array} \tag{6.8}$$

Suppose now that G is filtered-formal, and set $\mathfrak{m} = \mathfrak{m}(G;\mathbb{Q})$ and $\mathfrak{g} = \mathrm{gr}(G;\mathbb{Q})$. We may identify $\widehat{\mathfrak{g}} \cong \mathfrak{m}$. Let $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$ be the inclusion into the completion. Passing to solvable quotients, we obtain a morphism of filtered Lie algebras,

$$\varphi^{(i)}\colon\ \mathfrak{g}/\mathfrak{g}^{(i)} \longrightarrow \mathfrak{m}/\overline{\mathfrak{m}^{(i)}}\,. \tag{6.9}$$

Passing to the associated graded Lie algebras, we obtain the following diagram:

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{g}^{(i)} & \xrightarrow{\Psi_G^{(i)}} & \mathrm{gr}(G/G^{(i)};\mathbb{Q}) \\ \downarrow{\scriptstyle \mathrm{gr}(\varphi^{(i)})} & & \downarrow{\scriptstyle \cong} \\ \mathrm{gr}(\mathfrak{m}/\overline{\mathfrak{m}^{(i)}}) & \xrightarrow{\cong} & \mathrm{gr}(\mathfrak{m}(G/G^{(i)};\mathbb{Q}))\,. \end{array} \tag{6.10}$$

All the graded Lie algebras in this diagram are generated in degree 1, and all the morphisms induce the identity in this degree. Therefore, the diagram is commutative. Moreover, the right vertical arrow from (6.9) is an isomorphism by Quillen's isomorphism (3.21), while the lower horizontal arrow is an isomorphism by Theorem 6.1.2.

Recall that, by assumption, $\mathfrak{m} = \widehat{\mathfrak{g}}$; therefore, the inclusion of filtered Lie algebras $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$ induces a morphism between the following two exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathrm{gr}}(\overline{\mathfrak{m}^{(i)}}) & \longrightarrow & \mathrm{gr}(\mathfrak{m}) & \longrightarrow & \mathrm{gr}(\mathfrak{m})/\widetilde{\mathrm{gr}}(\overline{\mathfrak{m}^{(i)}}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle \cong} & & \uparrow & & \\ 0 & \longrightarrow & \mathfrak{g}^{(i)} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{g}^{(i)} & \longrightarrow & 0\,. \end{array} \tag{6.11}$$

Here $\widetilde{\mathrm{gr}}$ means taking the associated graded Lie algebra corresponding to the induced filtration. Using formulas (2.2) and (2.9), it can be shown that $\widetilde{\mathrm{gr}}(\overline{\mathfrak{m}^{(i)}}) = \mathfrak{g}^{(i)}$. Therefore,

the morphism $\mathfrak{g}/\mathfrak{g}^{(i)} \to \operatorname{gr}(\mathfrak{m})/\widetilde{\operatorname{gr}}(\overline{\mathfrak{m}^{(i)}})$ is an isomorphism. We also know that $\operatorname{gr}(\mathfrak{m}/\overline{\mathfrak{m}^{(i)}}) = \operatorname{gr}(\mathfrak{m})/\widetilde{\operatorname{gr}}(\overline{\mathfrak{m}^{(i)}})$. Hence, the map $\operatorname{gr}(\varphi^{(i)})$ is an isomorphism, and so, by (6.10), the map $\Psi_G^{(i)}$ is an isomorphism, too.

By Lemma 2.1.2, the map $\varphi^{(i)}$ induces an isomorphism of complete, filtered Lie algebras between the degree completion of $\mathfrak{g}/\mathfrak{g}^{(i)}$ and $\mathfrak{m}/\overline{\mathfrak{m}^{(i)}}$. As shown above, $\Psi_G^{(i)}$ is an isomorphism; hence, its degree completion is also an isomorphism. Composing with the isomorphism from Theorem 6.1.2, we obtain an isomorphism between the degree completion $\widehat{\operatorname{gr}}(G/G^{(i)};\mathbb{Q})$ and the Malcev Lie algebra $\mathfrak{m}(G/G^{(i)};\mathbb{Q})$. This shows that the solvable quotient $G/G^{(i)}$ is filtered-formal. □

Remark 6.1.6. As shown in [86, §3], the analogue of Theorem 6.1.5 does not hold if the ground field $\mathbb{Q}$ has characteristic $p > 0$. More precisely, there are pro-p groups G for which the morphisms $\Psi_G^{(i)}$ $(i \geq 2)$ are not isomorphisms.

Returning now to the setup from Lemma 3.1.10, let us compose the canonical projection $\operatorname{gr}(q_i)\colon \operatorname{gr}(G;\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G/G^{(i)};\mathbb{Q})$ with the epimorphism $\Phi_G\colon \mathfrak{h}(G;\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G;\mathbb{Q})$. We obtain in this fashion an epimorphism $\mathfrak{h}(G;\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G/G^{(i)};\mathbb{Q})$, which fits into the following commuting diagram:

$$\begin{array}{ccccc} \mathfrak{h}(G) & & \xrightarrow{\Phi_G} & & \operatorname{gr}(G) \\ & \swarrow \quad \searrow & & \swarrow & \downarrow \\ \mathfrak{h}(G/G^{(i)}) & & \operatorname{gr}(G/G^{(i)}) & & \\ & \searrow \quad \nearrow & & \nwarrow & \downarrow \\ \mathfrak{h}(G)/\mathfrak{h}(G)^{(i)} & & \longrightarrow & & \operatorname{gr}(G)/\operatorname{gr}(G)^{(i)}\,. \end{array} \tag{6.12}$$

Putting things together, we obtain the following corollary, which recasts Theorem 4.2 from [117] in a setting which is both functorial, and holds in wider generality. This corollary provides a way to detect non-1-formality of groups.

Corollary 6.1.7. *For each for $i \geq 2$, there is a natural epimorphism of graded $\mathbb{Q}$-Lie algebras,*

$$\Phi_G^{(i)}\colon \mathfrak{h}(G;\mathbb{Q})/\mathfrak{h}(G;\mathbb{Q})^{(i)} \twoheadrightarrow \operatorname{gr}(G/G^{(i)};\mathbb{Q})\,.$$

Moreover, if G is 1-formal, then $\Phi_G^{(i)}$ is an isomorphism.

Corollary 6.1.8. *Suppose the group G is 1-formal. Then, for each for $i \geq 2$, the solvable quotient $G/G^{(i)}$ is graded-formal if and only if $\mathfrak{h}(G;\mathbb{Q})^{(i)}$ vanishes.*

Proof. By Proposition 6.1.1, the canonical projection $q_i\colon G \to G/G^{(i)}$ induces an isomorphism $\mathfrak{h}(q_i)\colon \mathfrak{h}(G;\mathbb{Q}) \to \mathfrak{h}(G/G^{(i)};\mathbb{Q})$. Since we assume G is 1-formal, Corollary 6.1.7 guarantees that the map $\Phi_G^{(i)}\colon \mathfrak{h}(G;\mathbb{Q})/\mathfrak{h}(G;\mathbb{Q})^{(i)} \to \operatorname{gr}(G/G^{(i)};\mathbb{Q})$ is an isomorphism. The claim follows from the left square of diagram (6.12). □

6.2 Chen Lie algebras and Alexander invariants

In this section, we discuss the relationship between the Chen Lie algebra and the Alexander invariant of a finitely generated group.

6.2.1 Alexander invariants

Once again, let G be a finitely generated group. Let us consider the $\mathbb{C}$-vector space $H_1(G',\mathbb{C}) = G'/G'' \otimes \mathbb{C}$. This vector space can be viewed as a (finitely generated) module over the group algebra $\mathbb{C}[H]$, with the abelianization $H = G/G'$ acting on G'/G'' by conjugation. Following [106], we denote this module by $B_{\mathbb{C}}(G)$, or $B(G)$ for short, and call it the *Alexander invariant* of G. We refer to [106, 32, 117, 123] for ways to compute presentations for the module $B(G)$ in various degrees of generality.

The module $B = B(G)$ may be filtered by powers of the augmentation ideal, $I = \ker(\varepsilon\colon \mathbb{C}[H] \to \mathbb{C})$, where ε is the ring map defined by $\epsilon(h) = 1$ for all $h \in H$. The associated graded module,

$$\operatorname{gr}(B) = \bigoplus_{k\geq 0} I^k B/I^{k+1}B, \tag{6.13}$$

then, is a module over the graded ring $\operatorname{gr}(\mathbb{C}[H]) = \bigoplus_{k\geq 0} I^k/I^{k+1}$. We call this module the *associated graded Alexander invariant* of G.

Work of W. Massey [106] implies that the map $j\colon G'/G'' \to G/G''$ restricts to isomorphisms

$$I^k B \longrightarrow \Gamma_{k+2}(G/G'') \tag{6.14}$$

for all $k \geq 0$. Taking successive quotients of the respective filtrations and tensoring with $\mathbb{C}$, we obtain isomorphisms

$$\operatorname{gr}_k(j)\colon \operatorname{gr}_k(B(G)) \longrightarrow \operatorname{gr}_{k+2}(G/G'') \ \text{ for } k \geq 0. \tag{6.15}$$

Consequently, the Chen ranks of G can be expressed in terms of the Hilbert series of the graded module $\operatorname{gr}(B(G))$, as follows:

$$\sum_{k\geq 2} \theta_k(G) \cdot t^{k-2} = \operatorname{Hilb}(\operatorname{gr}(B(G)), t). \tag{6.16}$$

Remark 6.2.1. If the group $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ is a finitely presented, commutator-relators group, then the Hilbert series of the module $\operatorname{gr}(B(G))$ may be computed using the algorithm from [31, 33]. To start with, identify $\mathbb{C}[H]$ with $\Lambda = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. The Alexander invariant of G admits then a finite presentation for the form

$$\Lambda^{\binom{n}{3}} \oplus \Lambda^m \xrightarrow{\delta_3 + \nu_G} \Lambda^{\binom{n}{2}} \longrightarrow B(G) \longrightarrow 0\,, \tag{6.17}$$

Here, δ_i is the i-th differential in the standard Koszul resolution of $\mathbb{C}$ over Λ, and ν_G is a map satisfying $\delta_2 \circ \nu_G = D_G$, where D_G is the abelianization of the Jacobian matrix of Fox derivatives of the relators. Next, one computes a Gröbner basis for the module $B(G)$, in a suitable monomial ordering. An application of the standard tangent cone algorithm yields then a presentation for $\operatorname{gr}(B(G))$, from which ones computes the Hilbert series of $\operatorname{gr}(B(G))$. Finally, the Chen ranks of G are given by formula (6.16).

6.2.2 Another filtration on G'/G''

Next, we compare the module $\mathfrak{B}(\operatorname{gr}(G))$ to another, naturally defined S-module associated to the group G. Let $\operatorname{gr}^{\widetilde{\Gamma}}(G'/G'')$ be the associated graded Lie algebra of G'/G'' with respect

to the induced filtration

$$\widetilde{\Gamma}_k(G'/G'') := (G'/G'') \cap \Gamma_k(G/G''). \tag{6.18}$$

The terms of this filtration fit into short exact sequences

$$0 \longrightarrow \widetilde{\Gamma}_k(G'/G'') \longrightarrow \Gamma_k(G/G'') \longrightarrow \Gamma_k(G/G') \longrightarrow 0\,. \tag{6.19}$$

Noting that $\Gamma_k(G/G') = 0$ for $k \geq 2$, we deduce that

$$\widetilde{\Gamma}_k(G'/G'') = \Gamma_k(G/G''), \text{ for } k \geq 2. \tag{6.20}$$

As before, it is readily checked that the adjoint representation of $\operatorname{gr}_1(G/G'') = H \otimes \mathbb{C}$ on $\operatorname{gr}(G/G'')$ induces an S-action on $\operatorname{gr}^{\widetilde{\Gamma}}(G'/G'')$, preserving the grading. Hence, the Lie algebra $\mathfrak{C}(G) := \operatorname{gr}^{\widetilde{\Gamma}}(G'/G'')$ can also be viewed as a graded module over S, by setting

$$\mathfrak{C}(G)_k = \operatorname{gr}^{\widetilde{\Gamma}}_{k+2}(G'/G''). \tag{6.21}$$

Proposition 6.2.2. *The canonical morphism of graded Lie algebras* $\Psi\colon \operatorname{gr}(G)/\operatorname{gr}(G)'' \twoheadrightarrow \operatorname{gr}(G/G'')$ *from Theorem 6.1.5 induces an epimorphism of* S*-modules,*

$$\varphi\colon \mathfrak{B}(\operatorname{gr}(G)) \twoheadrightarrow \mathfrak{C}(G)\,.$$

Moreover, if G *is filtered-formal, then* φ *is an isomorphism.*

Proof. The map Φ fits into the following commutative diagram of graded Lie algebras,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{gr}(G)'/\operatorname{gr}(G)'' & \longrightarrow & \operatorname{gr}(G)/\operatorname{gr}(G)'' & \longrightarrow & \operatorname{gr}(G)/\operatorname{gr}(G)' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \operatorname{gr}^{\widetilde{\Gamma}}(G'/G'') & \xrightarrow{\operatorname{gr}(j)} & \operatorname{gr}(G/G'') & \longrightarrow & \operatorname{gr}(G/G') & \longrightarrow & 0. \end{array} \tag{6.22}$$

Thus, Ψ induces a morphism of graded Lie algebras, $\varphi\colon \operatorname{gr}(G)'/\operatorname{gr}(G)'' \to \operatorname{gr}^{\widetilde{\Gamma}}(G'/G'')$, as indicated above. By the Five Lemma, φ is surjective. Observe that $\operatorname{gr}(G)/\operatorname{gr}(G)' \cong \operatorname{gr}(G/G') \cong H \otimes \mathbb{C}$ acts on both the source and target of φ by adjoint representations. Hence,

upon regrading according to (5.16) and (6.21), the map $\varphi\colon \mathfrak{B}(\mathrm{gr}(G)) \twoheadrightarrow \mathfrak{C}(G)$ becomes a morphism of S-modules.

If G is filtered-formal, then, according to Theorem 6.1.5, the map Ψ is an isomorphism of graded Lie algebras. Hence, the induced map φ is an isomorphism of S-modules. □

6.2.3 Comparison with the associated graded Alexander invariant

Finally, we identify the associated graded Alexander invariant $\mathrm{gr}(B(G))$ with the S-module $\mathfrak{C}(G)$ defined above. To do that, we first identify the respective ground rings.

Choose a basis $\{x_1, \dots, x_n\}$ for the torsion-free part of $H = G_{\mathrm{ab}}$. We may then identify the group algebra $\mathrm{gr}(\mathbb{C}[H])$ with the polynomial algebra $R = \mathbb{C}[s_1, \dots, s_n]$, where s_i corresponds to $\overline{x_i - 1} \in I/I^2$, see Quillen [132]. On the other hand, we may also identify the symmetric algebra $S = \mathrm{Sym}(H \otimes \mathbb{C})$ with the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$. The desired ring isomorphism, $R \cong S$, is gotten by sending s_i to x_i.

Proposition 6.2.3. *Under the above identification $R \cong S$, the graded R-module $\mathrm{gr}(B(G))$ is canonically isomorphic to the graded S-module $\mathfrak{C}(G)$.*

Proof. Recall from §6.2.1 that the inclusion map $j\colon G'/G'' \to G/G''$ restricts to an isomorphism $I^k B(G) \to \Gamma_{k+2}(G/G'')$ for each $k \geq 0$. Using the induced filtration $\widetilde{\Gamma}$ from (6.18) and the identification (6.20), we obtain $\mathbb{C}$-linear isomorphisms $I^k B(G) \cong \widetilde{\Gamma}_{k+2}(G'/G'')$, for all $k \geq 0$. Taking the successive quotients of the respective filtrations and regrading according to (6.21), we obtain a $\mathbb{C}$-linear isomorphism $\mathrm{gr}(B(G)) \cong \mathfrak{C}(G)$.

Under the identification $\mathrm{gr}(\mathbb{C}[H]) \cong R$, the associated graded Alexander invariant of G may be viewed as a graded R-module, with R-action defined by

$$s_i(\overline{z}) = (\overline{x_i - 1})\overline{z} = \overline{x_i z x_i^{-1} - z} = \overline{[x_i, z]z - z} = \overline{[x_i, z]} + \overline{z} - \overline{z} = \overline{[x_i, z]}, \tag{6.23}$$

for all $z \in G'$. (In this computation, we follow the convention from [106], and view the Alexander invariant $B(G) = G'/G''$ as an additive group; however, when we consider the

induced filtration $\widetilde{\Gamma}$ on G'/G'', we view it as a multiplicative subgroup of G/G''.)

Finally, recall that $\mathfrak{C}_\bullet(G) = \mathrm{gr}^{\widetilde{\Gamma}}_{\bullet-2}(G'/G'')$ is an S-module, with S-action given by $x_i(\overline{z}) = \overline{[x_i, z]}$. Hence, the aforementioned isomorphism $R \cong S$ identifies the R-module $\mathrm{gr}(B(G))$ with the S-module $\mathfrak{C}(G)$. □

6.2.4 Discussion

In the 1-formal case, we obtain the following corollary, which can also be deduced from [42, Theorem 5.6].

Corollary 6.2.4. *Let G be a 1-formal group. Then $\mathrm{gr}(B(G)) \cong \mathfrak{B}(G)$, as modules over the polynomial ring $S = \mathrm{gr}(\mathbb{C}[H])$.*

Proof. Follows at once from Propositions 5.2.2, 6.2.2, and 6.2.3. □

Using those propositions once again, we obtain another corollary.

Corollary 6.2.5. *Let G be a finitely generated group. The following then hold.*

1. *$\theta_k(\mathrm{gr}(G)) \leq \theta_k(\mathfrak{h}(G))$, with equality if $k \leq 2$, or if G is graded-formal.*

2. *$\theta_k(G) \leq \theta_k(\mathrm{gr}(G))$, with equality if $k \leq 3$, or if G is filtered-formal.*

The graded-formality assumption from part (1) of the above corollary is clearly necessary for the equality $\theta_k(\mathrm{gr}(G)) = \theta_k(\mathfrak{h}(G))$ to hold for all k. On the other hand, it is not clear whether the filtered-formality hypothesis from part (2) is necessary for the equality $\theta_k(G) = \theta_k(\mathrm{gr}(G))$ to hold in general. In view of several computations (some of which are summarized in the next section), we are led to formulate the following question. Suppose G is a graded-formal group. Does the equality $\theta_k(G) = \theta_k(\mathrm{gr}(G))$ hold for all k?

6.3 Resonance varieties and Chen ranks

In this section, we detect the relationship among the Chen ranks, the Alexander invariants and the resonance varieties.

6.3.1 Chen ranks and Alexander invariants

Let G be a finitely generated group. The terms of the lower central series of G are defined inductively by $\Gamma_1 G = G$ and $\Gamma_i G = [G, \Gamma_{i-1} G]$ for $i \geq 2$. The *associated graded Lie algebra* of G is the locally finite graded vector space

$$\operatorname{gr}(G) = \bigoplus_{i \geq 1} (\Gamma_i G / \Gamma_{i+1} G) \otimes \mathbb{C}, \tag{6.24}$$

with Lie bracket $[\,,\,]\colon \operatorname{gr}_i(G) \times \operatorname{gr}_j(G) \to \operatorname{gr}_{i+j}(G)$ induced by the group commutator.

Following [27, 28], let us define the *Chen ranks* of G as the LCS ranks of G/G'', the quotient of G by its second derived subgroup:

$$\theta_k(G) := \dim_{\mathbb{C}}(\operatorname{gr}_k(G/G'')). \tag{6.25}$$

For the free group of rank n, Chen showed that

$$\theta_k(F_n) = (k-1)\binom{n+k-2}{k} \quad \text{for all } k \geq 2. \tag{6.26}$$

The (complex) *Alexander invariant* of G is defined as $B(G) := (G'/G'') \otimes \mathbb{C}$, with G/G' acting on the cosets of G'' via conjugation, that is, $gG' \cdot hG'' = ghg^{-1}G''$, for $g \in G$, $h \in G'$. Hence, $B = B(G)$ can be viewed as a module over the ring $R = \mathbb{C}[G/G']$, and may be filtered by the powers of the augmentation ideal $I := \ker(\varepsilon\colon R \to \mathbb{C})$, where $\epsilon(\sum n_g g) = \sum n_g$.

Let $\operatorname{gr}_I(B) := \bigoplus_{r \geq 0} I^r B / I^{r+1} B$ be the associated graded object, viewed as a graded module over the ring $S := \operatorname{gr}_I(R) \cong \operatorname{Sym}(G/G' \otimes \mathbb{C})$. In [106], W. Massey expressed the Chen ranks of G in terms of the Hilbert series of the Alexander invariant of G, as follows:

$$\sum_{k \geq 0} \theta_{k+2}(G) \cdot t^k = \operatorname{Hilb}(\operatorname{gr}_I(B(G)), t), \tag{6.27}$$

6.3.2 Chen ranks and resonance varieties

Let G be a finitely generated group, and suppose the cohomology algebra $A = H^*(G;\mathbb{C})$ is locally finite. The resonance varieties of the group G are defined to be $\mathcal{R}^i_t(G) := \mathcal{R}^i_t(A)$. The first resonance variety of G can be described as,

$$\mathcal{R}_d(G) = \{a \in H^1(G,\mathbb{C}) \mid H^1(A,\mathbb{C});\delta_a) \geq d\}. \tag{6.28}$$

Suppose G is a finitely presented, commutator-relators group. As shown in [110], for each $d \geq 1$, the resonance variety $R^1_d(G)$ coincides, at least away from the origin $0 \in H^1(G;\mathbb{C})$, with the support variety of the annihilator of d-th exterior power of the infinitesimal Alexander invariant; that is,

$$\mathcal{R}^1_d(G) = V\Big(\operatorname{Ann}\Big(\bigwedge^d \mathfrak{B}(\mathfrak{h}(G))\Big)\Big). \tag{6.29}$$

The first author conjectured in [140] that for $k \gg 0$, the Chen ranks of an arrangement group G are given by the *Chen ranks formula*

$$\theta_k(G) = \sum_{m\geq 2} h_m(G)\cdot\theta_k(F_m), \tag{6.30}$$

where $h_m(G)$ is the number of m-dimensional irreducible components of $\mathcal{R}^1_1(G)$. A positive answer to this conjecture was given in [34] for a class of 1-formal groups which includes arrangement groups.

Using Hilbert series of the Alexander invariants, the Chen ranks formula (6.30) translates into the equivalent statement that

$$\operatorname{Hilb}(\operatorname{gr}(B(G)),t) - \sum_{m\geq 2} h_m(G)\cdot\operatorname{Hilb}(\operatorname{gr}(B(F_m)),t) \tag{6.31}$$

is a polynomial (in the variable t).

Recently, D. Cohen and Schenck proved the Chen ranks conjecture. To state this result, recall that a subspace $U \subset H^1(G;\mathbb{C})$ is called *isotropic* if the cup product $U\wedge U \to H^2(G;\mathbb{C})$ is the zero map.

Theorem 6.3.1 ([34]). *Let G be a finitely presented, commutator-relators 1-formal group. Assume that the components of $\mathcal{R}_1(G)$ are isotropic, projectively disjoint, and reduced as a scheme. Then, for $k \gg 0$, the Chen ranks of G are given by formula 6.30.*

Proposition 6.3.2. *Theorem 6.3.1 is still true without the "commutator-relators" assumption.*

Proof. Let $G = F/R$ be a group satisfying all assumptions of Theorem 6.3.1 except for the requirement that $R \subset [F, F]$. By Corollary 4.3.4, there exists a commutator-relators group G_c, such that $\mathfrak{h}(G) \cong \mathfrak{h}(G_c)$. It follows that $\mathfrak{B}(G) \cong \mathfrak{B}(G_c)$; thus, $\mathcal{R}_1(G) \cong \mathcal{R}_1(G_c)$, and so $h_m(G) = h_m(G_c)$ for all $m > 0$. The group G_c may not be 1-formal. However, from [34], we have that

$$\bar{\theta}_k(G_c) = \sum_{m \geq 2} h_m(G_c) \cdot \theta_k(F_m),$$

where $\bar{\theta}_k(G_c) := \dim_k(\mathfrak{h}(G_c)/\mathfrak{h}(G_c)'')$ are the holonomy Chen ranks of G. By Corollary 6.2.4, we have that $\operatorname{gr}(B(G)) \cong \mathfrak{B}(G)$, for any finitely generated 1-formal group. Hence

$$\theta_k(G) = \bar{\theta}_k(G) = \bar{\theta}_k(G_c) = \sum_{m \geq 2} h_m(G_c) \cdot \theta_k(F_m) = \sum_{m \geq 2} h_m(G) \cdot \theta_k(F_m).$$

This finishes the proof. □

Example 6.3.3. The pure braid group P_n is an arrangement group, and thus satisfies the hypothesis of Theorem 6.3.1. In fact, we know from Proposition 7.3.1 that the resonance variety $\mathcal{R}^1_1(P_n)$ has $\binom{n}{3} + \binom{n}{4} = \binom{n+1}{4}$ irreducible components, all of dimension 2. Thus, the computation from (7.18) agrees with the one predicted by formula (6.30), for all $k \geq 3$.

As noted in [34], it is easy to find examples of non-1-formal groups for which the Chen ranks formula does not hold. For instance, if $G = F_2/\Gamma_3(F_2)$ is the Heisenberg group, then $\mathcal{R}^1_1(G) = H^1(G, \mathbb{C}) = \mathbb{C}^2$, and thus formula (6.30) would predict in this case that $\theta_k(G) = \theta_k(F_2)$ for k large enough, where in reality $\theta_k(G) = 0$ for $k \geq 3$. On the other hand, here is an example of a finitely presented, commutator-relators group which satisfies the Chen ranks formula, yet which is *not* 1-formal.

Example 6.3.4. Using the notation from [110], let $\mathcal{A} = \mathcal{A}(31425)$ be the 'horizontal' arrangement of 2-planes in $\mathbb{R}^4$ determined by the specified permutation, and let G be the fundamental group of its complement. From [110, Example 6.5], we know that $\mathcal{R}^1_1(G)$ is an irreducible cubic hypersurface in $H^1(G, \mathbb{C}) = \mathbb{C}^5$. Hence, by Theorem 5.1.2, the group G is not 1-formal (for an alternative argument, see [42, Example 8.2]). On the other hand, the singularity link determined by $\mathcal{A}$ has all linking numbers ± 1, and thus satisfies the Murasugi Conjecture, that is, $\theta_k(G) = \theta_k(F_4)$, for all $k \geq 2$, see [108, 117]. Therefore, the Chen ranks formula holds for the group G.

6.3.3 Products and coproducts

We now analyze the way the Chen ranks formula (6.30) behaves under (finite) products and coproducts of groups.

Lemma 6.3.5. *Let G_1 and G_2 be two finitely generated groups. The number of m-dimensional irreducible components of the corresponding first resonance varieties satisfies the following additivity formula,*

$$h_m(G_1 \times G_2) = h_m(G_1) + h_m(G_2). \tag{6.32}$$

Proof. We start by identifying the affine space $H^1(G_1 \times G_2; \mathbb{C})$ with $H^1(G_1; \mathbb{C}) \times H^1(G_2; \mathbb{C})$. Next, by Proposition 5.1.3, we have that

$$\mathcal{R}^1_1(G_1 \times G_2) = \mathcal{R}^1_1(G_1) \times \{0\} \cup \{0\} \times \mathcal{R}^1_1(G_2). \tag{6.33}$$

Suppose $\mathcal{R}^1_1(G_1) = \bigcup_{i=1}^s A_i$ and $\mathcal{R}^1_1(G_2) = \bigcup_{j=1}^t B_j$ are the decompositions into irreducible components for the respective varieties. Then $A_i \times \{0\}$ and $\{0\} \times B_j$ are irreducible subvarieties of $\mathcal{R}^1_1(G_1 \times G_2)$. Observe now that $\mathcal{R}^1_1(G_1) \times \{0\}$ and $\{0\} \times \mathcal{R}^1_1(G_2)$ intersect only at 0. It follows that

$$\mathcal{R}^1_1(G_1 \times G_2) = \bigcup_{i=1}^{s} A_i \times \{0\} \cup \bigcup_{j=1}^{t} \{0\} \times B_j \tag{6.34}$$

is the irreducible decomposition for the first resonance variety of $G_1 \times G_2$. The claimed additivity formula follows. □

Corollary 6.3.6. *If both G_1 and G_2 satisfy the Chen ranks formula, then $G_1 \times G_2$ also satisfies the Chen ranks formula.*

Proof. Follows at once from formulas (6.7) and (6.32). □

However, even if both G_1 and G_2 satisfy the Chen ranks formula, the free product $G_1 * G_2$ may not satisfy this formula. We illustrate this phenomenon with an infinite family of examples.

Example 6.3.7. Let $G_n = \mathbb{Z} * \mathbb{Z}^{n-1}$. Clearly, both factors of this free product satisfy the Chen ranks formula; in fact, both factors satisfy the hypothesis of Theorem 6.3.1. Moreover, G_n is 1-formal and $\mathcal{R}^1(G_n)$ is projectively disjoint and reduced as a scheme. Using Theorem 4.1(3) and Lemma 6.2 from [118], a short computation reveals that

$$\sum_{k\geq 2} \theta_k(G_n) t^k = t\, \frac{1-(1-t)^{n-1}}{(1-t)^n}. \tag{6.35}$$

On the other hand, if $n \geq 2$, then $\mathcal{R}^1_1(G_n) = H^1(G_n, \mathbb{C})$, by Proposition 5.1.4. Thus, formula (6.30) would say that $\theta_k(G_n) = \theta_k(F_n)$ for $k \gg 0$. However, comparing formulas (7.16) and (6.35), we find that

$$\theta_k(F_n) - \theta_k(G_n) = \sum_{i=2}^{k} \theta_i(F_{n-1}). \tag{6.36}$$

Hence, if $n \geq 3$, the group G_n does *not* satisfy the Chen ranks formula. Note that G_n also does not satisfy the isotropicity hypothesis of Theorem 6.3.1, since the restriction of the cup product to the factor $\mathbb{Z}^{n-1}$ is nonzero, again provided that $n \geq 3$.

Chapter 7

Pure virtual braid groups

The virtual braid groups were first introduced by Kauffman in 1999 in the context of the study of virtual knot theory. As in the theory of the classical braid groups, the kernel of the canonical epimorphism from virtual braid group to the corresponding symmetric group is called a pure virtual braid group. Bardakov gave presentations for the pure virtual braid groups and defined a class of subgroups known as the upper pure virtual braid groups. Bartholdi, Enriquez, Etingof, and Rains independently defined and studied these groups, which they called the quasitriangular groups and triangular groups respectively, as a group-theoretic version of the set of solutions to the Yang–Baxter equations. In this chapter, we investigate the resonance varieties, lower central series ranks, and Chen ranks of the pure virtual braid groups and their upper-triangular subgroups. As an application, we give a complete answer to the 1-formality question for this class of groups. This chapter is based on the work in my papers [143, 145] with Alex Suciu.

7.1 Pure braid groups and pure virtual braid groups

In this section, we look at the pure virtual braid groups and their upper-triangular subgroups from the point view of combinatorial group theory.

7.1.1 Backgrounds of virtual braids

Virtual knot theory, as introduced by Kauffman in [78], is an extension of classical knot theory. This new theory studies embeddings of knots in thickened surfaces of arbitrary genus, while the classical theory studies the embeddings of circles in thickened spheres. Another motivation comes from the representation of knots by Gauss diagrams. In [62], Goussarov, Polyak, and Viro showed that the usual knot theory embeds into virtual knot theory, by realizing any Gauss diagram by a virtual knot. Many knot invariants, such as knot groups, the bracket polynomial, and finite-type Vassiliev invariants can be extended to invariants of virtual knots, see [78, 62].

The virtual braid groups vB_n were introduced in [78] and further studied in [6, 9, 79, 7, 76]. As shown by Kamada in [76], any virtual link can be constructed as the closure of a virtual braid, which is unique up to certain Reidemeister-type moves. In this paper, we will be mostly interested in the kernel of the canonical epimorphism $vB_n \to S_n$, called the *pure virtual braid group*, vP_n, and a certain subgroup of this group, vP_n^+, which we call the *upper pure virtual braid group*.

In [11], Bartholdi, Enriquez, Etingof, and Rains independently defined and studied the group vP_n, which they called the n-th quasitriangular groups QTr_n, as a group-theoretic version of the set of solutions to the Yang–Baxter equations. Their work was developed in a deep way by P. Lee in [91]. The authors of [11, 91] construct a classifying space for vP_n with finitely many cells, and find a presentation for the cohomology algebra of vP_n, which they show is a Koszul algebra. They also obtain parallel results for a quotient group of vP_n, called the the n-th triangular group, which has the same generators as vP_n, and one more set of relations. It is readily seen that the triangular group Tr_n is isomorphic to vP_n^+.

7.1.2 Presentations and classifying spaces of P_n

We first review the results of the pure braid groups P_n. Let $\mathrm{Aut}(F_n)$ be the group of (right) automorphisms of the free group F_n on generators $x_1, \dots, x_n$. Magnus [101] showed that the map $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ which sends an automorphism to the induced map on the abelianization $(F_n)^{\mathrm{ab}} = \mathbb{Z}^n$ is surjective, with kernel denoted by IA_n. Furthermore, the kernel of this homomorphism, denoted by IA_n, is generated by automorphisms α_{ij} and α_{ijk} $(1 \leq i \neq j \neq k \leq n)$ which send x_i to $x_j x_i x_j^{-1}$ and $x_i x_j x_k x_j^{-1} x_k^{-1}$, respectively, and leave invariant the remaining generators of F_n.

An automorphism of the free group F_n is called a symmetric automorphism if it sends each generator x_i to a conjugate of $x_{\sigma(i)}$, for some permutation $\sigma \in \Sigma_n$. Recall that the Artin braid group B_n consists of those permutation-conjugacy automorphisms which fix the word $x_1 \cdots x_n \in F_n$. For each $1 \leq i < n$, let σ_i be the braid automorphism which sends x_i to $x_i x_{i+1} x_i^{-1}$ and x_{i+1} to x_i, while leaving the other generators of F_n fixed. As shown for instance in [18], the braid group B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$, subject to the well-known relations

$$\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & 1 \leq i \leq n-2, \\ \sigma_i \sigma_j = \sigma_j \sigma_i, & |i-j| \geq 2. \end{cases} \tag{R1}$$

On the other hand, the symmetric group S_n has a presentation with generators s_i for $1 \leq i \leq n-1$ and relations

$$\begin{cases} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & 1 \leq i \leq n-2, \\ s_i s_j = s_j s_i, & |i-j| \geq 2. \\ s_i^2 = 1, & 1 \leq i \leq n-1; \end{cases} \tag{R2}$$

The canonical projection from the braid group to the symmetric group, which sends the elementary braid σ_i to the transposition s_i, has kernel the *pure braid group* on n strings, $P_n = \ker(B_n \twoheadrightarrow S_n) = B_n \cap \mathrm{IA}_n$. A generating set for P_n are the n-braids

$$A_{ij} = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}$$

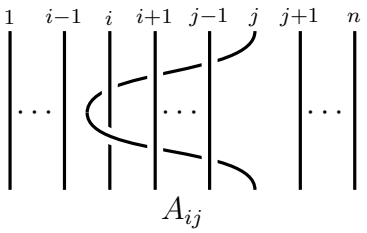


Figure 7.1: The pure braids A_{ij} for $i < j$.

for $1 \leq i < j \leq n$, (see Figure 7.1,) subject to the relations (see, e.g., [69])

$$A_{ij}^{A_{rs}} = \begin{cases} A_{ij} & \text{if } i < r < s < j \text{ or } r < s < i < j \\ A_{ij}^{A_{rj}^{-1}} & \text{if } r < i = s < j \\ A_{ij}^{A_{sj}^{-1} A_{rj}^{-1}} & \text{if } i = r < s < j \\ A_{ij}^{A_{sj} A_{rj} A_{sj}^{-1} A_{rj}^{-1}} & \text{if } r < i < s < j. \end{cases} \tag{7.1}$$

where $x^y := y^{-1}xy$ is the conjugation in a group.

As is well-known, the center of P_n $(n \geq 2)$ is infinite cyclic, and so we have a direct product decomposition of the form $P_n \cong \overline{P}_n \times \mathbb{Z}$. The first few groups in this series are easy to describe: $P_1 = \{1\}$, $P_2 = \mathbb{Z}$, and $P_3 \cong F_2 \times \mathbb{Z}$, where F_n denotes the free group on n generators.

The *configuration space* of n ordered points in a connected manifold M is defined to be

$$\mathrm{Conf}_n(M) := \{(x_1, \cdots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}. \tag{7.2}$$

There is a natural free right action of S_n on the configuration space $\mathrm{Conf}_n(M)$,

$$\mu : \mathrm{Conf}_n(M) \times S^n \to \mathrm{Conf}_n(M),$$

defined by permutation of coordinates, $\mu((x_1, \cdots, x_n), \alpha) = (x_1, \cdots, x_n) \cdot \alpha = (x_{\alpha(1)}, \cdots, x_{\alpha(n)})$. Denote the orbit space for the free action μ by $C_n(M) = \mathrm{Conf}_n(M)/S_n$. The configuration space $\mathrm{Conf}_n(\mathbb{C})$ is a classifying space for the Artin pure braid group P_n, while the space $C_n(\mathbb{C})$ is a classifying space for the Artin braid group B_n. See [69, §1.3] for more details.

7.1.3 Presentations and classifying spaces of vP_n

As shown by Bardakov in [7], the virtual braid group vB_n has presentation with generators σ_i and s_i for $i = 1, \ldots, n-1$, and relations (R1), (R2) and

$$\begin{cases} s_i\sigma_j = \sigma_j s_i, & |i-j| \geq 2, \\ \sigma_i s_{i+1} s_i = s_{i+1} s_i \sigma_{i+1}, & 1 \leq i \leq n-2, \end{cases} \tag{R3}$$

The *pure virtual braid group* vP_n has presentation with generators x_{ij} with $1 \leq i \neq j \leq n$ (see Figure 7.1 for a description of the corresponding virtual braids), and relations

$$\begin{aligned} x_{ij}x_{ik}x_{jk} &= x_{jk}x_{ik}x_{ij} && \text{for distinct } i,j,k, \\ x_{ij}x_{kl} &= x_{kl}x_{ij} && \text{for distinct } i,j,k,l. \end{aligned} \tag{7.3}$$

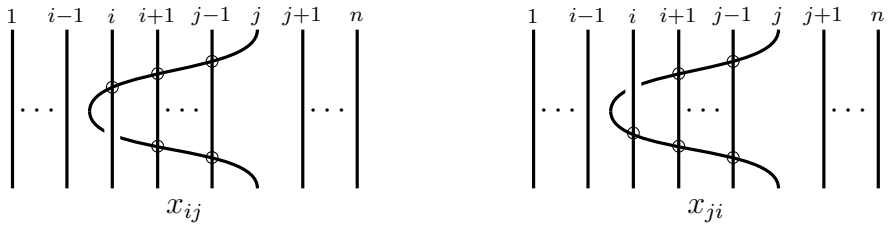


Figure 7.2: The virtual pure braids x_{ij} and x_{ji} for $i < j$.

The *upper-triangular pure virtual braid group*, vP_n^+, is the subgroup of vP_n generated by those elements x_{ij} with $1 \leq i < j \leq n$. Its defining relations are

$$\begin{aligned} x_{ij}x_{ik}x_{jk} &= x_{jk}x_{ik}x_{ij} && \text{for } i < j < k, \\ x_{ij}x_{kl} &= x_{kl}x_{ij} && \text{for } i \neq j \neq k \neq l,\ i < j, \text{ and } k < l. \end{aligned} \tag{7.4}$$

Of course, both vP_1 and vP_1^+ are the trivial group. It is readily seen that $vP_2^+ = \mathbb{Z}$, while $vP_3^+ \cong \mathbb{Z} * \mathbb{Z}^2$. Likewise, vP_2 is isomorphic to F_2.

A classifying space for the group vP_n^+ is identified in [11] as the quotient space of the $(n-1)$-dimensional permutahedron $\mathbf{Per}_n$ by actions of certain symmetric groups. More precisely, let $\mathbf{Per}_n$ be the convex hull of the orbit of a generic point in $\mathbb{R}^n$ under the permutation

action of the symmetric group S_n on its coordinates. Then $\mathbf{Per}_n$ is a polytope whose faces are indexed by all ordered partitions of the set $[n] = \{1, \ldots, n\}$; see Figure 7.3. For each $r \in [n]$, there is a natural action of S_r on the disjoint union of all $(n-r)$-dimensional faces, $C_1 \sqcup \cdots \sqcup C_r$. Similarly, a classifying space for vP_n can be constructed as a quotient space of $\mathbf{Per}_n \times S_n$.

7.1.4 Split monomorphisms

In [11], the group vP_n is called the *quasi-triangular group*, and is denoted by QTr_n, while the quotient group of QTr_n by the relations of the form $x_{ij} = x_{ji}$ for $i \neq j$ is called the *triangular group*, and is denoted by Tr_n.

Lemma 7.1.1. *The group Tr_n is isomorphic to vP_n^+.*

Proof. Let $\phi\colon \mathrm{Tr}_n \to vP_n^+$ be the homomorphism defined by $\phi(x_{ij}) = x_{ij}$ for $i < j$ and $\phi(x_{ij}) = x_{ji}$ for $i > j$, and let $\psi\colon vP_n^+ \to \mathrm{Tr}_n$ be the homomorphism defined by $\psi(x_{ij}) = x_{ij}$ for $i < j$. It is easy to show that ϕ and ψ are well-defined homomorphisms, and $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$. Thus, ϕ is an isomorphism. □

Corollary 7.1.2. *The inclusion $j_n\colon vP_n^+ \hookrightarrow vP_n$ is a split monomorphism.*

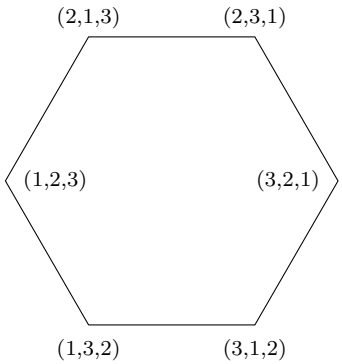
Proof. The split surjection is defined by the composition of the quotient surjection $vP_n \twoheadrightarrow \mathrm{Tr}_n$ and the map $\phi\colon \mathrm{Tr}_n \to vP_n^+$ from Lemma 7.1.1. □

There are several other split monomorphisms between the aforementioned groups.

Lemma 7.1.3. *For each $n \geq 2$, there are split monomorphisms $\iota_n\colon vP_n \to vP_{n+1}$ and $\iota_n^+\colon vP_n^+ \to vP_{n+1}^+$.*

Proof. The maps ι_n and ι_n^+ are defined by sending the generators of vP_n to the generators of vP_{n+1} with the same indices. The split surjection $\pi_n\colon vP_{n+1} \twoheadrightarrow vP_n$ sends x_{ij} to zero for $i = n+1$ or $j = n+1$, and sends x_{ij} to x_{ij} otherwise. The split surjection $\pi_n^+\colon vP_{n+1}^+ \twoheadrightarrow vP_n^+$ is defined similarly. □

Per$_3$



Per$_4$

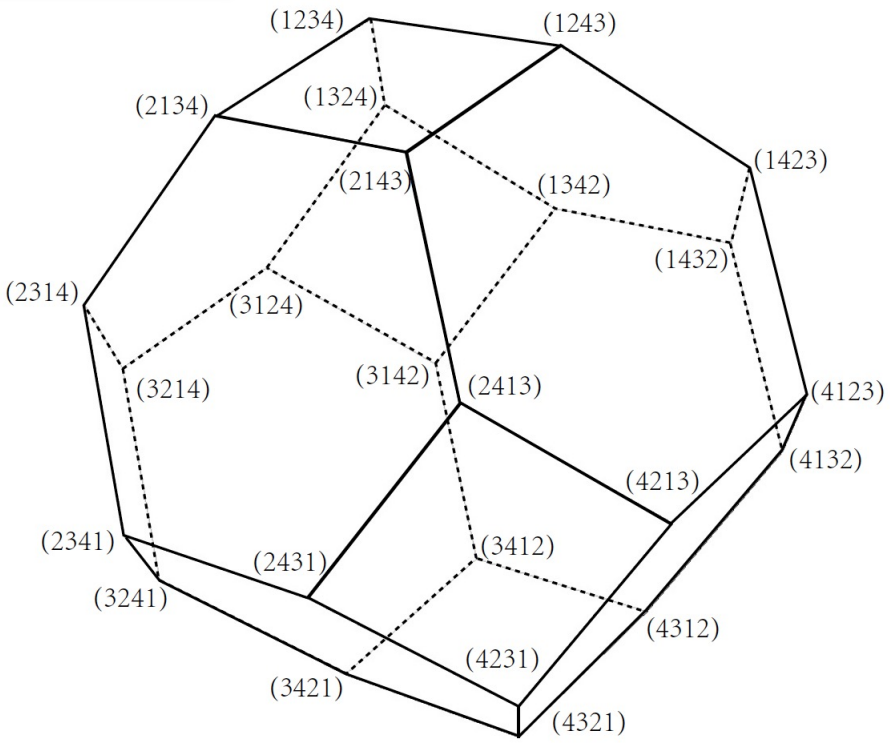


Figure 7.3: Permutahedrons.

As noted by Bardakov in [7, Lemma 6], the pure virtual braid group vP_n admits a semi-direct product decomposition of the form $vP_n \cong F_{q(n)} \rtimes vP_{n-1}$, where $q(1) = 2$ and $q(n)$ is infinite for $n \geq 2$. Furthermore, as shown in [11], there exits a monomorphism from P_n to vP_n.

7.1.5 A free product decomposition for vP_3

The pure virtual braid group vP_3 is generated by $x_{12}, x_{21}, x_{13},\ x_{31}, x_{23}, x_{32}$, subject to the relations

$$x_{12}x_{13}x_{23} = x_{23}x_{13}x_{12}, \qquad x_{21}x_{23}x_{13} = x_{13}x_{23}x_{21}, \qquad x_{13}x_{12}x_{32} = x_{32}x_{12}x_{13},$$

$$x_{31}x_{32}x_{12} = x_{12}x_{32}x_{31}, \qquad x_{23}x_{21}x_{31} = x_{31}x_{21}x_{23}, \qquad x_{32}x_{31}x_{21} = x_{21}x_{31}x_{32}.$$

The next lemma gives a free product decomposition for this group, which will play an important role in the proof that vP_3 is 1-formal.

Lemma 7.1.4. *There is a free product decomposition $vP_3 \cong \overline{P}_4 * \mathbb{Z}$.*

Proof. As shown in [10], if we set $a_1 = x_{13}x_{23}$, $b_1 = x_{13}x_{12}$, $b_2 = x_{21}x_{31}$, $a_2 = x_{32}x_{31}, c_1 = x_{13}x_{31}$, $c_2 = x_{13}$, then there is a free product decomposition, $vP_3 \cong G_3 * \mathbb{Z}$, where G_3 is generated by $\{a_1, a_2, b_1, b_2, c_1\}$, subject to the relations

$$[a_1, b_1] = [a_2, b_2] = 1, \quad b_1^{c_1} = b_1^{a_2}, \quad a_1^{c_1} = a_1^{b_2}, \quad b_2^{c_1} = b_2^{a_1 b_2}, \quad a_2^{c_1} = a_2^{b_1 a_2},$$

where $y^x = x^{-1}yx$. Replacing the generators in the presentation of G_3 by x_1, x_2, x_3, x_4, x_5^{-1}, respectively, and simplifying the relations, we obtain a new presentation for the group G_3, with generators $x_1, \ldots, x_5$ and relations

$$x_1x_3 = x_3x_1,\ x_2x_4 = x_4x_2,\ x_5x_3x_2 = x_3x_2x_5 = x_2x_5x_3,\ x_1x_4x_5 = x_4x_5x_1 = x_5x_1x_4.$$

On the other hand, as noted for instance in [31], the group $\overline{P}_4$ has a presentation with generators $z_1, \ldots, z_5$ and relations

$$z_2z_3 = z_3z_2,\ z_2^{-1}z_4z_2z_1 = z_1z_2^{-1}z_4z_2,\ z_5z_3z_1 = z_3z_1z_5 = z_1z_5z_3,\ z_5z_4z_2 = z_4z_2z_5 = z_2z_5z_4.$$

Define a homomorphism $\phi\colon G_3 \to \overline{P}_4$ by sending $x_1 \mapsto z_2$, $x_2 \mapsto z_1$, $x_3 \mapsto z_3$, $x_4 \mapsto z_2^{-1} z_4 z_2$ and $x_5 \mapsto z_5$. A routine check shows that ϕ is a well-defined homomorphism, with inverse $\psi\colon \overline{P}_4 \to G_3$ sending $z_1 \mapsto x_2$, $z_2 \mapsto x_1$, $z_3 \mapsto x_3$, $z_4 \mapsto x_1 x_4 x_1^{-1}$, and $z_5 \mapsto x_5$. This completes the proof. □

As a quick application of this lemma, we obtain the following corollary, which was first proved by Bardakov et al. [10] using a different method.

Corollary 7.1.5. *The pure virtual braid group vP_3 is a residually torsion-free nilpotent group.*

Proof. It is readily seen that two groups G_1 and G_2 are residually torsion-free nilpotent if and only their direct product, $G_1 \times G_2$, is residually torsion-free nilpotent. Now, as shown by Falk and Randell in [49], the pure braid groups P_n are residually torsion-free nilpotent. Hence, the subgroup $\overline{P}_4 \subset P_4$ is also residually torsion-free nilpotent.

On the other hand, Malcev [103] showed that if G_1 and G_2 are residually torsion-free nilpotent groups, then the free product $G_1 * G_2$ is also residually torsion-free nilpotent. The claim follows from the decomposition $vP_3 \cong \overline{P}_4 * \mathbb{Z}$. □

A more general question was asked by Bardakov and Bellingeri in [8]: Are the groups vP_n or vP_n^+ residually torsion-free nilpotent?

7.2 Cohomology rings and Hilbert series of P_n and vP_n

In this section we discuss what is known about the cohomology rings of the pure (virtual) braid groups, and the corresponding Hilbert series.

7.2.1 Hilbert series and generating functions

Recall that the (ordinary) *generating function* for a sequence of power series $\mathbf{P} = \{p_n(t)\}_{n\geq 1}$ is defined by $F(u,t) := \sum_{n=0}^{\infty} p_n(t) u^n$. Likewise, the *exponential generating function* for $\mathbf{P}$

is defined by $E(u,t) := \sum_{n=0}^{\infty} p_n(t)\frac{u^n}{n!}$.

Now let $\mathbf{G} = \{G_n\}_{n\geq 1}$ be a sequence of groups admitting classifying spaces $K(G_n, 1)$ with finitely many cells in each dimension. We then define the exponential generating function for the corresponding Poincaré polynomials by

$$\mathrm{Poin}(\mathbf{G}, u, t) := 1 + \sum_{n=1}^{\infty} \mathrm{Poin}(G_n, t)\frac{u^n}{n!}. \tag{7.5}$$

In particular, if we set $t = -1$, we obtain the exponential generating function for the Euler characteristics of the groups G_n, denoted by $\chi(\mathbf{G})$.

For instance, the Poincaré polynomial of a free group of rank n is $\mathrm{Poin}(F_n, t) = 1 + nt$. Thus, the exponential generating function for the sequence $\mathbf{F} = \{F_n\}_{n\geq 1}$ is $\mathrm{Poin}(\mathbf{F}, u, t) = (1 + tu)e^u$.

7.2.2 Pure braid groups

A classifying space for the pure braid group P_n is the configuration space $\mathrm{Conf}(\mathbb{C}, n)$ of n distinct points on the complex line. This space has the homotopy type of a finite, $(n-1)$-dimensional CW-complex. The cohomology algebras of the pure braid groups are computed by Arnold in [3].

Theorem 7.2.1 ([3]). *The cohomology ring $A_n = H^*(P_n; \mathbb{Z})$ is the skew-commutative ring generated by degree 1 elements e_{ij} $(1 \leq i < j \leq n)$, subject to the relations*

$$e_{ik}e_{jk} = e_{ij}(e_{jk} - e_{ik}) \ \text{ for } \ i < j < k. \tag{7.6}$$

Clearly, this algebra is quadratic. In fact, A_n is a Koszul algebra ([137]), that is to say, $\mathrm{Ext}^i_{A_n}(\mathbb{C}, \mathbb{C})_j = 0$ for $i \neq j$. Furthermore, the Poincaré polynomial of $\mathrm{Conf}(\mathbb{C}, n)$, or, equivalently, the Hilbert series of A_n, is given by

$$\mathrm{Poin}(P_n, t) = \prod_{k=1}^{n-1}(1 + kt) = \sum_{i=0}^{n-1} c(n, n-i)\, t^i, \tag{7.7}$$

where $c(n,m)$ are the (unsigned) Stirling numbers of the first kind, counting the number of permutations of n elements which contain exactly m permutation cycles. The associated graded Lie algebra of P_n is computed by Kohno [80] and Falk–Randell [48].

Theorem 7.2.2. *The graded Lie algebra* $\mathrm{gr}(P_n;\mathbb{Q})$ *is generated by* s_{ij} *for* $1 \le i \ne j \le n$, *subjects to the relations* $s_{ij} = s_{ji}$, $[s_{jk}, s_{ik} + s_{ij}] = 0$ *and* $[s_{ij}, s_{kl}] = 0$ *for* $i \ne j \ne l$.

In particular, the pure braid group P_n is graded-formal.

Proposition 7.2.3. *The exponential generating function for the Poincaré polynomials of the pure braid groups* P_n *is given by*

$$\mathrm{Poin}(\mathbf{P}, u, t) = \exp\left(-\frac{\log(1-tu)}{t}\right).$$

Proof. It is known (see, e.g. [139]) that the exponential generating function for the unsigned Stirling numbers $c(n,k)$ is given by

$$\exp(-x \cdot \log(1-z)) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c(n,k)x^k\frac{z^n}{n!}. \tag{7.8}$$

Setting $x = t^{-1}$ and $z = tu$, we obtain

$$\exp\left(-\frac{\log(1-tu)}{t}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c(n,k)t^{-k}\frac{(tu)^n}{n!} = 1 + \sum_{n=1}^{\infty}\sum_{i=0}^{n-1} c(n,n-i)t^i\frac{u^n}{n!}, \tag{7.9}$$

where we used $c(0,0) = 1$ and $c(n,0) = 0$ for $n \ge 1$. This completes the proof. □

7.2.3 Pure virtual braid groups

In [11], Bartholdi et al. describe classifying spaces for the pure virtual braid group vP_n and vP_n^+. Let us note here that both these spaces are finite, $(n-1)$-dimensional CW-complexes.

The following theorem provides presentations for the cohomology algebras of the pure virtual braid groups and their upper triangular subgroups.

Theorem 7.2.4 ([11, 91])**.** *For each* $n \ge 2$, *the following hold.*

1. *The cohomology algebra* $A_n = H^*(vP_n; \mathbb{C})$ *is the skew-commutative algebra generated by degree* 1 *elements* a_{ij} $(1 \leq i \neq j \leq n)$ *subject to the relations* $a_{ij}a_{ik} = a_{ij}a_{jk} - a_{ik}a_{kj}$, $a_{ik}a_{jk} = a_{ij}a_{jk} - a_{ji}a_{ik}$, *and* $a_{ij}a_{ji} = 0$ *for* i, j, k *all distinct.*

2. *The cohomology algebra* $A_n^+ = H^*(vP_n^+; \mathbb{C})$ *is the skew-commutative algebra generated by degree* 1 *elements* a_{ij} $(1 \leq i \neq j \leq n)$, *subject to the relations* $a_{ij} = -a_{ji}$ *and* $a_{ij}a_{jk} = a_{jk}a_{ki}$ *for* $i \neq j \neq k$.

Corollary 7.2.5. *The cohomology algebra* $H^*(vP_n^+; \mathbb{C})$ *has a simplified presentation with generators* e_{ij} *in degree* 1 *for* $1 \leq i < j \leq n$, *and relations* $e_{ij}(e_{ik} - e_{jk})$ *and* $(e_{ij} - e_{ik})e_{jk}$ *for* $i < j < k$.

Proof. Let $\widetilde{A}_n^+$ be the algebra given by the above presentation. The morphism $\phi\colon A_n^+ \to \widetilde{A}_n^+$ defined by $\phi(a_{ij}) = e_{ij}$ for $i < j$ and $\phi(a_{ij}) = -e_{ij}$ for $i > j$ is easily checked to be an isomorphism. □

In [11], Bartholdi et al. also showed that both A_n and A_n^+ are Koszul algebras, and computed the Hilbert series of these graded algebras, as follows:

$$\begin{aligned} \operatorname{Poin}(vP_n, t) &= \sum_{i=0}^{n-1} L(n, n-i)\, t^i, \\ \operatorname{Poin}(vP_n^+, t) &= \sum_{i=0}^{n-1} S(n, n-i)\, t^i. \end{aligned} \tag{7.10}$$

Here $L(n, n-i)$ are the Lah numbers, i.e., the number of ways of partitioning $[n]$ into $n-i$ nonempty ordered subsets, while $S(n, n-i)$ are the Stirling numbers of the second kind, i.e., the number of ways of partitioning $[n]$ into $n-i$ nonempty (unordered) sets. The polynomial $\operatorname{Poin}(vP_n^+, t)$ is the rank-generating function for the partition lattice Π_n, see e.g. [139, Exercise 3.10.4]. All roots of these polynomial are negative real numbers.

It is now readily seen that the exponential generating function for the polynomials $\operatorname{Poin}(vP_n, t)$ and $\operatorname{Poin}(vP_n^+, t)$ are $\exp\left(\frac{u}{1-tu}\right)$ and $\exp\left(\frac{\exp(tu)-1}{t}\right)$, respectively.

Finally, let us note that Dies and Nicas [40] showed that the Euler characteristic of vP_n is non-zero for all $n \geq 2$, while the Euler characteristic of vP_n^+ is non-zero for all $n \geq 3$, with one possible exception (and no exception if Wilf's conjecture is true).

7.3 Resonance varieties of P_n and vP_n

7.3.1 Pure braid groups

Since P_n admits a classifying space of dimension $n-1$, the resonance varieties $\mathcal{R}^i_d(P_n)$ are empty for $i \geq n$. In degree $i = 1$, the resonance varieties $\mathcal{R}^i_d(P_n)$ are either trivial, or a union of 2-dimensional subspaces.

Proposition 7.3.1 ([33]). *The first resonance variety of the pure braid group P_n has decomposition into irreducible components given by*

$$\mathcal{R}^1_1(P_n) = \bigcup_{1\leq i<j<k\leq n} L_{ijk} \cup \bigcup_{1\leq i<j<k<l\leq n} L_{ijkl},$$

where

$$L_{ijk} = \{x_{ij} + x_{ik} + x_{jk} = 0 \text{ and } x_{st} = 0 \text{ if } \{s,t\} \not\subset \{i,j,k\}\},$$
$$L_{ijkl} = \left\{ \begin{array}{l} \sum_{\{p,q\}\subset\{i,j,k,l\}} x_{pq} = 0,\ x_{ij} = x_{kl},\ x_{jk} = x_{il},\ x_{ik} = x_{jl}, \\ x_{st} = 0 \text{ if } \{s,t\} \not\subset \{i,j,k,l\} \end{array} \right\}.$$

Furthermore, $\mathcal{R}^1_d(P_n) = \{0\}$ for $2 \leq d \leq \binom{n}{2}$, and $\mathcal{R}^1_d(P_n) = \emptyset$ for $d > \binom{n}{2}$.

Recall that $P_n \cong \overline{P}_n \times \mathbb{Z}$, where $\overline{P}_n$ is the quotient of P_n by its (infinite cyclic) center. Thus, the resonance varieties of the group $\overline{P}_n$ can be described in a similar manner, using Proposition 5.1.3.

7.3.2 Resonance varieties of vP_3

A partial computation of the resonance varieties $\mathcal{R}^i_d(vP_3)$ was done in [10] for $i = 1$ and $d = 1, 5, 6$, as well as $i = 2$ and $d = 2, 6$. We use the preceding discussion to give a complete

computation of all these varieties.

Proposition 7.3.2. *For $d \geq 1$, the resonance varieties of the pure virtual braid group vP_3 are given by*

$$\mathcal{R}^i_d(vP_3) \cong \begin{cases} \mathcal{R}^1_{d-1}(\overline{P}_4) \times \mathbb{C}, & \text{for } i = 1, d \leq 5 \\ \{0\} & \text{for } i = 1, d = 6, \\ \mathcal{R}^1_{d-2}(\overline{P}_4) \times \mathbb{C} & \text{for } i = 2, d \leq 6 \\ \emptyset & \text{otherwise.} \end{cases}$$

Consequently, $\mathcal{R}^1_1(vP_3) = \mathbb{C}^6$, while $\mathcal{R}^1_2(vP_3)$ is a union of five 3-dimensional subspaces, pairwise intersecting in the 1-dimensional subspace $\mathcal{R}^1_3(vP_3) = \mathcal{R}^1_4(vP_3) = \mathcal{R}^1_5(vP_3)$.

Proof. By Lemma 7.1.4, we have an isomorphism $vP_3 \cong \overline{P}_4 * \mathbb{Z}$, which yields an isomorphism $H^1(vP_3; \mathbb{C}) \cong H^1(\overline{P}_4; \mathbb{C}) \oplus \mathbb{C}$. Under this identification, Proposition 5.1.4 shows that $\mathcal{R}^1_d(vP_3) \cong \mathcal{R}^1_{d-1}(\overline{P}_4) \times \mathbb{C}$ for $d \leq 5$, and $\mathcal{R}^1_6(vP_3) = \{0\}$.

The same proposition also shows that $\mathcal{R}^2_d(vP_3) \cong \mathcal{R}^2_d(\overline{P}_4) \times \mathbb{C}$. On the other hand, by Lemma 5.1.1, we have that $\mathcal{R}^2_d(\overline{P}_4) = \mathcal{R}^1_{d-2}(\overline{P}_4)$ for $d \leq 6$, since $\overline{P}_4$ admits a 2-dimensional classifying space, and $\chi(\overline{P}_4) = 2$. Finally, the description of the resonance varieties $\mathcal{R}^1_d(vP_3)$ for $d \leq 6$ follows from Proposition 7.3.1. □

Let $a_{12}, a_{13}, a_{23}, a_{21}, a_{31}, a_{32}$ be the basis of $H^1(vP_3, \mathbb{C})$ specified in Theorem 7.2.4, and let x_{ij} the corresponding coordinate functions on this affine space. Tracing through the isomorphisms $H^1(vP_3, \mathbb{C}) \cong H^1(\overline{P}_4, \mathbb{C}) \times \mathbb{C} \cong H^1(P_4, \mathbb{C})$, we see that the components of $\mathcal{R}^1_2(vP_3)$ have equations

$$\{x_{12} - x_{23} = x_{12} + x_{32} = x_{12} + x_{21} = 0\},$$

$$\{x_{13} + x_{23} = x_{12} + x_{32} = x_{21} + x_{31} = 0\},$$

$$\{x_{13} + x_{23} = x_{13} - x_{32} = x_{13} + x_{31} = 0\},$$

$$\{x_{12} + x_{13} = x_{12} + x_{21} = x_{12} - x_{31} = 0\},$$

$$\{x_{12}+x_{13}=x_{23}+x_{21}=x_{31}+x_{32}=0\},$$

while their common intersection is the line $\{x_{12}=-x_{21}=-x_{13}=x_{31}=x_{23}=-x_{32}\}$.

7.3.3 Resonance varieties of vP_4^+

We now switch to the upper-triangular group vP_4^+, and compute its degree 1 resonance varieties. Let $e_{12}, e_{13}, e_{23}, e_{14}, e_{24}, e_{34}$ be the basis of $H^1(vP_4^+, \mathbb{C})$ specified in Corollary 7.2.5, and let x_{ij} be the corresponding coordinate functions on this affine space.

Lemma 7.3.3. *The depth* 1 *resonance variety* $\mathcal{R}_1^1(vP_4^+)$ *is the irreducible,* 3*-dimensional subvariety of degree* 6 *inside* $H^1(vP_4^+, \mathbb{C}) = \mathbb{C}^6$ *defined by the equations*

$$\begin{cases} x_{12}x_{24}(x_{13}+x_{23})+x_{13}x_{34}(x_{12}-x_{23})-x_{24}x_{34}(x_{12}+x_{13})=0,\\ x_{12}x_{23}(x_{14}+x_{24})+x_{12}x_{34}(x_{23}-x_{14})+x_{14}x_{34}(x_{23}+x_{24})=0,\\ x_{13}x_{23}(x_{14}+x_{24})+x_{14}x_{24}(x_{13}+x_{23})+x_{34}(x_{13}x_{23}-x_{14}x_{24})=0,\\ x_{12}(x_{13}x_{14}-x_{23}x_{24})+x_{34}(x_{13}x_{23}-x_{14}x_{24})=0. \end{cases}$$

The depth 2 *resonance variety* $\mathcal{R}_2^1(vP_4^+)$ *consists of* 13 *lines in* $\mathbb{C}^6$*, spanned by the vectors*

$$\begin{array}{lll} e_{12},\ e_{13},\ e_{23}, & e_{14},\ e_{24},\ e_{34}, & e_{12}-e_{13}+e_{23},\\ e_{12}-e_{14}+e_{24}, & e_{13}-e_{14}+e_{34}, & e_{23}-e_{24}+e_{34},\\ e_{13}-e_{23}-e_{14}+e_{24}, & e_{12}+e_{23}-e_{14}+e_{34}, & e_{12}-e_{13}+e_{24}-e_{34}. \end{array}$$

Finally, $\mathcal{R}_d^1(vP_4^+)=\{0\}$ *if* $3\le d\le 6$*, and* $\mathcal{R}_d^1(vP_4^+)=\emptyset$ *if* $d\ge 7$*.*

Proof. Let $A = H^*(vP_4^+, \mathbb{C})$ be the cohomology algebra of vP_4^+, as described in Corollary 7.2.5. The differential $\delta^1\colon A^1\otimes S\to A^2\otimes S$ in the cochain complex (5.6) is then given by

$$\delta^1=\begin{pmatrix} -x_{34} & 0 & 0 & 0 & 0 & x_{12}\\ -x_{13}-x_{23} & x_{12}-x_{23} & x_{12}+x_{13} & 0 & 0 & 0\\ 0 & -x_{24} & 0 & 0 & x_{13} & 0\\ 0 & 0 & -x_{14} & x_{23} & 0 & 0\\ -x_{14}-x_{24} & 0 & 0 & x_{12}-x_{24} & x_{12}+x_{14} & 0\\ 0 & -x_{14}-x_{34} & 0 & x_{13}-x_{34} & 0 & x_{13}+x_{14}\\ 0 & 0 & -x_{24}-x_{34} & 0 & x_{23}-x_{34} & x_{23}+x_{24} \end{pmatrix}.$$

Computing with the aid of Macaulay2 [98] the elementary ideals of this matrix and finding their primary decomposition leads to the stated conclusions. □

7.3.4 Resonance varieties of vP_5^+

We now switch to the upper-triangular group vP_5^+, and compute its degree 1 resonance varieties. Let $e_{12}, e_{13}, \dots, e_{45}$ be the basis of $H^1(vP_5^+, \mathbb{C})$ specified in Corollary 7.2.5, and let x_{ij} be the corresponding coordinate functions on this affine space. Computing with the aid of Macaulay2 [98], we get the first resonance variety of vP_5^+.

Lemma 7.3.4. *The depth 1 resonance variety $\mathcal{R}_1^1(vP_5^+) \cong \bigcup_{i=1}^{15} C_i$ is the 4-dimensional subvariety inside $H^1(vP_5^+, \mathbb{C}) = \mathbb{C}^{10}$ with 15 irreducible components.*

$$
\begin{aligned}
C_1 :&\quad \{x_{24} - x_{45} = x_{23} - x_{35} = x_{13} = x_{14} = x_{34} = x_{12} + x_{15} = 0\}\\
C_2 :&\quad \{x_{45} = x_{25} + x_{35} = x_{14} = x_{24} + x_{34} = x_{12} + x_{13} = x_{15} = 0\}\\
C_3 :&\quad \{x_{24} = x_{23} + x_{25} = x_{14} = x_{34} - x_{45} = x_{12} = x_{13} + x_{15} = 0\}\\
C_4 :&\quad \{x_{45} = x_{24} = x_{25} = x_{14} + x_{34} = x_{12} - x_{23} = x_{15} + x_{35} = 0\}\\
C_5 :&\quad \{x_{35} + x_{45} = x_{25} = x_{23} + x_{24} = x_{13} + x_{14} = x_{12} = x_{15} = 0\}\\
C_6 :&\quad \{x_{45} = x_{35} = x_{13} + x_{23} = x_{14} + x_{24} = x_{34} = x_{15} + x_{25} = 0\}\\
C_7 :&\quad \{x_{35} = x_{25} + x_{45} = x_{13} = -x_{23} + x_{34} = x_{12} + x_{14} = x_{15} = 0\}\\
C_8 :&\quad \{x_{24} + x_{25} = x_{23} = x_{13} = x_{34} + x_{35} = x_{12} = x_{14} + x_{15} = 0\}\\
C_9 :&\quad \{x_{24} = x_{23} = x_{13} - x_{35} = x_{14} - x_{45} = x_{34} = x_{12} - x_{25} = 0\}\\
C_{10} :&\quad \{x_{35} = x_{25} = x_{23} = -x_{13} + x_{34} = x_{12} - x_{24} = x_{15} + x_{45} = 0\}
\end{aligned}
$$

C_{11} is defined by the equations

$$
\begin{cases}
x_{24} = x_{25} = x_{23} = x_{12} = 0\\
-x_{13}x_{14}x_{35} - x_{13}x_{34}x_{35} - x_{13}x_{14}x_{45} + x_{14}x_{34}x_{45} + x_{13}x_{35}x_{45} + x_{14}x_{35}x_{45} = 0\\
x_{13}x_{14}x_{15} + x_{13}x_{14}x_{35} + x_{13}x_{14}x_{45} - x_{13}x_{35}x_{45} - x_{14}x_{35}x_{45} - x_{15}x_{35}x_{45} = 0\\
x_{13}x_{34}x_{15} + x_{13}x_{34}x_{35} + x_{13}x_{34}x_{45} - x_{13}x_{15}x_{45} + x_{34}x_{15}x_{45} + x_{15}x_{35}x_{45} = 0\\
x_{14}x_{34}x_{15} + x_{13}x_{14}x_{35} + x_{13}x_{34}x_{35} + x_{14}x_{34}x_{35} + x_{14}x_{15}x_{35} + x_{34}x_{15}x_{35}\\
+x_{13}x_{14}x_{45} - x_{13}x_{35}x_{45} - x_{14}x_{35}x_{45} - x_{15}x_{35}x_{45} = 0
\end{cases}
$$

C_{12} is defined by the equations

$$
\begin{cases}
x_{35} = x_{23} = x_{13} = x_{34} = 0\\
x_{12}x_{14}x_{25} + x_{12}x_{24}x_{25} + x_{12}x_{14}x_{45} - x_{14}x_{24}x_{45} - x_{12}x_{25}x_{45} - x_{14}x_{25}x_{45} = 0\\
x_{14}x_{24}x_{15} + x_{14}x_{24}x_{25} + x_{14}x_{15}x_{25} + x_{24}x_{15}x_{25} + x_{14}x_{24}x_{45} - x_{15}x_{25}x_{45} = 0\\
x_{12}x_{24}x_{15} + x_{12}x_{24}x_{25} + x_{12}x_{24}x_{45} - x_{12}x_{15}x_{45} + x_{24}x_{15}x_{45} + x_{15}x_{25}x_{45} = 0\\
x_{12}x_{14}x_{15} - x_{12}x_{24}x_{25} + x_{14}x_{24}x_{45} - x_{15}x_{25}x_{45} = 0
\end{cases}
$$

C_{13} is defined by the equations

$$\begin{cases} x_{45} = x_{35} = x_{25} = x_{15} = 0 \\ -x_{13}x_{23}x_{14} - x_{13}x_{23}x_{24} - x_{13}x_{14}x_{24} - x_{23}x_{14}x_{24} - x_{13}x_{23}x_{34} + x_{14}x_{24}x_{34} = 0 \\ x_{12}x_{13}x_{14} - x_{13}x_{23}x_{14} - x_{12}x_{23}x_{24} - x_{13}x_{23}x_{24} - x_{13}x_{14}x_{24} - x_{23}x_{14}x_{24} = 0 \\ x_{12}x_{13}x_{24} + x_{12}x_{23}x_{24} + x_{12}x_{13}x_{34} - x_{13}x_{23}x_{34} - x_{12}x_{24}x_{34} - x_{13}x_{24}x_{34} = 0 \\ -x_{12}x_{23}x_{14} - x_{13}x_{23}x_{14} - x_{12}x_{23}x_{24} - x_{13}x_{23}x_{24} - x_{13}x_{14}x_{24} - x_{23}x_{14}x_{24} \\ -x_{12}x_{23}x_{34} - x_{13}x_{23}x_{34} + x_{12}x_{14}x_{34} - x_{23}x_{14}x_{34} = 0 \end{cases}$$

C_{14} is defined by the equations

$$\begin{cases} x_{45} = x_{24} = x_{14} = x_{34} = 0 \\ x_{12}x_{13}x_{25} + x_{12}x_{23}x_{25} + x_{12}x_{13}x_{35} - x_{13}x_{23}x_{35} - x_{12}x_{25}x_{35} - x_{13}x_{25}x_{35} = 0 \\ x_{13}x_{23}x_{15} + x_{13}x_{23}x_{25} + x_{13}x_{15}x_{25} + x_{23}x_{15}x_{25} + x_{13}x_{23}x_{35} - x_{15}x_{25}x_{35} = 0 \\ x_{12}x_{23}x_{15} + x_{12}x_{23}x_{25} + x_{12}x_{23}x_{35} - x_{12}x_{15}x_{35} + x_{23}x_{15}x_{35} + x_{15}x_{25}x_{35} = 0 \\ x_{12}x_{13}x_{15} - x_{12}x_{23}x_{25} + x_{13}x_{23}x_{35} - x_{15}x_{25}x_{35} = 0 \end{cases}$$

C_{15} is defined by the equations

$$\begin{cases} x_{13} = x_{14} = x_{12} = x_{15} = 0 \\ x_{23}x_{24}x_{25} + x_{23}x_{24}x_{35} + x_{23}x_{24}x_{45} - x_{23}x_{35}x_{45} - x_{24}x_{35}x_{45} - x_{25}x_{35}x_{45} = 0 \\ x_{24}x_{34}x_{25} + x_{24}x_{34}x_{35} + x_{24}x_{25}x_{35} + x_{34}x_{25}x_{35} + x_{24}x_{34}x_{45} - x_{25}x_{35}x_{45} = 0 \\ x_{23}x_{24}x_{35} + x_{23}x_{34}x_{35} + x_{23}x_{24}x_{45} - x_{24}x_{34}x_{45} - x_{23}x_{35}x_{45} - x_{24}x_{35}x_{45} = 0 \\ x_{23}x_{34}x_{25} - x_{23}x_{24}x_{35} - x_{23}x_{24}x_{45} + x_{23}x_{34}x_{45} + x_{24}x_{34}x_{45} - x_{23}x_{25}x_{45} \\ +x_{34}x_{25}x_{45} + x_{23}x_{35}x_{45} + x_{24}x_{35}x_{45} + x_{25}x_{35}x_{45} = 0 \end{cases}$$

Here, the degree of C_i equals 1 for $1 \leq i \leq 10$, equals 3 for $11 \leq i \leq 15$.

The degree 1, depth 1 resonance varieties of the virtual pure braid groups vP_n^+ and vP_n can be computed in a similar fashion, at least for small values of n. We record in the following table the dimensions and the degrees of these varieties, in the range we were able to carry out those computations.

n	2	3	4	5	6	7	8
$\dim(\mathcal{R}_1^1(vP_n^+))$	0	3	3	4	5	6	7
$\deg(\mathcal{R}_1^1(vP_n^+))$	1	1	6	40	15	21	28
$\dim(\mathcal{R}_1^1(vP_n))$	2	6	6				
$\deg(\mathcal{R}_1^1(vP_n))$	0	1	4				

(7.11)

7.4 Formality properties

7.4.1 Graded algebras associated to P_n, vP_n and vP_n^+

We start with the pure braid groups P_n. As shown by Kohno [80] and Falk–Randell [48], the graded Lie algebra $\mathrm{gr}(P_n)$ is generated by degree 1 elements s_{ij} for $1 \leq i \neq j \leq n$, subjects to the relations

$$s_{ij} = s_{ji},\ [s_{jk}, s_{ik} + s_{ij}] = 0,\ [s_{ij}, s_{kl}] = 0 \text{ for } i \neq j \neq l. \tag{7.12}$$

In particular, the pure braid group P_n is graded-formal. The universal enveloping algebra $U(\mathrm{gr}(P_n))$ is Koszul with Hilbert series $\prod_{k=1}^{n-1}(1-kt)^{-1}$. Furthermore, the ranks $\phi_k(P_n)$ can be computed from formulas (2.12) and (7.7), as follows:

$$\phi_k(P_n) = \sum_{s=1}^{n-1} \phi_k(F_s) = \frac{1}{k} \sum_{s=1}^{n-1} \sum_{d|k} \mu(k/d) s^d. \tag{7.13}$$

Next, we give a presentation for the holonomy Lie algebras of the pure virtual braid groups, using a method described in Corollary 4.3.3. The Lie algebra $\mathfrak{h}(vP_n)$ is generated by r_{ij}, $1 \leq i \neq j \leq n$, with relations

$$[r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}] = 0 \text{ for distinct } i, j, k, \tag{7.14}$$

and $[r_{ij}, r_{kl}] = 0$ for distinct i, j, k, l. The Lie algebra $\mathfrak{h}(vP_n^+)$ is the quotient Lie algebra of $\mathfrak{h}(vP_n)$ by the ideal generated by $r_{ij} + r_{ji}$ for distinct $i \neq j$. Similarly as Corollary 7.2.5, the Lie algebra $\mathfrak{h}(vP_n^+)$ has a simplified presentation with generators r_{ij} for $i < j$ and the corresponding relations of $\mathfrak{h}(vP_n)$.

Theorem 7.4.1 ([11, 91])**.** *The Lie algebra $\mathfrak{h}(vP_n)$ is isomorphic to the Lie algebra $\mathrm{gr}(vP_n)$. Likewise, the Lie algebra $\mathfrak{h}(vP_n^+)$ is isomorphic to the Lie algebra $\mathrm{gr}(vP_n^+)$.*

As shown in [11], there exist isomorphisms of graded algebras, $H^*(vP_n^+; \mathbb{C}) \cong U(\mathrm{gr}(vP_n^+))^!$ and $H^*(vP_n; \mathbb{C}) \cong U(\mathrm{gr}(vP_n))^!$. Hence, the LCS ranks $\phi_k(vP_n^+)$ and $\phi_k(vP_n)$ can be computed by means of Lemma 3.1.3, and formulas (2.12) and (7.10).

7.4.2 Formality properties of vP_n and vP_n^+

Recall that the pure virtual braid groups vP_n and vP_n^+ are graded-formal for all $n \geq 1$. Furthermore, $vP_2^+ = \mathbb{Z}$ and $vP_2 = F_2$, and so both are 1-formal groups.

Lemma 7.4.2. *The groups vP_3^+ and vP_3 are both 1-formal.*

Proof. From Propositions 3.1.21 and 3.3.12, the free product of two 1-formal groups is 1-formal. Hence $vP_3^+ \cong \mathbb{Z}^2 * \mathbb{Z}$ is also 1-formal.

Since the pure braid group $P_4 \cong \overline{P}_4 \times \mathbb{Z}$ is 1-formal, Theorem 3.3.10 ensures that the subgroup $\overline{P}_4$ is also 1-formal. On the other hand, we know from Lemma 7.1.4 that $vP_3 \cong \overline{P}_4 * \mathbb{Z}$. Thus, by, Theorem 3.3.10, the group vP_3 is 1-formal. □

Lemma 7.4.3. *The group vP_4^+ is not 1-formal.*

Proof. As shown in Lemma 7.3.3, the resonance variety $\mathcal{R}_1^1(vP_4^+)$ is an irreducible subvariety of $H^1(vP_4^+, \mathbb{C})$. Thus, this variety doesn't decompose into a finite union of linear subspaces, and so, by Theorem 5.1.2, the group vP_4^+ is not 1-formal. □

Theorem 7.4.4. *The groups vP_n^+ and vP_n are not 1-formal for $n \geq 4$.*

Proof. As shown in Lemma 7.1.3, there is a split injection from vP_n^+ to vP_{n+1}^+. Since vP_4^+ is not 1-formal, Theorem 3.3.10, then, insures that the groups vP_n^+ are not 1-formal for $n \geq 4$.

By Corollary 7.1.2, there is a split monomorphism $vP_n^+ \to vP_n$. From Theorem 7.4.4, we know that the group vP_n^+ is not 1-formal for $n \geq 4$. Therefore, by Theorem 3.3.10, the group vP_n is not 1-formal for $n \geq 4$. □

Corollary 7.4.5. *The groups vP_n^+ and vP_n are not filtered formal for $n \geq 4$.*

To summarize, the groups vP_n and vP_n^+ are always graded-formal. Furthermore, they are 1-formal (equivalently, filtered-formal) if and only if $n \leq 3$. This completes the proof of Theorem 1.2.7 from the Introduction.

7.5 Chen ranks of P_n and vP_n

7.5.1 Chen ranks of the free groups

As shown in [27], the Chen ranks of the free group F_n are given by $\theta_1(F_n) = n$ and

$$\theta_k(F_n) = \binom{n+k-2}{k}(k-1) \quad \text{for} \quad k \geq 2. \tag{7.15}$$

Equivalently, by Massey's formula (6.16), the Hilbert series for the associated graded Alexander invariant of F_n is given by

$$\operatorname{Hilb}(\operatorname{gr}(B(F_n)), t) = \frac{1}{t^2} \cdot \left(1 - \frac{1-nt}{(1-t)^n}\right), \tag{7.16}$$

an identity which can also be verified directly, by using the fact that $B(F_n)$ is the cokernel of the third boundary map of the Koszul resolution $\bigwedge(\mathbb{Z}^n) \otimes \mathbb{C}[\mathbb{Z}^n]$.

From formula (7.16), we see that the generating and exponential generating functions for the Hilbert series of the associated graded Alexander invariants of the sequence of free groups $\mathbf{F} = \{F_n\}_{n\geq 1}$ are given by

$$\begin{aligned} \sum_{n=1}^{\infty} \operatorname{Hilb}(\operatorname{gr}(B(F_n)), t) \cdot u^n &= \frac{u^2}{(1-u)(1-t-u)^2}, \\ \sum_{n=1}^{\infty} \operatorname{Hilb}(\operatorname{gr}(B(F_n)), t) \cdot \frac{u^n}{n!} &= \frac{e^u}{t^2} + \frac{e^{u/(1-t)}}{t^2}\left(\frac{tu}{1-t} - 1\right). \end{aligned} \tag{7.17}$$

7.5.2 Chen ranks of the pure braid groups

A comprehensive algorithm for computing the Chen ranks of finitely presented groups was developed in [31, 33], leading to the following expressions for the Chen ranks of the pure braid groups:

$$\theta_1(P_n) = \binom{n}{2}, \ \theta_2(P_n) = \binom{n}{3}, \ \text{and} \ \theta_k(P_n) = (k-1)\binom{n+1}{4} \ \text{ for } k \geq 3, \tag{7.18}$$

or, equivalently,

$$\operatorname{Hilb}(\operatorname{gr}(B(P_n)), t) = \binom{n+1}{4}\frac{1}{(1-t)^2} - \binom{n}{4}. \tag{7.19}$$

It follows that the two generating functions for the Hilbert series of the associated graded Alexander invariants of the sequence of pure braid groups $\mathbf{P} = \{P_n\}_{n\geq 1}$ are given by

$$\sum_{n=1}^{\infty} \operatorname{Hilb}(\operatorname{gr}(B(P_n)), t) \cdot u^n = \frac{u^3}{(1-u)^5}\left(\frac{1}{(1-t)^2} - u\right), \tag{7.20}$$
$$\sum_{n=1}^{\infty} \operatorname{Hilb}(\operatorname{gr}(B(P_n)), t) \cdot \frac{u^n}{n!} = \frac{e^u u^3}{24}\left(\frac{u+4}{(1-t)^2} - u\right).$$

7.5.3 Chen ranks of vP_3 and vP_3^+

We now return to the pure virtual braid groups, and study their Chen ranks. Recall that $vP_2^+ = \mathbb{Z}$ and $vP_2 = F_2$, so we may as well assume $n \geq 3$. We start with the case $n = 3$.

Proposition 7.5.1. *The groups vP_3^+ and vP_3 do not satisfy the Chen ranks formula, despite the fact that they are both 1-formal, and their first resonance varieties are projectively disjoint and reduced as schemes.*

Proof. Recall that $vP_3^+ \cong \mathbb{Z}^2 * \mathbb{Z}$. Thus, the claim for vP_3^+ is handled by the argument from Example 6.3.7.

Next, recall that $vP_3 \cong \overline{P}_4 * \mathbb{Z}$. We know from Lemma 7.4.2 that vP_3 is 1-formal. Furthermore, we know from Proposition 7.3.2 that $\mathcal{R}_1^1(vP_3) = H^1(vP_3, \mathbb{C})$. Clearly, this variety is projectively disjoint and reduced as a scheme. Using the algorithm described in Remark 6.2.1, we find that the Hilbert series of the associated graded Alexander invariants of vP_3 is given by

$$\operatorname{Hilb}(\operatorname{gr}(B(vP_3)), t) = (9 - 20t + 15t^2 - 4t^4 + t^5)/(1-t)^6. \tag{7.21}$$

On the other hand, as noted above, $\mathcal{R}_1^1(vP_3) = \mathbb{C}^6$. Using (7.16) and (7.21), we compute

$$\operatorname{Hilb}(\operatorname{gr}(B(vP_3)), t) - \operatorname{Hilb}(\operatorname{gr}(B(F_6)), t) = (-6 + 6t^3 - 5t^4 + t^5)/(1-t)^6. \tag{7.22}$$

Since this expression is *not* a polynomial in t, we conclude that formula (6.30) does not hold for vP_3, and this ends the proof. □

Remark 7.5.2. The resonance varieties of vP_3^+ and vP_3 are not isotropic, since both groups have non-vanishing cup products stemming from the subgroups $\mathbb{Z}^2$ and $\overline{P}_4$, respectively. Thus, once again, the groups vP_3^+ and vP_3 illustrate the necessity of the isotropicity hypothesis from Theorem 6.3.1.

7.5.4 Holonomy Chen ranks of vP_n and vP_n^+

We conclude with a summary of what else we know about the Chen ranks of the pure virtual braid groups, as well as the Chen ranks of the respective holonomy Lie algebras and associated graded Lie algebras.

Proposition 7.5.3. *The following equalities of Chen ranks hold.*

1. $\theta_k(\mathfrak{h}(vP_n^+)) = \theta_k(\mathrm{gr}(vP_n^+))$ *and* $\theta_k(\mathfrak{h}(vP_n)) = \theta_k(\mathrm{gr}(vP_n))$, *for all* n *and* k.
2. $\theta_k(\mathfrak{h}(vP_n^+)) = \theta_k(vP_n^+)$ *for* $n \leq 6$ *and all* k.
3. $\theta_k(\mathfrak{h}(vP_n)) = \theta_k(vP_n)$ *for* $n \leq 3$ *and all* k.

Proof. (1) Recall from Theorem 7.4.1 that the pure virtual braid groups vP_n and vP_n^+ are graded-formal. Therefore, by Corollary 6.2.5, claim (2) holds.

For $n \leq 3$, claims (2) and (3) follow from the 1-formality of the groups vP_n^+ and vP_n in that range, and Corollary 6.2.5.

Using now the algorithms described in Remarks 6.2.1 and 5.2.3, a Macaulay2 [98] computation reveals that

$$\sum_{k\geq 2} \theta_k(vP_4^+)t^{k-2} = \sum_{k\geq 2} \theta_k(\mathfrak{h}(vP_4^+))t^{k-2} = (8-3t+t^2)/(1-t)^4, \tag{7.23}$$

$$\sum_{k\geq 2} \theta_k(vP_5^+)t^{k-2} = \sum_{k\geq 2} \theta_k(\mathfrak{h}(vP_5^+))t^{k-2} = (20+15t+5t^2)/(1-t)^4,$$

$$\sum_{k\geq 2} \theta_k(vP_6^+)t^{k-2} = \sum_{k\geq 2} \theta_k(\mathfrak{h}(vP_6^+))t^{k-2} = (40+35t-40t^2-20t^3)/(1-t)^5.$$

This establishes claim (2) for $4 \leq n \leq 6$, thereby completing the proof. □

It would be interesting to decide whether the equalities in parts (2) and (3) of the above proposition hold for all n and all k.

Finally, let us address the validity of the Chen ranks formula (6.30) for the pure virtual braid groups on $n \geq 4$ strings. We know from Lemma 7.3.3 that $\mathcal{R}_1^1(vP_4^+)$ has a single irreducible component of dimension 3. Lemma 7.3.3 shows that $\mathcal{R}_1^1(vP_5^+)$ has 15 irreducible components, all of dimension 4. Using now (7.23), it is readily seen that the Chen ranks formula does not hold for either vP_4^+ or vP_5^+. Based on this evidence, and some further computations, we expect that the Chen ranks formula does not hold for any of the groups vP_n and vP_n^+ with $n \geq 4$. At last, we list some more computation results for the Hilbert series of the infinitesimal Alexander invariants using Macaulay2 [98].

$$\begin{aligned}
\operatorname{Hilb}(\mathfrak{B}(vP_7^+), t) &= \frac{70 + 70t - 210t^2 + 35t^3 + 56t^4}{(1-t)^6} \\
&= 70 + 490t + 1680t^2 + 4165t^3 + 8596t^4 + 15771t^5 + 26656t^6 + \cdots
\end{aligned}$$

$$\begin{aligned}
\operatorname{Hilb}(\mathfrak{B}(vP_8^+), t) &= \frac{112 + 126t - 644t^2 + 476t^3 + 84t^4 - 126t^5}{(1-t)^7} \\
&= 112 + 910t + 3374t^2 + 8904t^3 + 19488t^4 + 37898t^5 + 67914t^6 + \cdots
\end{aligned}$$

$$\begin{aligned}
\operatorname{Hilb}(\mathfrak{B}(vP_9^+), t) &= \frac{168 + 210t - 1554t^2 + 2016t^3 - 588t^4 - 462t^5 + 246t^6}{(1-t)^8} \\
&= 168 + 1554t + 6174t^2 + 17304t^3 + 40236t^4 + 83286t^5 + 159090t^6 + \cdots
\end{aligned}$$

$$\begin{aligned}
\operatorname{Hilb}(\mathfrak{B}(vP_{10}^+), t) &= \frac{240 + 330t - 3240t^2 + 6000t^3 - 4080t^4 - 90t^5 + 1320t^6 - 435t^7}{(1-t)^9} \\
&= 240 + 2490t + 10530t^2 + 31290t^3 + 77370t^4 + 170820t^5 + 348540t^6 + \cdots
\end{aligned}$$

Let $h_i = \operatorname{Hilb}(\mathfrak{B}(vP_i), t)$, then we have

$$h_4 = \frac{30 - 16t - 39t^2 - 59t^3 + 276t^4 - 290t^5 + 61t^6 + 100t^7 - 75t^8 + 16t^9}{(1-t)^6}$$
$$= 30 + 164t + 495t^2 + 1051t^3 + 1987t^4 + 3487t^5 + 5727t^6 + \cdots$$

$$h_5 = \frac{70 + 80t - 265t^2 - 355t^3 + 1430t^4 - 1460t^5 + 305t^6 + 500t^7 - 375t^8 + 80t^9}{(1-t)^6}$$
$$= 70 + 500t + 1685t^2 + 3655t^3 + 7035t^4 + 12545t^5 + 20805t^6 + \cdots$$

$$h_6 = \frac{135 + 380t - 795t^2 - 1155t^3 + 4345t^4 - 4390t^5 + 915t^6 + 1500t^7 - 1125t^8 + 240t^9}{(1-t)^6}$$
$$= 135 + 1190t + 4320t^2 + 9615t^3 + 19010t^4 + 34805t^5 + 59100t^6 + \cdots$$

The Hilbert series $h_7 = \mathrm{Hilb}(\mathfrak{B}(vP_7), t)$ is given by

$$\frac{231 + 805t - 2800t^2 - 1015t^3 + 13027t^4 - 20461t^5 + 12390t^6 + 1365t^7 - 6125t^8 + 3185t^9 - 560t^{10}}{(1-t)^7}$$
$$= 231 + 2422t + 9303t^2 + 21329t^3 + 43652t^4 + 82880t^5 + 146013t^6 + \cdots$$

Chapter 8

Pure welded braid groups

The group of basis-conjugating automorphisms of the free group of rank n is called the pure welded braid group wP_n. McCool gave a presentation for the pure welded braid group (also known as the McCool group), with a subgroup called the upper McCool group wP_n^+. Using a Gröbner basis computation, we find a simple presentation for the infinitesimal Alexander invariant of this group. As an application, we compute the resonance varieties and the Chen ranks of the upper McCool groups. This computation reveals that, unlike for the pure braid group P_n and the full McCool group wP_n, the Chen ranks conjecture does not hold for wP_n^+, for any $n \geq 4$. As an application, we show that wP_n^+ is not isomorphic to P_n in that range, thereby answering a question of Cohen et al [29]. This chapter is based on the work in papers [143, 146].

8.1 The McCool groups

8.1.1 McCool groups

An automorphism of the free group F_n is called a symmetric automorphism if it sends each generator x_i to a conjugate of $x_{\sigma(i)}$, for some permutation $\sigma \in \Sigma_n$. The set of all such automorphisms forms a subgroup wB_n of $\mathrm{Aut}(F_n)$ is also known as the *braid-permutation group*,

[29, 54, 30] or the *welded braid group*, [6]. The welded braid group wB_n has presentation with generators σ_i and s_i for $i = 1, \dots, n-1$, and relations (R1), (R2), (R3) and

$$s_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i s_{i+1}, \quad 1 \leq i \leq n-2. \tag{R4}$$

The group of basis-conjugating automorphisms, $wP_n = \ker(wB_n \twoheadrightarrow S_n)$ is also known as the pure welded braid group, or the *McCool group*, and can be realized as the pure motion group of n unknotted, unlinked circles in S^3. This group is generated by the automorphisms α_{ij}, for all $1 \leq i \neq j \leq n$. It is generated by the automorphisms α_{ij} $(i \neq j)$ which send x_i to $x_j x_i x_j^{-1}$ and leave invariant the other generators of F_n, together with the automorphisms τ_{ij} $(i < j)$ which interchange x_i and x_j. McCool [112] gave a presentation of the *basis-conjugating group* (*McCool group*) wP_n.

Theorem 8.1.1 ([112])**.** *The McCool group wP_n are generated by α_{ij} for $i \neq j$ corresponding to relations $\alpha_{ij}\alpha_{kj}\alpha_{ik} = \alpha_{ik}\alpha_{ij}\alpha_{kj}$ for i, j, k distinct; $[\alpha_{kj}, \alpha_{st}] = 1$ if $\{j, k\} \cap \{s, t\} = \emptyset$; $[\alpha_{ij}, \alpha_{kj}] = 1$ for i, j, k distinct.*

Proposition 8.1.2. *There exist monomorphisms and epimorphisms making the following diagram commute.*

$$\begin{array}{ccccc} B_n & \xhookrightarrow{\varphi_n} & vB_n & \xrightarrow{\pi_n} & wB_n \\ \uparrow & & \uparrow & & \uparrow \\ P_n & \hookrightarrow & vP_n & \twoheadrightarrow & wP_n \end{array}$$

Furthermore, the compositions of the horizontal homomorphisms are also injective.

Proof. There are natural inclusions $\varphi_n \colon B_n \hookrightarrow vB_n$ and $\psi_n \colon B_n \hookrightarrow wB_n$ that send σ_i to σ_i, as well as a canonical projection $\pi_n \colon vB_n \twoheadrightarrow wB_n$, that matches the generators σ_i and s_i of the respective groups. By construction, we have that $\pi_n \circ \varphi_n = \psi_n$.

We claim that these homomorphisms restrict to homomorphisms between the respective pure-braid like groups. Indeed, as shown in [11], the homomorphism φ_n restricts to a map $P_n \hookrightarrow vP_n$, given by

$$A_{ij} \mapsto \alpha_{j-1,j} \dots \alpha_{i+1,j}\alpha_{i,j}\alpha_{j,i}(\alpha_{j-1,j} \dots \alpha_{i+1,j})^{-1}. \tag{8.1}$$

Clearly, the projection π_n restricts to a map $vP_n \twoheadrightarrow wP_n$ that sends α_{ij} to α_{ij}. Using these observations, together with work of Bardakov [7], we see that the homomorphism ψ_n restricts to an injective map $P_n \hookrightarrow wP_n$. □

The welded braid group wB_n is the fundamental group of the 'untwisted ring space', which consists of all configurations of n parallel rings (i.e., unknotted circles) in $\mathbb{R}^3$, see Figure 1.2. However, as shown in [20], this space is not aspherical. The pure welded braid group wP_n can be viewed as the pure motion group of n unknotted, unlinked circles in $\mathbb{R}^3$, cf. [61]. The group wP_n^+ is the fundamental group of the subspace consisting of all configurations of circles of unequal diameters in the 'untwisted ring space', see [20, 12].

In [75], Jensen, McCammond, and Meier computed the cohomology ring of wP_n, thereby verifying a long-standing conjecture of Brownstein and Lee.

Theorem 8.1.3 ([75]). *For each $n \geq 2$, the ring $H^*(wP_n; \mathbb{Z})$ is the graded-commutative ring generated by degree 1 elements e_{ij} $(1 \leq i \neq j \leq n)$, subject to the relations $e_{ij}e_{ji} = 0$ for $i \neq j$ and $e_{ji}e_{kj} = e_{ki}(e_{kj} - e_{ij})$ for distinct i, j, k. The Hilbert series of this ring is $(1 + nt)^{n-1}$.*

As shown in [36], the cohomology algebra $H^*(wP_n; \mathbb{Q})$ is not Koszul for $n \geq 4$. The graded Lie algebra $\mathrm{gr}(wP_n)$ is generated by x_{ij} for $1 \leq i \neq j \leq n$ with relations $[x_{ij}, x_{st}] = 0$ for $\{i, j\} \cap \{s, t\} = \emptyset$, $[x_{ik}, x_{ij} + x_{kj}] = 0$ and $[x_{ij}, x_{kj}] = 0$ for i, j, k distinct.

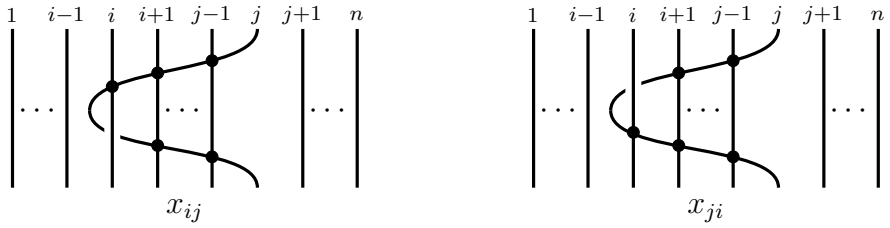


Figure 8.1: The pure welded braids x_{ij} and x_{ji} for $i < j$.

8.1.2 Upper McCool groups

The *upper-triangular McCool group*, wP_n^+, is the subgroup of the McCool group generated by the automorphisms α_{ij} with $i < j$. In [29], Cohen, Pakhianathan, Vershinin, and Wu computed the cohomology ring of wP_n^+ and found presentations for the associated graded Lie algebras of wP_n^+ as the quotient of the free Lie algebra on $x_{21}, \ldots, x_{n,n-1}$ by the ideal generated by $[x_{kj}, x_{st}] = 0$ if $\{j,k\} \cap \{s,t\} = \emptyset$, $[x_{kj}, x_{sj}] = 0$ if $\{s,k\} \cap \{j\} = \emptyset$, and $[x_{ik}, x_{ij} + x_{kj}] = 0$ if $j < k < i$.

Work of Berceanu and Papadima [15] shows that the McCool groups as well as their upper subgroups are 1-formal.

McCool gave a presentation of the *basis-conjugating group* (*McCool group*) wP_n with generators x_{ij} for $i \neq j$ ([112]). The subgroup of wP_n generated by x_{ij} for $i > j$ is called the *upper triangular McCool group*, denoted by wP_n^+. The integral cohomology of wP_n^+ was computed by Cohen, Pakhianathan, Vershinin, and Wu, as follows.

Theorem 8.1.4 ([29]). *The cohomology algebra $A = H^*(wP_n^+; \mathbb{C}) = E/I$ is a graded, graded-commutative (associative) algebra, generated by degree 1 elements u_{ij} for $1 \leq j < i \leq n$, subject to the relations $u_{ij}(u_{ik} - u_{jk}) = 0$ for $k < j < i$.*

8.2 The infinitesimal Alexander invariant of wP_n^+

In this section, we give a presentation for the infinitesimal Alexander invariant of the upper McCool groups. We also simplify this presentation to a reduced presentation.

8.2.1 Basis of free modules

We will use Theorem 8.1.4 to compute a presentation for the infinitesimal Alexander invariant $\mathfrak{B}_n := \mathfrak{B}(wP_n^+)$. We first choose an order for the basis of $H^1(wP_n^+; \mathbb{C})$ by setting

$$u_{ij} \succ u_{kl}, \text{ if } i > k, \text{ or if } i = k \text{ and } j > l. \tag{8.2}$$

Let $\mathbf{x} = \{x_{ij} \mid 1 \leq j < i \leq n\}$ be the dual of the basis $\{u_{ij}\}$ of $H^1(wP_n^+; \mathbb{C})$, and let $S = \mathbb{Q}[\mathbf{x}]$ be the coordinate ring of $H^1(wP_n; \mathbb{C})$. The relations

$$\{r^*_{ijk} := (u_{ik} - u_{jk})u_{ij} \mid 1 \leq k < j < i \leq n\} \tag{8.3}$$

of the cohomology algebra $A = E/I$ from Theorem 8.1.4 give a basis for I^2, as well as for the free S-module $I^2 \otimes S = S^{\binom{n}{3}}$. Choose an order of the basis of the free S-module $I^2 \otimes S = S^{\binom{n}{3}}$ given by

$$r^*_{lst} \succ r^*_{ijk}, \text{ if } i > l, \text{ or if } i = l \text{ and } j > s, \text{ or if } i = l, j = s \text{ and } k > t. \tag{8.4}$$

That is,

$$\{r_{n,n-1,n-2} \succ \cdots \succ r_{654} \succ r_{653} \succ r_{652} \succ r_{651} \succ r_{643} \succ \cdots \succ r_{431} \succ r_{421} \succ r_{321}\} \tag{8.5}$$

The set $\{u_{st}u_{lk}u_{ij} \mid u_{ij} \succ u_{lk} \succ u_{st}\}$ gives a basis for E^3, and a basis for the free S-module $E^3 \otimes S = S^{\binom{N}{3}}$ for $N = \binom{n}{2}$.

8.2.2 A presentation for $\mathfrak{B}_n$

We first give a detailed description of the composite $\Phi\colon I^2 \otimes S \to E^3 \otimes S$, using the above basis.

Lemma 8.2.1. *The S-linear map $\Phi\colon I^2 \otimes S \to E^3 \otimes S$ is given by*

$$\Phi(r^*_{ijk}) = -(x_{ik} + x_{jk}) \cdot u_{jk}u_{ik}u_{ij} + \sum_{s>t, \{s,t\} \not\subseteq \{i,j,k\}} x_{st} \cdot u_{st}(u_{jk} - u_{ik})u_{ij}. \tag{8.6}$$

Proof. Recall that Φ is the composition of the differential $d^2\colon E^2 \to E^3$ from (5.3) with the inclusion $\iota\colon I^2 \to E^2$. Hence,

$$\Phi(r^*_{ijk}) = d^2(r^*_{ijk} \otimes 1) = \sum_{1 \leq t < s \leq n}^{n} u_{st}r_{ijk} \otimes x_{st} = \sum_{1 \leq t < s \leq n}^{n} u_{st}u_{ij}(u_{ik} - u_{jk}) \otimes x_{st} \tag{8.7}$$

Simplifying the last formula using graded-commutativity yields (8.6). □

From formula (8.6), we can see that each entry of the matrix of Φ is of the form $x_{ik}+x_{jk}$ or x_{st} for $\{s,t\} \not\subseteq \{i,j,k\}$, $t<s$ and $k<j<i$. For our purposes here, we are interested in the transpose of the presentation matrix of Φ.

By Lemmas 5.3.1 and 8.2.1, the dual of $\Phi\colon I^2\otimes S \to E^3\otimes S$ gives a presentation for the S-module $\mathfrak{B}_n$. The infinitesimal Alexander invariant $\mathfrak{B}_n$ has presentation

$$(E^3)^*\otimes S \xrightarrow{\Phi^*} (I^2)^*\otimes S \longrightarrow \mathfrak{B}_n. \tag{8.8}$$

8.2.3 A reduced presentation of $\mathfrak{B}_n$

In this section, we simplify the presentation of the S-module $\mathfrak{B}_n = \mathfrak{B}(wP_n^+)$ from (8.8). We also single out some properties of the simplified presentation. Let $\{r_{ijk} \mid 1\le k<j<i\le n\}$ be the dual basis of $I^2\otimes S$ from (8.3).

Lemma 8.2.2. *The submodule* $\operatorname{im}\Phi^*$ *of* $(I^2)^*\otimes S = S^{\binom{n}{3}}$ *is generated by* $\mathcal{B}=\bigcup \mathcal{B}_{ijk}$, *where* $\mathcal{B}_{ijk}$, $1\le k<j<i\le n$, *contains the following elements: (We give two notations for each class of elements. If there is no confusion, we use the short notation.)*

$$\begin{aligned}
\mathbf{g}_1 &= \mathbf{g}^{(1)}_{ijl_2k} = (-x_{jk}-x_{l_2k})\cdot r_{ijl_2} + x_{jl_2}\cdot r_{ijk} & \mathbf{h}_3 &= \mathbf{h}^{(3)}_{ijl_2k} = x_{l_2k}\cdot r_{ijk}\\
\mathbf{g}_2 &= \mathbf{g}^{(2)}_{ijl_2k} = x_{jk}\cdot r_{ijl_2} + x_{il_2}\cdot r_{ijk} & \mathbf{h}_4 &= \mathbf{h}^{(2)}_{il_3jk} = x_{l_3k}\cdot r_{ijk}\\
\mathbf{g}_3 &= \mathbf{g}^{(1)}_{il_3jk} = -x_{jk}\cdot r_{il_3j} + x_{il_3}\cdot r_{ijk} & \mathbf{h}_5 &= \mathbf{h}^{(3)}_{il_3jk} = x_{l_3j}\cdot r_{ijk}\\
\mathbf{g}_4 &= \mathbf{g}^{(1)}_{l_4ijk} = x_{jk}\cdot r_{l_4ij} + x_{l_4i}\cdot r_{ijk} & \mathbf{h}_6 &= \mathbf{h}^{(2)}_{l_4ijk} = x_{l_4k}\cdot r_{ijk}\\
\mathbf{h}_1 &= \mathbf{h}_{ijkl_1} = (x_{il_1}+x_{jl_1}+x_{kl_1})\cdot r_{ijk} & \mathbf{h}_7 &= \mathbf{h}^{(3)}_{l_4ijk} = x_{l_4j}\cdot r_{ijk}\\
\mathbf{h}_2 &= \mathbf{h}_{ijk} = (x_{ik}+x_{jk})\cdot r_{ijk} & \mathbf{h}_{st} &= \mathbf{h}_{stijk} = x_{st}\cdot r_{ijk}
\end{aligned} \tag{8.9}$$

where $1\le l_1<k<l_2<j<l_3<i<l_4\le n$ *and* $\{s,t\}\cap\{i,j,k\}=\emptyset$.

Proof. Write Φ^*_q for the restriction of Φ^* to the subspace spanned by the basis vectors of cardinality $q := \sharp\{i,j,k,l,s,t\}$. The map Φ^* can then be decomposed as the block-matrix $\Phi^*_3\oplus\Phi^*_4\oplus\Phi^*_5\oplus\Phi^*_6$. We now analyze formula (8.6) case by case, according to the cardinality $q=3,4,5,6$.

When $q = 3$, we have $l = i, s = j, t = k$. Then $\Phi_3^*((u_{jk}u_{ik}u_{ij})^*) = -(x_{ik} + x_{jk}) \cdot r_{ijk}$, and so Φ_3^* contributes elements of the form $\mathbf{h}_2$ to $\mathcal{B}$.

When $q = 4$, suppose $i > j > k > l$; there are $\binom{6}{3} - \binom{4}{3} = 16$ possible combinations:

$$\Phi_4^*((u_{kl}u_{jl}u_{il})^*) = 0$$

$$\Phi_4^*((u_{kl}u_{jl}u_{ik})^*) = -x_{jl} \cdot r_{ikl}$$

$$\Phi_4^*((u_{kl}u_{jl}u_{ij})^*) = x_{kl} \cdot r_{ijl}$$

$$\Phi_4^*((u_{kl}u_{jk}u_{il})^*) = x_{il} \cdot r_{jkl}$$

$$\Phi_4^*((u_{kl}u_{jk}u_{ik})^*) = -x_{jk} \cdot r_{ikl} + x_{ik} \cdot r_{jkl}$$

$$\Phi_4^*((u_{kl}u_{jk}u_{ij})^*) = x_{kl} \cdot r_{ijk} + x_{ij} \cdot r_{jkl}$$

$$\Phi_4^*((u_{kl}u_{il}u_{ij})^*) = -x_{kl} \cdot r_{ijl}$$

$$\Phi_4^*((u_{kl}u_{ik}u_{ij})^*) = -x_{kl} \cdot r_{ijk} + x_{ij} \cdot r_{ikl}$$

$$\Phi_4^*((u_{jl}u_{jk}u_{il})^*) = -x_{il} \cdot r_{jkl}$$

$$\Phi_4^*((u_{jl}u_{jk}u_{ik})^*) = -x_{ik} \cdot r_{jkl}$$

$$\Phi_4^*((u_{jl}u_{jk}u_{ij})^*) = x_{jl} \cdot r_{ijk} - x_{jk} \cdot r_{ijl} - x_{ij} \cdot r_{jkl}$$

$$\Phi_4^*((u_{jl}u_{il}u_{ik})^*) = -x_{jl} \cdot r_{ikl}$$

$$\Phi_4^*((u_{jl}u_{ik}u_{ij})^*) = -x_{jl} \cdot r_{ijk} - x_{ik} \cdot r_{ijl}$$

$$\Phi_4^*((u_{jk}u_{il}u_{ik})^*) = -x_{jk} \cdot r_{ikl}$$

$$\Phi_4^*((u_{jk}u_{il}u_{ij})^*) = -x_{il} \cdot r_{ijk} - x_{jk} \cdot r_{ijl}$$

$$\Phi_4^*((u_{il}u_{ik}u_{ij})^*) = -x_{il} \cdot r_{ijk} + x_{ik} \cdot r_{ijl} - x_{ij} \cdot r_{ikl}$$

The image of Φ_4^* is generated by $(x_{il} + x_{jl} + x_{kl}) \cdot r_{ijk}$ and

$x_{kl} \cdot r_{ijl}$	$x_{jl} \cdot r_{ikl}$	$x_{il} \cdot r_{jkl}$
$(-x_{jl} - x_{kl}) \cdot r_{ijk} + x_{jk} \cdot r_{ijl}$	$x_{jk} \cdot r_{ikl}$	$x_{ik} \cdot r_{jkl}$
$x_{jl} \cdot r_{ijk} + x_{ik} \cdot r_{ijl}$	$-x_{kl} \cdot r_{ijk} + x_{ij} \cdot r_{ikl}$	$x_{kl} \cdot r_{ijk} + x_{ij} \cdot r_{jkl}$

Hence, the image of Φ_4^* contributes $\mathbf{h}_1 = (x_{il_1} + x_{jl_1} + x_{kl_1}) \cdot r_{ijk}$ for $l_1 \leq k - 1$, contributes

$\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{h}_3$ for $k < l_2 < j < i$, $\mathbf{g}_3$, $\mathbf{h}_4$, $\mathbf{h}_5$ for $k < j < l_3 < i$, and $\mathbf{g}_4$, $\mathbf{h}_6$, $\mathbf{h}_7$ for $k < j < i < l_4$.

When $q = 5$, the only possible situation for which $\Phi_5^* \neq 0$ is when $l = j$, or $l = i$, or $s = k$, or $s = l$. Suppose $u_{ij} > u_{lk} > u_{st}$. Using formula (8.6) again, we then have

$$\Phi_5^*((u_{st}u_{lk}u_{ij})^*) = \begin{cases} x_{st} \cdot r_{ijk} & l = j \\ -x_{st} \cdot r_{ijk} & l = i \\ x_{ij} \cdot r_{lkt} & s = k \\ -x_{ij} \cdot r_{lkt} & s = l \\ 0 & otherwise \end{cases}$$

Hence, the map Φ_5^* will contribute $\mathbf{h}_{st} = x_{st} \cdot r_{ijk}$ for $\{s,t\} \cap \{i,j,k\} = \emptyset$ to $\mathcal{B}$.

When $q = 6$, we have $\Phi_6^*((u_{st}u_{lk}u_{ij})^*) = 0$. This completes the proof. □

Lemma 8.2.2 gives a minimal presentation for the infinitesimal Alexander invariant $\mathfrak{B}_n$, of the form

$$S^m \xrightarrow{\Psi} S^{\binom{n}{3}} \longrightarrow \mathfrak{B}_n. \tag{8.10}$$

Corollary 8.2.3. *The matrix of Ψ is block lower triangular, with diagonal vector v_{ijk} for $1 \leq k < j < i \leq n$. The vector v_{ijk} has $m_{ijk} = \binom{n}{2} - 2k$ entries given by $x_{il} + x_{jl} + x_{kl}$ for $1 \leq l \leq k-1$, $x_{ik} + x_{jk}$, and x_{st} for $\{s,t\} \not\subset \{i,j,k,l\}$, $1 \leq l \leq k-1$. Here $m = \frac{1}{12}n(40 - 73n + 43n^2 - 11n^3 + n^4)$.*

Proof. By Lemma 8.2.2, the matrix of Ψ has the claimed form. We find that $m_{ijk} = \binom{n}{2} - 2k$. The matrix of Ψ is a block triangular matrix, we have that

$$\begin{aligned} m &= \sum_{1 \leq k < j < i \leq n} (m_{ijk}) \\ &= \sum_{k=1}^{n} \binom{n-k}{2} (m_{ijk}) \\ &= \sum_{k=1}^{n} \binom{n-k}{2} \left(\binom{n}{2} - 2k \right) \\ &= \binom{n}{2} \left[\binom{n-3}{3} + \binom{n-3}{2} + \binom{n-3}{1} \right] - 2\binom{n+1}{4} \end{aligned}$$

$$= \frac{n}{12}(40 - 73n + 43n^2 - 11n^3 + n^4).$$

This finishes the proof. □

Example 8.2.4. The matrix Ψ for $n = 4$ looks like

$$\begin{pmatrix} v_{432} & 0 & 0 & 0 \\ * & v_{431} & 0 & 0 \\ * & * & v_{421} & 0 \\ * & * & * & v_{321} \end{pmatrix} = \begin{pmatrix} x_{41}+x_{31}+x_{21} & 0 & 0 & 0 \\ x_{42}+x_{32} & 0 & 0 & 0 \\ 0 & x_{21} & 0 & 0 \\ -x_{31}-x_{21} & x_{32} & 0 & 0 \\ 0 & x_{41}+x_{31} & 0 & 0 \\ x_{31} & x_{42} & 0 & 0 \\ 0 & 0 & x_{31} & 0 \\ 0 & 0 & x_{32} & 0 \\ 0 & 0 & x_{41}+x_{21} & 0 \\ -x_{21} & 0 & x_{43} & 0 \\ 0 & 0 & 0 & x_{31}+x_{21} \\ 0 & 0 & 0 & x_{41} \\ 0 & 0 & 0 & x_{42} \\ x_{21} & 0 & 0 & x_{43} \end{pmatrix}.$$

Example 8.2.5. The matrix Ψ for $n = 5$ looks like

$$\begin{pmatrix} v_{543} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & v_{542} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & v_{541} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & v_{532} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & v_{531} & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & v_{521} & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & v_{432} & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & v_{431} & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 & v_{421} & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 & 0 & v_{321} \end{pmatrix}$$

where $v_{543} = \begin{pmatrix} x_{51}+x_{41}+x_{31} \\ x_{52}+x_{42}+x_{32} \\ x_{53}+x_{43} \end{pmatrix}$ $v_{542} = \begin{pmatrix} x_{31} \\ x_{32} \\ x_{43} \\ x_{51}+x_{41}+x_{21} \\ x_{52}+x_{42} \\ x_{53} \end{pmatrix}$ $v_{541} = \begin{pmatrix} x_{21} \\ x_{31} \\ x_{32} \\ x_{42} \\ x_{43} \\ x_{51}+x_{41} \\ x_{52} \\ x_{53} \end{pmatrix}$ $v_{532} = \begin{pmatrix} x_{41} \\ x_{42} \\ x_{43} \\ x_{51}+x_{31}+x_{21} \\ x_{52}+x_{32} \\ x_{54} \end{pmatrix}$

$v_{531} = \begin{pmatrix} x_{21} \\ x_{32} \\ x_{41} \\ x_{42} \\ x_{43} \\ x_{51}+x_{31} \\ x_{52} \\ x_{54} \end{pmatrix}$ $v_{521} = \begin{pmatrix} x_{31} \\ x_{32} \\ x_{41} \\ x_{42} \\ x_{43} \\ x_{51}+x_{21} \\ x_{53} \\ x_{54} \end{pmatrix}$ $v_{432} = \begin{pmatrix} x_{41}+x_{31}+x_{21} \\ x_{42}+x_{32} \\ x_{51} \\ x_{52} \\ x_{53} \\ x_{54} \end{pmatrix}$ $v_{431} = \begin{pmatrix} x_{21} \\ x_{32} \\ x_{41}+x_{31} \\ x_{42} \\ x_{51} \\ x_{52} \\ x_{53} \\ x_{54} \end{pmatrix}$ $v_{421} =$

$$\begin{pmatrix} x_{31} \\ x_{32} \\ x_{41}+x_{21} \\ x_{43} \\ x_{51} \\ x_{52} \\ x_{53} \\ x_{54} \end{pmatrix} v_{321} = \begin{pmatrix} x_{31}+x_{21} \\ x_{41} \\ x_{42} \\ x_{43} \\ x_{51} \\ x_{52} \\ x_{53} \\ x_{54} \end{pmatrix}$$

8.3 A Gröbner basis for $\mathfrak{B}(wP_n^+)$

In this section, we compute a Gröbner basis for the infinitesimal Alexander invariant, which will be crucial in the computation of the resonance varieties and the Chen ranks of the upper McCool groups.

8.3.1 Gröbner basis for modules

Before proceeding, we recall some background material on Gröbner basis for modules, following Eisenbud [45]. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring and F be a free R-module with basis $\{e_1, \ldots, e_r\}$. A *monomial* in F is an element of form $m = \mathbf{x}^\alpha e_i$ and a *term* in F is an element of the form $c \cdot \mathbf{x}^\alpha e_i$, where $c \in \mathbb{C}$ is the coefficient.

A *monomial order* on F is a total order $\succ$ on the monomials of F such that if m_1 and m_2 are monomials of F and $n \neq 1$ is a monomial in R, then $m_1 > m_2$ implies $nm_1 > nm_2 > m_2$.

Example 8.3.1. A graded reverse lexicographic ('grevlex') order on R is defined by ordering $\mathbf{x}^\alpha > \mathbf{x}^\beta$ if $\deg(\alpha) > \deg(\beta)$, or $\deg(\alpha) = \deg(\beta)$ and the right-most entry in $\alpha - \beta$ is negative. One way to extend a 'grevlex' order on R to a monomial order on F is to declare $\mathbf{x}^\alpha e_i > \mathbf{x}^\beta e_j$ if $i < j$, or if $i = j$ and $\mathbf{x}^\alpha > \mathbf{x}^\beta$.

Given a monomial order $>$ on F and $f \in F$, the *initial term* of f is the largest term of f, denoted by $\mathrm{in}_>(f)$. If I is a submodule of F, then $\mathrm{in}_> I$ denotes the submodule generated by $\{\mathrm{in}_>(f) \mid f \in I\}$. If $g_1, \ldots, g_s$ generate I such that $\{\mathrm{in}_>(g_1), \ldots, \mathrm{in}_>(g_s)\}$ generates $\mathrm{in}_>(F)$, then we will call $\{g_1, \ldots, g_s\}$ a *Gröbner basis* for module I.

Theorem 8.3.2 ([45], Theorem15.26)**.** *Let $M = F/I$ be a finitely generated, graded R-module given by generators and relations, where F is a free R-module with a homogeneous*

basis and I is a submodule generated by homogeneous elements. Then

$$\mathrm{Hilb}(M,t) = \mathrm{Hilb}(F/\operatorname{in}(I),t).$$

A Gröbner basis $g_1,\dots,g_s$ such that $\operatorname{in}(g_i)$ does not divide $\operatorname{in}(g_i)$ for any $i \neq j$ is called a *minimal* Gröbner basis. A Gröbner basis $g_1,\dots,g_s$ such that $\operatorname{in}(g_i)$ does not divide any term of g_i for any $i \neq j$ is called a *reduced* Gröbner basis.

Let $\{g_1,\dots,g_t\}$ be a set of nonzero elements in F. Let $\bigoplus R\epsilon_i$ be the free module on $\{\epsilon_1,\dots,\epsilon_t\}$ corresponding to $\{g_1,\dots,g_t\}$. If $\operatorname{in}_>(g_i)$ and $\operatorname{in}_>(g_j)$ involve the same basis element of F, then define

$$\mathfrak{S}(g_i,g_j) := \frac{\operatorname{in}_>(g_j)}{\gcd(\operatorname{in}_>(g_i),\operatorname{in}_>(g_j))}\cdot g_i - \frac{\operatorname{in}_>(g_i)}{\gcd(\operatorname{in}_>(g_i),\operatorname{in}_>(g_j))}\cdot g_j \tag{8.11}$$

Using the division algorithm, $\mathfrak{S}(g_i,g_j)$ has a standard expression

$$\mathfrak{S}(g_i,g_j) = \sum f_k^{ij}\cdot g_k + h_{ij}, \tag{8.12}$$

where $\operatorname{in}_>(f_k^{ij}g_k) < \mathrm{LCM}(\operatorname{in}_>(g_i),\operatorname{in}_>(g_j))$. We say that the $\mathfrak{S}$-polynomial of g_i and g_j *vanishes*, if $h_{ij}=0$. If $\operatorname{in}_>(g_i)$ and $\operatorname{in}_>(g_j)$ involve distinct basis elements of F, then set $h_{ij}=0$.

Theorem 8.3.3 (Buchberger's criterion)**.** *The elements $g_1,\dots,g_t$ form a Gröbner basis if and only if $h_{ij}=0$ for all i and j.*

Let M be an S-module with a finite presentation, $S^m \xrightarrow{\varphi} S^n \to M \to 0$, i.e., $M = S^n/J$ where $J=\operatorname{im}(\varphi)$. Choosing basis $\{r_1,\dots,r_m\}$ and $\{e_1,\dots,e_n\}$ for the free modules S^m and S^n, respectively, we can view the S-linear map φ as an $m\times n$ matrix Ω, and J as the module generated by the entries of the matrix $\Omega\cdot(e_1,\dots,e_n)^{\mathsf{T}}$. A Gröbner basis for the module J is also called a Gröbner basis for the matrix Ω.

Lemma 8.3.4. *Let Ω be a presentation matrix for the S-module M, and let $\mathbf{G}$ be a Gröbner basis for Ω. Then* $\operatorname{rank}(\Omega|_a) = \operatorname{rank}(\mathbf{G}|_a)$. *Furthermore, if $\mathbf{G}$ is a block triangular matrix*

with diagonal blocks $\mathbf{G}_{ii}$*, then*

$$\mathbf{V}(E_0(\mathbf{G})) \subseteq \bigcup_i \mathbf{V}(E_0(\mathbf{G}_{ii})).$$

Proof. The $\mathfrak{S}$-polynomials arise from elementary row operations, and these operations do not change the rank of a matrix. The second statement follows from (3) in Lemma 5.3.2. □

8.3.2 A Gröbner basis for $\mathfrak{B}_n$

Let $S = \mathbb{C}[\mathbf{x}]$ be the coordinate ring of $H^1(wP_n; \mathbb{C})$ with variables ordered as

$$x_{ij} > x_{kl}, \text{ if } i > k, \text{ or if } i = k \text{ and } j > l. \tag{8.13}$$

We use the graded reverse lexicographical monomial order (grevlex) on the polynomial ring S. We use the basis and orders of basis from §8.2.1. Recall the presentation of $\mathfrak{B}_n$ in (8.10) $S^m \xrightarrow{\Psi} S^{\binom{n}{3}} \to \mathfrak{B}_n$. From Lemma 8.2.2, the module $\mathrm{im}(\Psi)$ is generated by $\mathcal{B} = \bigcup \mathcal{B}_{ijk}$.

Theorem 8.3.5. *A reduced Gröbner basis for* $\mathrm{im}(\Psi)$ *is given by* $\mathcal{G} = \bigcup \mathcal{G}_{ijk}$*, where*

$$\mathcal{G}_{ijk} = \mathcal{B}_{ijk} \cup \{x_{kl}x_{ks} \cdot r_{ijk}, x_{jt}x_{ks} \cdot r_{ijk} \mid 1 \le s \le l \le k-1, 1 \le t \le k\}.$$

Proof. We use two steps to prove that $\mathcal{G}$ is a Gröbner basis for $\mathrm{im}(\Psi)$.

Step 1. We first show that

$$\mathcal{B}_1 = \{x_{kl}x_{ks} \cdot r_{ijk}, x_{jt}x_{ks} \cdot r_{ijk} \mid 1 \le s \le l \le k-1, 1 \le t \le k\} \tag{8.14}$$

are elements in the submodule $\mathrm{im}(\Psi)$. From Lemma 8.2.2, we know that the following

elements are in the module $\mathrm{im}(\Psi)$ for $n \geq i > j > k > l \geq 1$ and $k > v \geq 1$:

$$\begin{aligned}
\mathbf{f}_1 &= (-x_{jl} - x_{kl}) \cdot r_{ijk} + x_{jk} \cdot r_{ijl} \\
\mathbf{f}_2 &= x_{jl} \cdot r_{ijk} + x_{ik} \cdot r_{ijl} \\
\mathbf{f}_3 &= -x_{kl} \cdot r_{ijk} + x_{ij} \cdot r_{ikl} \\
\mathbf{f}_4 &= x_{kl} \cdot r_{ijl} \\
\mathbf{f}_5 &= x_{kv} \cdot r_{ijl} \text{ for } v \neq l \\
\mathbf{f}_6 &= x_{jk} \cdot r_{ikl} \\
\mathbf{f}_7 &= x_{jl} \cdot r_{ikl} \\
\mathbf{f}_8 &= x_{jv} \cdot r_{ikl} \text{ for } v \neq l
\end{aligned}$$

Direct computation shows that

$$\begin{aligned}
x_{kl}^2 \cdot r_{ijk} &= (x_{jk} + x_{jl})\mathbf{f}_4 - x_{kl}(\mathbf{f}_1 + \mathbf{f}_2) \\
x_{kl}x_{kv} \cdot r_{ijk} &= (x_{jk} + x_{jl})\mathbf{f}_5 - x_{kv}(\mathbf{f}_1 + \mathbf{f}_2) \\
x_{jk}x_{kl} \cdot r_{ijk} &= x_{ij}\mathbf{f}_6 - x_{jk}\mathbf{f}_3 \\
x_{jl}x_{kl} \cdot r_{ijk} &= x_{ij}\mathbf{f}_7 - x_{jl}\mathbf{f}_3 \\
x_{jv}x_{kl} \cdot r_{ijk} &= x_{ij}\mathbf{f}_8 - x_{jv}\mathbf{f}_3
\end{aligned}$$

from which we conclude that $\mathcal{B}_1 \subset \mathrm{im}(\Psi)$.

Step 2. In view of Lemma 8.2.2, we need to check the vanishing of the $\mathfrak{S}$-polynomials among

$$\mathcal{B}_2 = \{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4\}, \tag{8.15}$$

and the $\mathfrak{S}$-polynomials between $\mathcal{G}_2$ and

$$\mathcal{B}_3 = \mathcal{B}_1 \cup \{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \mathbf{h}_4, \mathbf{h}_5, \mathbf{h}_6, \mathbf{h}_7, \mathbf{h}_{st}\}. \tag{8.16}$$

We have

$$\begin{aligned}\mathfrak{S}(\mathbf{g}_1,\mathbf{g}_2) &= (x_{il_2}x_{jk}+x_{jl_2}x_{jk}+x_{il_2}x_{l_2k})\cdot r_{ijl_2}\\ &= (x_{il_2}+x_{jl_2})(x_{jk}+x_{l_2k})\cdot r_{ijl_2}-x_{jl_2}x_{l_2k}\cdot r_{ijl_2}\\ &= (x_{jk}+x_{l_2k})\mathbf{h}_{ijl_2}-x_{jl_2}x_{l_2k}\cdot r_{ijl_2}\end{aligned} \tag{8.17}$$

and this "vanishes" as in the expression (8.12).

In order to verify the vanishing of all the $\mathfrak{S}$-polynomials, we relabel the index for (8.9) here, by writing $l_1=1$, $k=2$, $l_2=3$, $j=4$, $l_3=5$, $i=6$, $l_4=7$.

$$\begin{aligned}\mathbf{g}_1 &= (-x_{42}-x_{32})\cdot r_{643}+x_{43}\cdot r_{642} & \mathbf{h}_3 &= x_{32}\cdot r_{642}\\ \mathbf{g}_2 &= x_{42}\cdot r_{643}+x_{63}\cdot r_{642} & \mathbf{h}_4 &= x_{52}\cdot r_{642}\\ \mathbf{g}_3 &= -x_{42}\cdot r_{654}+x_{65}\cdot r_{642} & \mathbf{h}_5 &= x_{54}\cdot r_{642}\\ \mathbf{g}_4 &= x_{42}\cdot r_{764}+x_{76}\cdot r_{642} & \mathbf{h}_6 &= x_{72}\cdot r_{642}\\ \mathbf{h}_1 &= (x_{61}+x_{41}+x_{21})\cdot r_{642} & \mathbf{h}_7 &= x_{74}\cdot r_{642}\\ \mathbf{h}_2 &= (x_{62}+x_{42})\cdot r_{642} & \mathbf{h}_{st} &= x_{st}\cdot r_{642}.\end{aligned}$$

Here, $\mathbf{h}_{st}$ can be $\mathbf{h}_{75}$, $\mathbf{h}_{73}$, $\mathbf{h}_{71}$, $\mathbf{h}_{53}$, $\mathbf{h}_{51}$, $\mathbf{h}_{31}$. The reason we relabel the index here is that we can reduce the computation to wP_7^+. Then, we can calculate the $\mathfrak{S}$-polynomials with Macaulay 2 [98].

The $\mathfrak{S}$-polynomials $\mathfrak{S}(\mathbf{h}_i,\mathbf{h}_j)$ vanishes, since they are linear after factor out r_{642}. The vanishing of the other $\mathfrak{S}$-polynomials are based on the following classes of reasons:

(i) The monomials in a $\mathfrak{S}$-polynomial can factor out $\mathbf{h}_4$, $\mathbf{h}_5$, $\mathbf{h}_6$, $\mathbf{h}_7$ or $\mathbf{h}_{s,t}$ in (8.9).

(ii) The monomials in a $\mathfrak{S}$-polynomial are contained in $\mathcal{B}_1$ in (8.14).

(iii) Rewrite the $\mathfrak{S}$-polynomial first, then use (i) and (ii).

Next, we list the $\mathfrak{S}$-polynomials $\mathfrak{S}(\mathbf{g}_i, \mathbf{g}_j)$ and $\mathfrak{S}(\mathbf{g}_i, \mathbf{h}_j)$ and the reason for vanishing.

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_{31}) = x_{42}x_{31}{\cdot}r_{764} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_3) = x_{42}x_{32}{\cdot}r_{764} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_{51}) = x_{51}x_{42}{\cdot}r_{764} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_4) = x_{52}x_{42}{\cdot}r_{764} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_{53}) = x_{53}x_{42}{\cdot}r_{764} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_5) = x_{54}x_{42}{\cdot}r_{764} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_1) = (x_{61}x_{42} + x_{42}x_{41} + x_{42}x_{21}){\cdot}r_{764} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_2) = (x_{62}x_{42} + x_{42}^2){\cdot}r_{764} \quad \cdots\cdots \text{(ii)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_{71}) = x_{71}x_{42}{\cdot}r_{764} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_6) = x_{72}x_{42}{\cdot}r_{764} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_{73}) = x_{73}x_{42}{\cdot}r_{764} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_7) = x_{74}x_{42}{\cdot}r_{764} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{h}_{75}) = x_{75}x_{42}{\cdot}r_{764} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_{31}) = x_{42}x_{31}{\cdot}r_{654} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_3) = x_{42}x_{32}{\cdot}r_{654} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_{51}) = x_{51}x_{42}{\cdot}r_{654} \quad \cdots\cdots \text{(ii)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_4) = x_{52}x_{42}{\cdot}r_{654} \quad \cdots\cdots \text{(ii)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_{53}) = x_{53}x_{42}{\cdot}r_{654} \quad \cdots\cdots \text{(ii)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_5) = x_{54}x_{42}{\cdot}r_{654} \quad \cdots\cdots \text{(ii)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_1) = (x_{61}x_{42} + x_{42}x_{41} + x_{42}x_{21}){\cdot}r_{654} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_2) = (x_{62}x_{42} + x_{42}^2){\cdot}r_{654} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_{71}) = x_{71}x_{42}{\cdot}r_{654} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_6) = x_{72}x_{42}{\cdot}r_{654} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_{73}) = x_{73}x_{42}{\cdot}r_{654} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_7) = x_{74}x_{42}{\cdot}r_{654} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{h}_{75}) = x_{75}x_{42}{\cdot}r_{654} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{g}_4) = x_{65}x_{42}{\cdot}r_{764} + x_{76}x_{42}{\cdot}r_{654} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_{31}) = x_{42}x_{31}{\cdot}r_{643} \quad \cdots\cdots \text{(ii)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_3) = x_{42}x_{32}{\cdot}r_{643} \quad \cdots\cdots \text{(ii)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_{51}) = x_{51}x_{42}{\cdot}r_{643} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_4) = x_{52}x_{42}{\cdot}r_{643} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_{53}) = x_{53}x_{42}{\cdot}r_{643} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_5) = x_{54}x_{42}{\cdot}r_{643} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_1) = (x_{61}x_{42} + x_{42}x_{41} + x_{42}x_{21}){\cdot}r_{643} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_2) = (x_{62}x_{42} + x_{42}^2){\cdot}r_{643} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_{71}) = x_{71}x_{42}{\cdot}r_{643} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_6) = x_{72}x_{42}{\cdot}r_{643} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_{73}) = x_{73}x_{42}{\cdot}r_{643} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_7) = x_{74}x_{42}{\cdot}r_{643} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{h}_{75}) = x_{75}x_{42}{\cdot}r_{643} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{g}_3) = x_{63}x_{42}{\cdot}r_{654} + x_{65}x_{42}{\cdot}r_{643} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{g}_4) = x_{63}x_{42}{\cdot}r_{764} - x_{76}x_{42}{\cdot}r_{643} \quad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_{31}) = x_{32}x_{31}{\cdot}r_{643} \quad \cdots\cdots \text{(ii)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_3) = x_{32}^2{\cdot}r_{643} \quad \cdots\cdots \text{(ii)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_{51}) = x_{51}x_{32}{\cdot}r_{643} \quad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_4) = x_{52}x_{32}\cdot r_{643} \qquad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_{53}) = x_{53}x_{32}\cdot r_{643} \qquad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_5) = x_{54}x_{32}\cdot r_{643} \qquad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_1) = (x_{61}x_{32} + x_{41}x_{32} + x_{32}x_{21})\cdot r_{643} \qquad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_2) = x_{62}x_{32}\cdot r_{643} \qquad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_{71}) = x_{71}x_{32}\cdot r_{643} \qquad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_6) = x_{72}x_{32}\cdot r_{643} \qquad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_{73}) = x_{73}x_{32}\cdot r_{643} \qquad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_7) = x_{74}x_{32}\cdot r_{643} \qquad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{h}_{75}) = x_{75}x_{32}\cdot r_{643} \qquad \cdots\cdots \text{(i)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{g}_2) = (x_{63}x_{42} + x_{43}x_{42} + x_{63}x_{32})\cdot r_{643} \qquad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{g}_3) = x_{43}x_{42}\cdot r_{654} - (x_{65}x_{42} + x_{65}x_{32})\cdot r_{643} \qquad \cdots\cdots \text{(iii)}$$

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{g}_4) = x_{43}x_{42}\cdot r_{764} + (x_{76}x_{42} + x_{76}x_{32})\cdot r_{643} \qquad \cdots\cdots \text{(iii)}$$

Next, we list the rest $\mathfrak{S}$-polynomials.

$$\mathfrak{S}(\mathbf{g}_1, \mathbf{g}_1') = x_{43'}(x_{42} + x_{32}) \cdot r_{643} - x_{43}(x_{42} + x_{3'2}) \cdot r_{643'}$$

$$\mathfrak{S}(\mathbf{g}_2, \mathbf{g}_2') = x_{63'}x_{42} \cdot r_{643} - x_{63}x_{42} \cdot r_{643'}$$

$$\mathfrak{S}(\mathbf{g}_3, \mathbf{g}_3') = -x_{65'}x_{42} \cdot r_{654} + x_{65}x_{42} \cdot r_{65'4}$$

$$\mathfrak{S}(\mathbf{g}_4, \mathbf{g}_4') = x_{7'6}x_{42} \cdot r_{764} - x_{76}x_{42} \cdot r_{7'64}$$

We can reindex $(6433'2)$ by (64321), reindex $(655'42)$ by (65432), and reindex $(77'642)$ by (76542), and check the vanishing of the $\mathfrak{S}$-polynomials using the criterion (iii).

It is now easy to check that this Gröbner basis $\mathcal{G}$ of $\mathrm{im}(\Psi)$ is reduced. □

Recall from Corollary 8.2.3 that $\Omega(n)$ is the presentation matrix for the presentation of $\mathfrak{B}_n$ in (8.10).

Corollary 8.3.6. *The Gröbner basis of $\Omega(n)$ is given by a block lower triangular matrix $\mathbf{G}(n)$ with diagonal vector w_{ijk} for $1 \leq k < j < i \leq n$. Here, the vector w_{ijk} is constructed from v_{ijk} by adding elements $\{x_{kl}x_{ks}, x_{jt}x_{ks} \mid 1 \leq s \leq l \leq k-1, 1 \leq t \leq k\}$. Here,* $\dim(w_{ijk}) = \binom{n}{2} + \binom{k-1}{2} + (k-3)k$.

Proof. The dimension of w_{ijk} is determined by counting its quadratic elements and using the formula $\dim(v_{ijk}) = \binom{n}{2} - 2k$ from Corollary 8.2.3. □

Example 8.3.7. The matrix $\mathbf{G}(4)$ is

$$\begin{pmatrix} w_{432} & 0 & 0 & 0 \\ * & w_{431} & 0 & 0 \\ * & * & w_{421} & 0 \\ * & * & * & w_{321} \end{pmatrix} = \begin{pmatrix} x_{41}+x_{31}+x_{21} & 0 & 0 & 0 \\ x_{42}+x_{32} & 0 & 0 & 0 \\ \boxed{x_{21}x_{21}} & 0 & 0 & 0 \\ \boxed{x_{21}x_{31}} & 0 & 0 & 0 \\ \boxed{x_{21}x_{32}} & 0 & 0 & 0 \\ 0 & x_{21} & 0 & 0 \\ -x_{31}-x_{21} & x_{32} & 0 & 0 \\ 0 & x_{41}+x_{31} & 0 & 0 \\ x_{31} & x_{42} & 0 & 0 \\ 0 & 0 & x_{31} & 0 \\ 0 & 0 & x_{32} & 0 \\ 0 & 0 & x_{41}+x_{21} & 0 \\ -x_{21} & 0 & x_{43} & 0 \\ 0 & 0 & 0 & x_{31}+x_{21} \\ 0 & 0 & 0 & x_{41} \\ 0 & 0 & 0 & x_{42} \\ x_{21} & 0 & 0 & x_{43} \end{pmatrix}$$

Example 8.3.8. The matrix $\mathbf{G}(5)$ looks like

$$\begin{pmatrix} w_{543} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & w_{542} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & w_{541} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & w_{532} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & w_{531} & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & w_{521} & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & w_{432} & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & w_{431} & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 & w_{421} & 0 \\ * & * & 0 & * & 0 & 0 & * & 0 & 0 & w_{321} \end{pmatrix}$$

where $w_{543} = \begin{pmatrix} x_{51}+x_{41}+x_{31} \\ x_{52}+x_{42}+x_{32} \\ x_{53}+x_{43} \\ a_{31}^2 \\ a_{32}a_{31} \\ a_{41}a_{31} \\ a_{42}a_{31} \\ a_{43}a_{31} \\ a_{32}^2 \\ a_{41}a_{32} \\ a_{42}a_{32} \\ a_{43}a_{32} \end{pmatrix}$ $w_{542} = \begin{pmatrix} x_{31} \\ x_{32} \\ x_{43} \\ x_{51}+x_{41}+x_{21} \\ x_{52}+x_{42} \\ x_{53} \\ a_{21}^2 \\ a_{41}a_{21} \\ a_{42}a_{21} \end{pmatrix}$ $w_{541} = \begin{pmatrix} x_{21} \\ x_{31} \\ x_{32} \\ x_{42} \\ x_{43} \\ x_{51}+x_{41} \\ x_{52} \\ x_{53} \end{pmatrix}$ $w_{532} = \begin{pmatrix} x_{41} \\ x_{42} \\ x_{43} \\ x_{51}+x_{31}+x_{21} \\ x_{52}+x_{32} \\ x_{54} \\ a_{21}^2 \\ a_{31}a_{21} \\ a_{32}a_{21} \end{pmatrix}$

$w_{531} = \begin{pmatrix} x_{21} \\ x_{32} \\ x_{41} \\ x_{42} \\ x_{43} \\ x_{51}+x_{31} \\ x_{52} \\ x_{54} \end{pmatrix}$ $w_{521} = \begin{pmatrix} x_{31} \\ x_{32} \\ x_{41} \\ x_{42} \\ x_{43} \\ x_{51}+x_{21} \\ x_{53} \\ x_{54} \end{pmatrix}$ $w_{432} = \begin{pmatrix} x_{41}+x_{31}+x_{21} \\ x_{42}+x_{32} \\ x_{51} \\ x_{52} \\ x_{53} \\ x_{54} \\ a_{21}^2 \\ a_{31}a_{21} \\ a_{32}a_{21} \end{pmatrix}$ $w_{431} = \begin{pmatrix} x_{21} \\ x_{32} \\ x_{41}+x_{31} \\ x_{42} \\ x_{51} \\ x_{52} \\ x_{53} \\ x_{54} \end{pmatrix}$ $w_{421} = \begin{pmatrix} x_{31} \\ x_{32} \\ x_{41}+x_{21} \\ x_{43} \\ x_{51} \\ x_{52} \\ x_{53} \\ x_{54} \end{pmatrix}$ $w_{321} = \begin{pmatrix} x_{31}+x_{21} \\ x_{41} \\ x_{42} \\ x_{43} \\ x_{51} \\ x_{52} \\ x_{53} \\ x_{54} \end{pmatrix}$

8.4 The first resonance variety of wP_n^+

After completing the theoretical setup and performing the Gröbner basis computation for the module $\mathfrak{B}_n$, we are now ready to compute the first resonance varieties of the upper McCool groups.

8.4.1 Resonance varieties of the McCool groups

In [38], D. Cohen computed the first resonance varieties of the McCool groups.

Theorem 8.4.1 ([38]). *The resonance varieties $\mathcal{R}_1(wP_n)$ are given by*

$$\mathcal{R}_1(wP_n) = \bigcup_{1\leq i<j\leq n} C_{ij} \cup \bigcup_{1\leq i<j<k\leq n} C_{ijk}$$

where C_{ij} and C_{ijk} are certain linear subspaces of of $H^1(wP_n;\mathbb{C})$ of dimension 2 and 3, respectively.

Now we are ready to describe the first resonance varieties of the upper McCool groups wP_n^+.

Theorem 8.4.2. *The resonance varieties $\mathcal{R}_1(wP_n^+)$ are given by*

$$\mathcal{R}_1(wP_n^+) = \bigcup_{2\leq j<i\leq n} L_{ij},$$

where $L_{ij} = \mathbb{C}^j \subset \mathbb{C}^{\binom{n}{2}}$ is defined by the linear equations

$$\begin{cases} x_{i,l} + x_{j,l} = 0, \text{ for } 1 \leq l \leq j-1; \\ x_{i,l} = 0 \text{ for } j+1 \leq l \leq i-1; \\ x_{s,t} = 0 \text{ for } s \neq i, s \neq j, 1 \leq t < s. \end{cases} \tag{8.18}$$

Proof. Let $L = \bigcup_{2\leq j<i\leq n} L_{ij}$ be the subspace of $H^1(wP_n^+;\mathbb{C}) = \mathbb{C}^{\binom{n}{2}}$ with defining equations for each component L_{ij} given by (8.18). We will show that $L = \mathcal{R}_1(wP_n^+)$, thereby proving the claim of the theorem.

In order to prove that $L \subseteq \mathcal{R}_1(wP_n^+)$, we need to check that $L_{ij} \subseteq \mathcal{R}_1(wP_n^+)$ for all $i > j$. If $a \in L_{ij}$ is non-zero, then (8.18) implies that a is of the form

$$a = \sum_{l=1}^{j-1} a_{il}(u_{il} - u_{jl}) + a_{ij}u_{ij}. \tag{8.19}$$

By Lemma 5.1.5, we only need to check that the evaluation of Φ at a is not injective. By formula (5.11), we have that

$$\begin{aligned} \Phi|_a(r_{ijk}) &= \left(\sum_{l=1}^{j-1} a_{il}(u_{il} - u_{jl}) + a_{ij}u_{ij})(u_{ik} - u_{jk})u_{ij} \right) \\ &= \sum_{l=1, l\neq k}^{j-1} a_{il}(u_{il} - u_{jl})(u_{ik} - u_{jk})u_{ij}. \end{aligned} \tag{8.20}$$

Suppose $\mathbf{I}$ is a non-empty subset of $\{1,\dots,j-1\}$ such that $a_{il}\neq 0$ for $l\in\mathbf{I}$. Then

$$\Phi|_a\Big(\sum_{l\in\mathbf{I}}\prod_{t\neq l,t\in\mathbf{I}}a_{it}r_{ijl}\Big)=0,$$

and hence $\Phi|_a$ is not injective. Likewise, if $a_{il}=0$ for all $1\leq l\leq j-1$, i.e., if $a=a_{ij}u_{ij}$, then $\Phi|_a(r_{ijk})=0$, and so $\Phi|_a$ is not injective.

For the reverse inclusion, we use the Gröbner basis of the infinitesimal Alexander invariant $\mathfrak{B}_n$ provided by Theorem 8.3.5. The equation $w_{ijk}=0$ gives a linear space L_{ijk} such that $L_{ijk}\subset L_{i,j,j-1}$ and $L_{i,j,j-1}=L_{ij}$ defined by equations (8.18). By Lemma 5.3.2, we have that $\mathcal{R}_1(wP_n^+)\subseteq L$, and the proof is finished. □

The next proposition lists some of the basic properties of the (first) resonance varieties of the upper McCool groups.

Proposition 8.4.3. *Let L_{ij} be the components of the resonance variety of wP_n^+.*

1. *L_{ij} has basis $\{u_{jl}-u_{il},u_{ij},\ \text{for } 1\leq l\leq j-1\}$.*
2. *$L_{ij}\cap L_{k,l}=\{0\}$ if $i\neq k$ and $j\neq l$.*
3. *L_{ij} is isotropic (i.e., the cup product map $L_{ij}\wedge L_{ij}\to H^2(G;\mathbb{C})$ vanishes) if and only if $\dim L_{ij}=2$.*
4. *$\mathcal{R}_1(wP_n^+)=\mathcal{R}_1(wP_{n+1}^+)\cap H^1(wP_n^+;\mathbb{C})$.*

Proof. By Theorem 8.4.2, as a vector space, L_{ij} is the subspace of $H^1(wP_n^+;\mathbb{C})=\mathbb{C}^{\binom{n}{2}}$ defined by equations (8.18). By formula (8.19), L_{ij} has basis $\{u_{jl}-u_{il},u_{ij},\ \text{for } 1\leq l\leq j-1\}$. Using this basis, the other claims are readily verified. □

Lemma 8.4.4 (Corollary 5.3 [118])**.** *If $\alpha\colon G_1\to G_2$ is an epimorphism, then the induced monomorphism, $\alpha^*\colon H^1(G_2;\mathbb{C})\to H^1(G_1;\mathbb{C})$, takes $\mathcal{R}_1(G_2,\mathbb{C})$ to $\mathcal{R}_1(G_1,\mathbb{C})$.*

There is a split injection $\iota\colon wP_n^+\to wP_{n+1}^+$. However, using the first resonance varieties, we can show the following property.

Corollary 8.4.5. *For $n \geq 4$, there is no split monomorphism from wP_n^+ to wP_n.*

Proof. Suppose ι has a splitting epimorphism $\alpha\colon wP_n \twoheadrightarrow wP_n^+$. By Lemma 8.4.4, the epimorphism α induces a monomorphism $\alpha^* : H^1(wP_n^+;\mathbb{C}) \to H^1(wP_n;\mathbb{C})$ takes $\mathcal{R}_1(wP_n^+,\mathbb{C})$ to $\mathcal{R}_1(wP_n,\mathbb{C})$.

From Theorem 8.4.1, the first resonance $\mathcal{R}_1(wP_n)$ is a union of dimension 2 and 3 linear spaces. On the other side, the first resonance $\mathcal{R}_1(wP_n^+)$ contains dimension $n-1$ linear spaces. Hence, for $n \geq 5$, there is no epimorphism from wP_n to wP_n^+.

For $n = 4$, from Proposition 8.4.3, $L_{43} \subset \mathcal{R}_1(wP_4^+)$ is not isotropic. However, from [34] all components in $\mathcal{R}_1(wP_4)$ are isotropic. For any $a, b \in L_{43}$, such that $a \cup b \neq 0$, we have $\alpha^*(a) \cup \alpha^*(b) = \alpha^*(a \cup b) \neq 0$, by the injectivity of α^*. Hence, the monomorphism α^* must take non-isotropic component to non-isotropic component. This finish the proof. □

8.5 The Chen ranks of wP_n^+

We compute the Hilbert series of the infinitesimal Alexander invariant and the Chen ranks of the upper McCool groups.

8.5.1 Hilbert series of monomial ideals

The following comes from in [45, §15.1.1]. By Theorem 8.3.2, the Hilbert series of any graded module can be reduced to the computation of the Hilbert series of monomial module, that is $M = F/I$ such that I is monomials in F. Then $M = \bigoplus S/I_j$. Since the Hilbert function is additive, we only need to treat the case $M = S/I$ such that I is a monomial ideal.

Let I be the monomial ideal of S generated by $\{m_1, \ldots, m_t\}$. Choose a 'perfect' monomial $p \in F$, with degree d and let J be the monomial ideal generated by $\{p, m_1, \ldots, m_t\}$. Let I' be the ideal generated by $\{m_1/\gcd(m_1,p), \ldots, m_t/\gcd(m_t,p)\}$. We then have a short exact

sequence of graded modules

$$0 \longrightarrow S/I'(-d) \longrightarrow S/I \longrightarrow S/J \longrightarrow 0\,. \tag{8.21}$$

Lemma 8.5.1 ([45]). *Under the above notations, we have an equation of Hilbert series*

$$\mathrm{Hilb}(S/I,t) = \mathrm{Hilb}(S/J,t) + t^d\,\mathrm{Hilb}(S/I').$$

Remark 8.5.2. In the algorithm, choosing a perfect monomial $p \in F$ can make ideals I' and J have less generators than I. By choosing perfect p we can make the process very fast.

Example 8.5.3. Let $S = \mathbb{C}[x_{43}, x_{42}, x_{41}, x_{32}, x_{31}, x_{21}]$, and order the variables as $x_{43} > x_{42} > x_{41} > x_{32} > x_{31} > x_{21}$. Using the grevlex order on monomials, let us compute $\mathrm{Hilb}(S/I)$, where I is the ideal generated by $x_{41}, x_{42}, x_{32}x_{21}, x_{31}x_{21}, x_{21}^2$. Choose $p = x_{21}$ as the perfect monomial. Then $J = \{x_{41}, x_{42}, x_{21}\}$ and $I' = \{x_{41}, x_{42}, x_{32}, x_{31}, x_{21}\}$. Hence,

$$\mathrm{Hilb}(S/I,t) = \mathrm{Hilb}(S/J,t) + t\cdot\mathrm{Hilb}(S/I',t) = \frac{1}{(1-t)^3} + t\cdot\frac{1}{1-t}.$$

8.5.2 The Hilbert series of $\mathfrak{B}_n$

We are ready to compute the Hilbert series of the infinitesimal Alexander invariant of the upper McCool groups now.

Theorem 8.5.4. *The Hilbert series of the infinitesimal Alexander invariant $\mathfrak{B}_n$ is given by*

$$\mathrm{Hilb}(\mathfrak{B}_n) = \sum_{s=2}^{n-1}\binom{s}{2}\frac{1}{(1-t)^{n-s+1}} + \binom{n}{4}\frac{t}{1-t}. \tag{8.22}$$

Proof. This computation is an application of the method from [45, §15.1.1]. Since we already found a Gröbner basis $\mathcal{G}$ for $\mathrm{im}(\Psi)$, by Theorem 8.3.2, we only need to compute the Hilbert series of the resulting monomial ideals $\mathrm{in}_>(\mathrm{im}(\Psi)) = \langle \mathrm{in}_>(\mathcal{G})\rangle$.

Recall from Theorem 8.3.5 and Lemma 8.2.2 that

$$\mathrm{in}_>(\mathcal{G}_{ijk}) = \left\{ \begin{array}{ll} x_{ks}x_{kl}\cdot r_{ijk},\ x_{jt}x_{kl}\cdot r_{ijk}, & 1\le l\le s\le k-1, 1\le t\le k; \\ x_{ik}\cdot r_{ijk},\ x_{il}\cdot r_{ijk},\ x_{ab}\cdot r_{ijk}, & \{a,b\}\not\subset\{i,j,k,l\}, 1\le l\le k-1. \end{array} \right\}$$

Denote $I_{ijk} = \{x_{ks}x_{kl},\ x_{jt}x_{kl}, 1 \leq l \leq s \leq k-1, 1 \leq t \leq k; x_{ik},\ x_{il},\ x_{ab}, \{a,b\} \not\subset \{i,j,k,l\}\}$ Using Lemma 8.5.1, a straightforward computation shows that the Hilbert series of the corresponding monomial ideals are given by

$$\mathrm{Hilb}(S/\langle I_{ijk}\rangle) = \frac{1}{(1-t)^k} + \frac{kt}{1-t}.$$

Hence, the Hilbert series of $\mathfrak{B}_n$ is given by

$$\begin{aligned}\mathrm{Hilb}(\mathfrak{B}_n) &= \sum_{i>j>k} \mathrm{Hilb}(S/\langle I_{ijk}\rangle) \\ &= \sum_{k=1}^{n-2} \binom{n-k}{2} \left(\frac{1}{(1-t)^{k+1}} + \frac{kt}{1-t}\right).\end{aligned}$$

Setting $s = n-k$ completes the proof. □

8.5.3 The Chen ranks

We may compute the Chen ranks of wP_n^+ from the Hilbert series of its infinitesimal Alexander invariant, which is provided by Theorem 8.5.4.

Theorem 8.5.5. *The Chen ranks θ_k of the Chen group of wP_n^+ are given by $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, $\theta_3 = 2\binom{n+1}{4}$, and*

$$\theta_k = \binom{n+k-2}{k+1} + \theta_{k-1} = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}$$

for $k \geq 4$.

Proof. We need to find the coefficient of t^k in formula (8.22). Let

$$f(t) = \sum_{s=2}^{n-1} \binom{s}{2}(1-t)^{-n+s-1} + \binom{n}{4} t(1-t)^{-1}.$$

Computing derivatives, we find that

$$f^{(k)}(t) = \sum_{s=2}^{n-1} \binom{s}{2} \prod_{i=1}^{k}(n-s+i)(1-t)^{-n+s-k-1} + k!\binom{n}{4}(1-t)^{-k-1}.$$

Hence, the Chen ranks are given by

$$\theta_{k+2} = \frac{1}{k!} f^{(k)}(0) = \sum_{s=2}^{n-1} \binom{s}{2} \prod_{i=1}^{k} (n-s+i) + k! \binom{n}{4}.$$

Simplifying this expression, we obtain the claimed recurrence formula. □

Corollary 8.5.6. *The pure braid group P_n, the upper McCool group wP_n^+, and the product group $\Pi_n = \prod_{i=1}^{n-1} F_i$ are pairwise non-isomorphic for $n \geq 4$, although they all do have the same LCS ranks and the same Betti numbers.*

Proof. By [31], the fourth Chen ranks of P_n and Π_n are given by $\theta_4(P_n) = 3\binom{n+1}{4}$ and $\theta_4(\Pi_n) = 3\binom{n+2}{5}$. On the other hand, from Theorem 8.5.5, we see that

$$\theta_4(wP_n^+) = 2\binom{n+1}{4} + \binom{n+2}{5}. \tag{8.23}$$

Comparing these ranks completes the proof. □

8.5.4 The Chen ranks formula

In [34], Cohen and Schenck showed that the first resonance varieties of the McCool groups satisfy the hypotheses of Theorem 6.3.1, and that the Chen ranks of these groups are given by

$$\theta_k(wP_n) = (k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}, \quad \text{for } k \gg 0. \tag{8.24}$$

.

By Proposition 8.4.3, the components of $\mathcal{R}_1(wP_n^+)$ are not isotropic. We can also verify that the Chen ranks formula (6.30) does not hold for the group wP_n^+, as soon as $n \geq 4$.

Chapter 9

More Examples

In the previous two chapters, we already applied our techniques to the pure welded braid groups and the pure virtual braid groups. In this chapter, we study some other interesting groups, including torsion-free nilpotent groups, 1-relator groups, and the fundamental groups of orientable Seifert manifolds. In our current and future work, we will investigate these algebraic and geometric invariants for the pure braid groups on compact Riemann surfaces and the picture groups from quiver representations. This chapter is based on the work in paper [143].

9.1 Torsion-free nilpotent groups

In this section we study the graded and filtered formality properties of a well-known class of groups: that of finitely generated, torsion-free nilpotent groups. In the process, we prove Theorem 1.2.10 from the Introduction.

9.1.1 Nilpotent groups and Lie algebras

We start by reviewing the construction of the Malcev Lie algebra of a finitely generated, torsion-free nilpotent group G (see Cenkl and Porter [26], Lambe and Priddy [88], and

Malcev [104] for more details). There is a refinement of the upper central series of such a group,

$$G = G_1 > G_2 > \cdots > G_n > G_{n+1} = 1, \tag{9.1}$$

with each subgroup $G_i < G$ a normal subgroup of G_{i+1}, and each quotient G_i/G_{i+1} an infinite cyclic group. (The integer n is an invariant of the group, called the length of G.) Using this fact, we can choose a *Malcev basis* $\{u_1, \ldots, u_n\}$ for G, which satisfies $G_i = \langle G_{i+1}, u_i \rangle$. Consequently, each element $u \in G$ can be written uniquely as $u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n}$.

Using this basis, the group G, as a set, can be identified with $\mathbb{Z}^n$ via the map sending $u_1^{a_1} \cdots u_n^{a_n}$ to $a = (a_1, \ldots, a_n)$. The multiplication in G then takes the form

$$u_1^{a_1} \cdots u_n^{a_n} \cdot u_1^{b_1} \cdots u_n^{b_n} = u_1^{\rho_1(a,b)} \cdots u_n^{\rho_n(a,b)}, \tag{9.2}$$

where $\rho_i \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ is a rational polynomial function, for each $1 \le i \le n$. This procedure identifies the group G with the group $(\mathbb{Z}^n, \rho)$, with multiplication the map $\rho = (\rho_1, \ldots, \rho_n) \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$. Thus, we can define a simply-connected nilpotent Lie group $G \otimes \mathbb{Q} = (\mathbb{Q}^n, \rho)$ by extending the domain of ρ, which is called the *Malcev completion* of G.

The discrete group G is a subgroup of the real Lie group $G \otimes \mathbb{R}$. The quotient space, $M = (G \otimes \mathbb{R})/G$, is a compact manifold, called a *nilmanifold*. As shown in [104], the Lie algebra of M is isomorphic to $\mathfrak{m}(G; \mathbb{R})$. It is readily apparent that the nilmanifold M is an Eilenberg–MacLane space of type $K(G, 1)$. As shown by Nomizu, the cohomology ring $H^*(M, \mathbb{R})$ is isomorphic to the cohomology ring of the Lie algebra $\mathfrak{m}(G; \mathbb{R})$.

The polynomial functions ρ_i have the form

$$\rho_i(a, b) = a_i + b_i + \tau_i(a_1, \ldots, a_{i-1}, b_1, \ldots, b_{i-1}). \tag{9.3}$$

Denote by $\sigma = (\sigma_1, \ldots, \sigma_n)$ the quadratic part of ρ. Then $\mathbb{Q}^n$ can be given a Lie algebra structure, with bracket $[a, b] = \sigma(a, b) - \sigma(b, a)$. As shown in [88], this Lie algebra is isomorphic to the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{Q})$.

The group $(\mathbb{Z}^n, \rho)$ has canonical basis $\{e_i\}_{i=1}^n$, where e_i is the i-th standard basis vector. Then the Malcev Lie algebra $\mathfrak{m}(G;\mathbb{Q}) = (\mathbb{Q}^n, [\,,\,])$ has Lie bracket given by $[e_i, e_j] = \sum_{k=1}^n s_{i,j}^k e_k$, where $s_{i,j}^k = b_k(e_i, e_j) - b_k(e_j, e_i)$.

The Chevalley–Eilenberg complex $\bigwedge^*(\mathfrak{m}(G;\mathbb{Q}))$ is a minimal model for $M = K(G,1)$. Clearly, this model is generated in degree 1; thus, it is also a 1-minimal model for G. As shown by Hasegawa in [68], the nilmanifold M is formal if and only if M is a torus.

9.1.2 Nilpotent groups and filtered formality

Let G be a finitely generated, torsion-free nilpotent group, and let $\mathfrak{m} = \mathfrak{m}(G;\mathbb{Q})$ be its Malcev Lie algebra, as described above. Note that $\mathrm{gr}(\mathfrak{m}) = \mathbb{Q}^n$ has the same basis $e_1, \dots, e_n$ as $\mathfrak{m}$, but, as we shall see, the Lie bracket on $\mathrm{gr}(\mathfrak{m})$ may be different. The Lie algebra $\mathfrak{m}$ (and thus, the group G) is filtered-formal if and only if $\mathfrak{m} \cong \widehat{\mathrm{gr}}(\mathfrak{m}) = \mathrm{gr}(\mathfrak{m})$, as filtered Lie algebras. In general, though, this isomorphism need not preserve the chosen basis.

Example 9.1.1. For any finitely generated free group F, the k-step, free nilpotent group $F/\Gamma_{k+1}F$ is filtered-formal. Indeed, F is 1-formal, and thus filtered-formal. Hence, by Theorem 3.3.8, each nilpotent quotient of F is also filtered-formal. In fact, as shown in [109, Corollary 2.14], $\mathfrak{m}(F/\Gamma_{k+1}F) \cong \mathbf{L}/(\Gamma_{k+1}\mathbf{L})$, where $\mathbf{L} = \mathfrak{lie}(F)$.

Example 9.1.2. Let G be the 3-step, rank 2 free nilpotent group $F_2/\Gamma_4 F_2$. Identifying G with $\mathbb{Z}^5$ as a set, then G has a presentation with generators $x_1, \dots, x_5$ and relations $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$, $[x_2, x_3] = x_5$, and x_4, x_5 central. Let $\{z_1, \dots, z_5\}$ be the corresponding basis for $\mathrm{gr}(G;\mathbb{Q}) = \mathbb{Q}^2 \oplus \mathbb{Q} \oplus \mathbb{Q}^2$. The Lie brackets are then given by $[z_1, z_2] = z_3$, $[z_1, z_3] = z_4$, $[z_2, z_3] = z_5$, and $[z_i, z_j] = 0$, otherwise (see [88, 26]). A direct computation by (9.3) shows that

$$\rho_i(a,b) = a_i + b_i, \text{ for } i = 1, 2,$$

$$\rho_3(a,b) = a_3 + b_3 - a_2 b_1,$$

$$\rho_4(a,b) = a_4 + b_4 - a_3b_1 + a_2\binom{b_1}{2},$$

$$\rho_5(a,b) = a_5 + b_5 - a_3b_2 + a_2(1+b_2)b_1.$$

Then the Malcev Lie algebra $\mathfrak{m}(G;\mathbb{Q}) = \mathbb{Q}^5$ has Lie brackets given by $[e_1,e_2] = e_3 - e_4/2 - e_5$, $[e_1,e_3] = e_4$, $[e_2,e_3] = e_5$, and $[e_i,e_j] = 0$, otherwise. An isomorphism between $\mathfrak{m}(G;\mathbb{Q})$ and $\mathrm{gr}(G;\mathbb{Q})$ is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & \frac{1}{2} & * & * \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

However, it is readily checked that the identity map of $\mathbb{Q}^5$ is not a Lie algebra isomorphism between $\mathfrak{m} = \mathfrak{m}(G;\mathbb{Q})$ and $\mathrm{gr}(\mathfrak{m})$. Moreover, the differential of the 1-minimal model $\mathcal{M}(G) = \bigwedge^*(\mathfrak{m})$ is not homogeneous on the Hirsch weights, although $\mathfrak{m}$ (and G) are filtered-formal.

Now consider a finite-dimensional, nilpotent Lie algebra $\mathfrak{m}$ over a field $\mathbb{Q}$ of characteristic 0. The filtered-formality of such a Lie algebra coincides with the notions of 'Carnot', 'naturally graded', 'homogeneous' and 'quasi-cyclic' which appear in [37, 77, 92]. In this context, Cornulier proves in [37, Theorem 3.14] that the Carnot property for $\mathfrak{m}$ is equivalent to the Carnot property for $\mathfrak{m} \otimes_{\mathbb{Q}} \mathbb{K}$.

9.1.3 Torsion-free nilpotent groups and filtered formality

We now study in more detail the filtered-formality properties of torsion-free nilpotent groups. We start by singling out a rather large class of groups which enjoy this property.

Theorem 9.1.3. *Let G be a finitely generated, torsion-free, 2-step nilpotent group. If G_{ab} is torsion-free, then G is filtered-formal.*

Proof. The lower central series of our group takes the form $G = \Gamma_1 G > \Gamma_2 G > \Gamma_3 G = 1$. Let $\{x_1,\dots,x_n\}$ be a basis for $G/\Gamma_2 G = \mathbb{Z}^n$, and let $\{y_1,\dots,y_m\}$ be a basis for $\Gamma_2 G = \mathbb{Z}^m$. Then,

as shown for instance by Igusa and Orr in [72, Lemma 6.1], the group G has presentation

$$G = \Big\langle x_1, \dots, x_n, y_1, \dots, y_m \;\Big|\; [x_i, x_j] = \prod_{k=1}^{m} y_k^{c_{i,j}^k},\ [y_i, y_j] = 1, \text{for } i < j; [x_i, y_j] = 1 \Big\rangle. \tag{9.4}$$

Let $a, b \in \mathbb{Z}^{n+m}$. A routine computation shows that

$$\begin{aligned} \rho_i(a,b) &= a_i + b_i, && \text{for } 1 \le i \le n, \\ \rho_{n+k}(a,b) &= a_{n+k} + b_{n+k} - \sum_{j=1}^{k} \sum_{i=j+1}^{n} c_{j,i}^k a_i b_j, && \text{for } 1 \le k \le m. \end{aligned} \tag{9.5}$$

Set $c_{j,i}^k = -c_{i,j}^k$ if $j > i$. It follows that the Malcev Lie algebra $\mathfrak{m}(G;\mathbb{Q}) = (\mathbb{Q}^{n+m}, [\ ,\ ])$ has Lie bracket given on generators by $[e_i, e_j] = \sum_{k=1}^{m} c_{i,j}^k e_{n+k}$ for $1 \le i \ne j \le n$, and zero otherwise.

Turning now to the associated graded Lie algebra of our group, we have an additive decomposition, $\operatorname{gr}(G;\mathbb{Q}) = \operatorname{gr}_1(G;\mathbb{Q}) \oplus \operatorname{gr}_2(G;\mathbb{Q}) = \mathbb{Q}^n \oplus \mathbb{Q}^m$, where the first factor has basis $\{e_1, \dots, e_n\}$, the second factor has basis $\{e_{n+1}, \dots, e_{n+m}\}$, and the Lie bracket is given as above. Therefore, $\mathfrak{m}(G) \cong \operatorname{gr}(G;\mathbb{Q})$, as filtered Lie algebras. Hence, G is filtered-formal. □

It is known that all nilpotent Lie algebras of dimension 4 or less are filtered-formal, see for instance [37]. In general, though, finitely generated, torsion-free nilpotent groups need not be filtered-formal. We illustrate this phenomenon with two examples: the first one extracted from the work of Cornulier [37], and the second one adapted from the work of Lambe and Priddy [88]. In both examples, the nilpotent Lie algebra $\mathfrak{m}$ in question may be realized as the Malcev Lie algebra of a finitely generated, torsion-free nilpotent group G.

Example 9.1.4. Let $\mathfrak{m}$ be the 5-dimensional $\mathbb{Q}$-Lie algebra with non-zero Lie brackets given by $[e_1, e_3] = e_4$ and $[e_1, e_4] = [e_2, e_3] = e_5$. It is readily checked that the center of $\mathfrak{m}$ is 1-dimensional, generated by e_5, while the center of $\operatorname{gr}(\mathfrak{m})$ is 2-dimensional, generated by e_2 and e_5. Therefore, $\mathfrak{m} \ncong \operatorname{gr}(\mathfrak{m})$, and so $\mathfrak{m}$ is not filtered-formal. It follows that the nilpotent group G is not filtered-formal, either. It readily checked that the 1-minimal model $\mathcal{M}(G) = \bigwedge^*(\mathfrak{m})$ does not have positive Hirsch weights, nevertheless, $\mathcal{M}(G)$ has positive weights, given by the index of the chosen basis.

Example 9.1.5. Let $\mathfrak{m}$ be the 7-dimensional $\mathbb{Q}$-Lie algebra with non-zero Lie brackets given on basis elements by $[e_2, e_3] = e_6$, $[e_2, e_4] = e_7$, $[e_2, e_5] = -e_7$, $[e_3, e_4] = e_7$, and $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq 6$. Then $\mathrm{gr}(\mathfrak{m})$ has the same additive basis as $\mathfrak{m}$, with non-zero brackets given by $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq 6$. Once again, we claim that $\mathfrak{m} \ncong \mathrm{gr}(\mathfrak{m})$, and so both $\mathfrak{m}$ and G are not filtered-formal. In this case, though, we cannot use the indexing of the basis to put positive weights on $\mathcal{M}(G)$.

To prove the claim that $\mathfrak{m} \ncong \mathrm{gr}(\mathfrak{m})$, we suppose $\phi\colon \mathfrak{m} \to \mathrm{gr}(\mathfrak{m})$ is an isomorphism of the underlying vector spaces, preserving Lie brackets. Choose a basis $\{z_1, \ldots, z_7\}$ for $\mathrm{gr}(\mathfrak{m}) = \mathbb{Q}^7$. Then ϕ is given by a matrix $A = (a_{ij})$ such that $a_{ij} = 0$ for $3 \leq i \leq 7$ and $i > j \geq 1$.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ 0 & 0 & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46} & a_{47} \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} & a_{57} \\ 0 & 0 & 0 & 0 & 0 & a_{66} & a_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} \end{pmatrix}. \tag{9.6}$$

Since $[\phi(e_2), \phi(e_3)] = a_{21}a_{33}z_3 + a_{21}a_{34}z_4 + a_{21}a_{35}z_6 + a_{21}a_{36}z_7$ and $\phi(e_6) = a_{66}z_6 + a_{67}z_7$, we must have $a_{21}a_{33} = 0$ and $a_{21}a_{35} = a_{66}$. Moreover, since $\det(A) \neq 0$, we must also have $a_{ii} \neq 0$ for $i \geq 3$. But this is impossible, and the claim is proved. Another quick proof is pointed out by Cornulier: Plainly, $\mathrm{gr}(\mathfrak{m})$ is metabelian, (i.e., its derived subalgebra is abelian), while $\mathfrak{m}$ is not metabelian, thus the claim follows.

9.1.4 Graded formality and Koszulness

Carlson and Toledo [24] classified finitely generated, 1-formal, nilpotent groups with first Betti number 5 or less, while Plantiko [127] gave sufficient conditions for the associated graded Lie algebras of such groups to be non-quadratic. The following proposition follows from Theorem 4.1 in [127] and Lemma 2.4 in [24].

Proposition 9.1.6 ([24, 127]). *Let $G = F/R$ be a finitely presented, torsion-free, nilpotent group. If there exists a non-zero decomposable element u in the kernel of the cup product $H^1(G;\mathbb{Q}) \wedge H^1(G;\mathbb{Q}) \to H^2(G;\mathbb{Q})$, i.e., $u = v \wedge w$ for $v, w \in H^1(G;\mathbb{Q})$, then G is not graded-formal.*

Example 9.1.7. Let $U_n(\mathbb{R})$ be the nilpotent Lie group of upper triangular matrices with 1's along the diagonal. The quotient $M = U_n(\mathbb{R})/U_n(\mathbb{Z})$ is a nilmanifold of dimension $N = n(n-1)/2$. The unipotent group $U_n(\mathbb{Z})$ has canonical basis $\{u_{ij} \mid 1 \leq i < j \leq n\}$, where u_{ij} is the matrix obtained from the identity matrix by putting 1 in position (i, j). Moreover, $U_n(\mathbb{Z}) \cong (\mathbb{Z}^N, \rho)$, where $\rho_{ij}(a, b) = a_{ij} + b_{ij} + \sum_{i<k<j} a_{ik} b_{kj}$, see [88]. The unipotent group $U_n(\mathbb{Z})$ is filtered-formal; nevertheless, Proposition 9.1.6 shows that this group is not graded-formal for $n \geq 3$.

Proposition 9.1.8. *Let G be a finitely generated, torsion-free, nilpotent group, and suppose G is filtered-formal. Then G is abelian if and only if the algebra $U(\mathrm{gr}(G;\mathbb{Q}))$ is Koszul.*

Proof. We only need to deal with the proof of the non-trivial direction. If the algebra $U = U(\mathrm{gr}(G;\mathbb{Q}))$ is Koszul, then the Lie algebra $\mathrm{gr}(G;\mathbb{Q})$ is quadratic, i.e., the group G is graded-formal. Under the assumption that G is filtered-formal, we then have that G is 1-formal.

Let M be the nilmanifold with fundamental group G. Then M is also 1-formal. By Nomizu's theorem, the cohomology ring $A = H^*(M;\mathbb{Q})$ is isomorphic to the Yoneda algebra $\mathrm{Ext}^*_U(\mathbb{Q}, \mathbb{Q})$. On the other hand, since U is Koszul, the Yoneda algebra is isomorphic to $U^!$, which is also Koszul. Hence, A is a Koszul algebra. As shown by Papadima and Yuzvinsky [126], if M is 1-formal and if A is Koszul, then M is formal. By [68], this happens if and only if M is a torus. This completes the proof. □

Corollary 9.1.9. *Let G be a finitely generated, torsion-free, 2-step nilpotent group. If G_{ab} is torsion-free, then $U(\mathrm{gr}(G;\mathbb{Q}))$ is not Koszul.*

Example 9.1.10. Let $G = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4], [x_1, x_2][x_3, x_4] \rangle$. The group G is a 2-step, commutator-relators nilpotent group. Hence, by the above corollary, the enveloping algebra $U(\mathfrak{h}(G; \mathbb{Q}))$ is not Koszul. In fact, $U(\mathfrak{h}(G; \mathbb{Q}))^!$ is isomorphic to the quadratic algebra from Example 2.2.9, which is not Koszul.

9.2 One-relator groups and link groups

We start this section with the notion of mild (or inert) presentation of a group, due to J. Labute and D. Anick, and its relevance to the associated graded Lie algebra. We then continue with various applications to two important classes of finitely presented groups: one-relator groups and fundamental groups of link complements.

9.2.1 Mild presentations

Let F be a finitely generated free group, with generating set $\mathbf{x} = \{x_1, \dots, x_n\}$. The *weight* of a word $r \in F$ is defined as $\omega(r) = \sup\{k \mid r \in \Gamma_k F\}$. Since F is residually nilpotent, $\omega(r)$ is finite. The image of r in $\operatorname{gr}_{\omega(r)}(F)$ is called the *initial form* of r, and is denoted by $\operatorname{in}(r)$.

Let $G = F/R$ be a quotient of F, with presentation $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$, where $\mathbf{r} = \{r_1, \dots, r_m\}$. Let $\operatorname{in}_{\mathbb{Q}}(\mathbf{r})$ be the ideal of the free $\mathbb{Q}$-Lie algebra $\mathfrak{lie}(\mathbf{x})$ generated by $\{\operatorname{in}(r_1), \dots, \operatorname{in}(r_m)\}$. Clearly, this is a homogeneous ideal; thus, the quotient

$$L(G; \mathbb{Q}) := \mathfrak{lie}(\mathbf{x}) / \operatorname{in}_{\mathbb{Q}}(\mathbf{r}) \tag{9.7}$$

is a graded Lie algebra. As noted by Labute in [84], the ideal $\operatorname{in}_{\mathbb{Q}}(\mathbf{r})$ is contained in $\operatorname{gr}^{\widetilde{\Gamma}}(R; \mathbb{Q})$, where $\widetilde{\Gamma}_k R = \Gamma_k F \cap R$ is the induced filtration on R. Hence, there exists an epimorphism $L(G; \mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G; \mathbb{Q})$.

Proposition 9.2.1. *Let G be a commutator-relators group, and let $\mathfrak{h}(G; \mathbb{Q})$ be its holonomy Lie algebra. Then the canonical projection $\Phi_G \colon \mathfrak{h}(G; \mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G; \mathbb{Q})$ factors through an epimorphism $\mathfrak{h}(G; \mathbb{Q}) \twoheadrightarrow L(G; \mathbb{Q})$.*

Proof. Let $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$ be a commutator-relators presentation for our group. By Corollary 4.3.3, the holonomy Lie algebra $\mathfrak{h}(G;\mathbb{Q})$ admits a presentation of the form $\mathfrak{lie}(\mathbf{x})/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the degree 2 part of $M(r) - 1$, for all $r \in \mathbf{r}$. On the other hand, $\mathrm{in}(r)$ is the smallest degree homogeneous part of $M(r) - 1$. Hence, $\mathfrak{a} \subseteq \mathrm{in}_{\mathbb{Q}}(\mathbf{r})$, and this complete the proof. □

Following [2, 84], we say that a group G is a *mildly presented group* (over $\mathbb{Q}$) if it admits a presentation $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$ such that the quotient $\mathrm{in}_{\mathbb{Q}}(\mathbf{r})/[\mathrm{in}_{\mathbb{Q}}(\mathbf{r}), \mathrm{in}_{\mathbb{Q}}(\mathbf{r})]$, viewed as a $U(L(G;\mathbb{Q}))$-module via the adjoint representation of $L(G;\mathbb{Q})$, is a free module on the images of $\mathrm{in}(r_1), \dots, \mathrm{in}(r_m)$. As shown by Anick in [2], a presentation $G = \langle x_1, \dots, x_n \mid r_1, \dots r_m \rangle$ is mild if and only if

$$\mathrm{Hilb}(U(L(G;\mathbb{Q})), t) = \left(1 - nt + \sum_{i=1}^{m} t^{\omega(r_i)}\right)^{-1}. \tag{9.8}$$

Theorem 9.2.2 (Labute [83, 84])**.** *Let G be a finitely-presented group.*

1. *If G is mildly presented, then $\mathrm{gr}(G;\mathbb{Q}) = L(G;\mathbb{Q})$.*

2. *If G has a single relator r, then G is mildly presented. Moreover, for each $k \geq 1$, the LCS rank $\phi_k = \dim_{\mathbb{Q}} \mathrm{gr}(G;\mathbb{Q})$ is given by*

$$\phi_k = \frac{1}{k} \sum_{d \mid k} \mu(k/d) \left[\sum_{0 \leq i \leq [d/e]} (-1)^i \frac{d}{d+i-ei} \binom{d+i-ie}{i} n^{d-ei} \right], \tag{9.9}$$

where μ is the Möbius function and $e = \omega(r)$.

Labute states this theorem over $\mathbb{Z}$, but his proof works for any commutative PID with unity. There is an example in [84] showing that the mildness condition is crucial for part (1) of the theorem to hold. We give now a much simpler example to illustrate this phenomenon.

Example 9.2.3. Let $G = \langle x_1, x_2, x_3 \mid x_3, x_3[x_1, x_2] \rangle$. Clearly, $G \cong \langle x_1, x_2 \mid [x_1, x_2] \rangle$, which is a mild presentation. However, the Lie algebra $\mathfrak{lie}(x_1, x_2, x_3)/\mathrm{ideal}(x_3)$ is not isomorphic to $\mathrm{gr}(G;\mathbb{Q}) = \mathfrak{lie}(x_1, x_2)/\mathrm{ideal}([x_1, x_2])$. Hence, the first presentation is not a mild.

Lemma 9.2.4. *Let G be a group admitting a mild presentation $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ such that $r_i \in [F, F]$ for $1 \leq i \leq m$. If G is graded-formal, then the LCS ranks of G are given by*

$$\phi_k(G) = \frac{1}{k} \sum_{d \mid k} \mu(k/d) \frac{\left(n + \sqrt{n^2 - 4m}\right)^d + \left(n - \sqrt{n^2 - 4m}\right)^d}{(2m)^d}. \tag{9.10}$$

Moreover, if the enveloping algebra $U = U(\mathrm{gr}(G; \mathbb{C}))$ is Koszul, then

$$\mathrm{Hilb}(\mathrm{Ext}_U(\mathbb{C}; \mathbb{C}), t) = 1 + nt + mt^2.$$

Proof. Since G has a mild presentation, $\mathrm{gr}(G)$ is isomorphic to the Lie algebra $\mathfrak{L}(G)$ associated to this presentation. Furthermore, since G is a graded-formal, and all the relators r_i are commutators, we have that $\omega(r_i) = 2$ for $1 \leq i \leq m$. Using now the Poincaré–Birkhoff–Witt theorem and formula (9.8), we find that $\mathrm{Hilb}(U(\mathrm{gr}(G)), t) \cdot (1 - nt + mt^2) = 1$. Hence, the LCS ranks formula follows from Lemma 3.1.4.

Now suppose $U = U(\mathrm{gr}(G))$ is a Koszul algebra. Then $\mathrm{Ext}_U(\mathbb{C}; \mathbb{C}) = U^!$, and the expression for the Hilbert series of $\mathrm{Ext}_U(\mathbb{C}; \mathbb{C})$ follows from (2.12). □

9.2.2 Mildness and graded formality

We now use Labute's work on the associated graded Lie algebra and our presentation of the holonomy Lie algebra to give two graded-formality criteria.

Corollary 9.2.5. *Let G be a group admitting a mild presentation $\langle \mathbf{x} \mid \mathbf{r} \rangle$. If $\omega(r) \leq 2$ for each $r \in \mathbf{r}$, then G is graded-formal.*

Proof. By Theorem 9.2.2, the associated graded Lie algebra $\mathrm{gr}(H; \mathbb{Q})$ has a presentation of the form $\mathfrak{lie}(\mathbf{x})/\operatorname{in}_{\mathbb{Q}}(\mathbf{r})$, with $\operatorname{in}_{\mathbb{Q}}(\mathbf{r})$ a homogeneous ideal generated in degrees 1 and 2. Using the degree 1 relations to eliminate superfluous generators, we arrive at a presentation with only quadratic relations. The desired conclusion follows from Lemma 3.1.14. □

An important sufficient condition for mildness of a presentation was given by Anick [2]. Recall that ι denotes the canonical injection from the free Lie algebra $\mathfrak{lie}(\mathbf{x})$ into

$\mathbb{Q}\langle\mathbf{x}\rangle$. Fix an ordering on the set $\{\mathbf{x}\}$. The set of monomials in the homogeneous elements $\iota(\mathrm{in}(r_1)), \dots, \iota(\mathrm{in}(r_m))$ inherits the lexicographic order. Let w_i be the highest term of $\iota(\mathrm{in}(r_i))$ for $1 \leq i \leq m$. Suppose that (i) no w_i equals zero; (ii) no w_i is a submonomial of any w_j for $i \neq j$, i.e., $w_j = uw_iv$ cannot occur; and (iii) no w_i overlaps with any w_j, i.e., $w_i = uv$ and $w_j = vw$ cannot occur unless $v = 1$, or $u = w = 1$. Then, the set $\{r_1, \dots, r_n\}$ is mild (over $\mathbb{Q}$). We use this criterion to provide an example of a finitely-presented group G which is graded-formal, but not filtered-formal.

Example 9.2.6. Let G be the group with generators $x_1, \dots, x_4$ and relators $r_1 = [x_2, x_3]$, $r_2 = [x_1, x_4]$, and $r_3 = [x_1, x_3][x_2, x_4]$. Ordering the generators as $x_1 \succ x_2 \succ x_3 \succ x_4$, we find that the highest terms for $\{\iota(\mathrm{in}(r_1)), \iota(\mathrm{in}(r_2)), \iota(\mathrm{in}(r_3))\}$ are $\{x_2x_3, x_1x_4, x_1x_3\}$, and these words satisfy the above conditions of Anick. Thus, by Theorem 9.2.2, the Lie algebra $\mathrm{gr}(G;\mathbb{Q})$ is the quotient of $\mathfrak{lie}(x_1, \dots, x_4)$ by the ideal generated by $[x_2, x_3]$, $[x_1, x_4]$, and $[x_1, x_3] + [x_2, x_4]$. Hence, $\mathfrak{h}(G;\mathbb{Q}) \cong \mathrm{gr}(G;\mathbb{Q})$, that is, G is graded-formal. On the other hand, using the Tangent Cone theorem of Dimca et al. [42], one can show that the group G is not 1-formal. Therefore, G is not filtered-formal.

9.2.3 Non-mild presentations

Again, let G_n denote any one of the pure braid-like groups P_n, vP_n, or vP_n^+. Recall that G_n is graded-formal, and $\mathrm{gr}(G_n) \cong \mathfrak{L}(G_n)$. However, as we show next, the groups G_n are not mildly presented, except for small n.

Proposition 9.2.7. *The pure braid groups P_n and the pure virtual braid groups vP_n and vP_n^+ admit mild presentations if and only if $n \leq 3$.*

Proof. Let G_n denote any of the aforementioned groups. Then G_n is a commutator-relators group, and the universal enveloping algebra of the associated graded Lie algebra is Koszul. From formulas (7.7) and (7.10), for $n \leq 3$, Anick's criterion (9.8) is satisfied. Hence, G_n has a mild presentation for $n \leq 3$.

Now suppose $n \geq 4$. Using formulas (7.7) and (7.10) once again, we see that the third Betti numbers of these groups are given by

$$b_3(P_n) = s(n, n-3), \quad b_3(vP_n) = L(n, n-3), \quad b_3(vP_n^+) = S(n, n-3). \tag{9.11}$$

Thus, $\dim H^3(G_n, \mathbb{C}) > 0$ for $n \geq 4$. The claim now follows from Proposition 9.2.4 and the fact that $H^*(G_n, \mathbb{C}) = U(\mathrm{gr}(G_n, \mathbb{C}))^! = \mathrm{Ext}_U(\mathbb{C}, \mathbb{C})$. □

9.2.4 One-relator groups

If the group G admits a finite presentation with a single relator, much more can be said.

Corollary 9.2.8. *Let $G = \langle \mathbf{x} \mid r \rangle$ be a 1-relator group.*

1. *If r is a commutator relator, then $\mathfrak{h}(G; \mathbb{Q}) = \mathfrak{lie}(\mathbf{x})/\mathrm{ideal}(M_2(r))$.*

2. *If r is not a commutator relator, then $\mathfrak{h}(G; \mathbb{Q}) = \mathfrak{lie}(y_1, \dots, y_{n-1})$.*

Proof. Part (1) follows from Corollary 4.3.3. When r is not a commutator relator, the Jacobian matrix $J_G = (\epsilon(\partial_i r))$ has rank 1. Part (2) then follows from Theorem 4.3.1. □

Corollary 9.2.9. *Let $G = \langle x_1, \dots x_n \mid r \rangle$ be a 1-relator group, and let $\mathfrak{h} = \mathfrak{h}(G; \mathbb{Q})$. Then*

$$\mathrm{Hilb}(U(\mathfrak{h}); t) = \begin{cases} 1/(1-(n-1)t) & \text{if } \omega(r) = 1, \\ 1/(1-nt+t^2) & \text{if } \omega(r) = 2, \\ 1/(1-nt) & \text{if } \omega(r) \geq 3. \end{cases} \tag{9.12}$$

Proof. Let $\mathbf{x} = \{x_1, \dots, x_n\}$. By Corollary 9.2.8, the universal enveloping algebra $U(\mathfrak{h})$ is isomorphic to either $T(y_1, \dots, y_{n-1})$ if $\omega(r) = 1$, or to $T(\mathbf{x})/\mathrm{ideal}(M_2(r))$ if $\omega(r) = 2$, or to $T(\mathbf{x})$ if $\omega(r) \geq 3$. The claim now follows from Proposition 2.2.7 and Corollary 2.2.8. □

Theorem 9.2.10. *Let $G = \langle \mathbf{x} \mid r \rangle$ be a group defined by a single relation. Then G is graded-formal if and only if $\omega(r) \leq 2$.*

Proof. By Theorem 9.2.2, the given presentation of G is mild. The weight $\omega(r)$ can also be computed as $\omega(r) = \inf\{|I| \mid M(r)_I \neq 0\}$. If $\omega(r) \leq 2$, then, by Corollary 9.2.8, we have that

$$\mathfrak{h}(G;\mathbb{Q}) \cong \operatorname{gr}(G;\mathbb{Q}) \cong \mathfrak{lie}(\mathbf{x})/\operatorname{ideal}(\operatorname{in}(r)), \tag{9.13}$$

and so G is graded-formal.

On the other hand, if $\omega(r) \geq 3$, then $\mathfrak{h}(G;\mathbb{Q}) = \mathfrak{lie}(\mathbf{x})$. However, $\operatorname{gr}(G;\mathbb{Q}) = \mathfrak{lie}(\mathbf{x})/\operatorname{ideal}(\operatorname{in}(r))$. Hence, G is not graded-formal. □

Example 9.2.11. Let $G = \langle x_1, x_2 \mid r\rangle$, where $r = [x_1, [x_1, x_2]]$. Clearly, $\omega(r) = 3$. Hence, G is not graded-formal.

However, even if a one-relator group has weight 2 relation, the group need not be filtered-formal, as we show in the next example.

Example 9.2.12. Let $G = \langle x_1, \dots, x_5 \mid [x_1, x_2][x_3, [x_4, x_5]] = 1\rangle$. By Theorem 9.2.2, the Lie algebra $\operatorname{gr}(G;\mathbb{Q})$ has presentation $\mathfrak{lie}(x_1, \dots, x_5)/\operatorname{ideal}([x_1, x_2])$. Thus, by Corollary 9.2.5, the group G is graded-formal. On the other hand, Remark 4.2.8 shows that G admits a non-trivial triple Massey product of the form $\langle u_3, u_4, u_5\rangle$. Thus, G is not 1-formal, and so G is not filtered-formal.

We now determine the ranks of the (rational) Chen Lie algebra associated to an arbitrary finitely presented, 1-relator, 1-formal group, thereby extending a result of Papadima and Suciu from [117].

Proposition 9.2.13. *Let $G = F/\langle r\rangle$ be a one-relator group, where $F = \langle x_1, \dots, x_n\rangle$, and suppose G is 1-formal. Then*

$$\operatorname{Hilb}(\operatorname{gr}(G/G'';\mathbb{Q}), t) = \begin{cases} 1 + nt - \dfrac{1 - nt + t^2}{(1-t)^n} & \text{if } r \in [F, F], \\[2ex] 1 + (n-1)t - \dfrac{1 - (n-1)t}{(1-t)^{n-1}} & \text{otherwise.} \end{cases}$$

Proof. The first claim is proved in [117, Theorem 7.3]. To prove the second claim, recall that $\mathrm{gr}(G/G'';\mathbb{Q}) \cong \mathfrak{h}(G;\mathbb{Q})/\mathfrak{h}(G;\mathbb{Q})''$, for any 1-formal group G. Now, since we are assuming that $r \notin [F,F]$, Theorem 4.3.1 implies that $\mathfrak{h}(G;\mathbb{Q}) \cong \mathfrak{lie}(y_1,\dots,y_{n-1})$. The claim follows from Chen's formula (6.5) and the fact that $\mathrm{gr}(F/F'',\mathbb{Q}) = \mathfrak{lie}(x_1,\dots,x_n)/\mathfrak{lie}''(x_1,\dots,x_n)$, □

9.2.5 Link groups

We conclude this section with a very well-studied class of groups which occurs in low-dimensional topology. Let $L = (L_1,\dots,L_n)$ be an n-component link in S^3. The complement of the link, $X = S^3 \setminus \bigcup_{i=1}^n L_i$, has the homotopy type of a connected, finite, 2-dimensional CW-complex with $b_1(X) = n$ and $b_2(X) = n-1$. The link group, $G = \pi_1(X)$, carries crucial information about the homotopy type of X: if $n = 1$ (i.e., the link is a knot), or if $n > 1$ and L is a not a split link, then X is a $K(G,1)$.

Every link L as above arises as a closed-up braid $\widehat{\beta}$. That is, there is a braid β in the Artin braid group B_k such that L is isotopic to the link obtained from β by joining the top and bottom of each strand. The link group, then, has presentation $G = \langle x_1,\dots x_k \mid \beta(x_i) = x_i (i = 1,\dots,k)\rangle$, where B_k is now viewed as a subgroup of $\mathrm{Aut}(F_k)$ via the Artin embedding. If β belongs to the pure braid group $P_n \subset B_n$, then $L = \widehat{\beta}$ is an n-component link, called a *pure braid link*.

Associated to an n-component link L there is a linking graph Γ, with vertex set $\{1,\dots,n\}$, and an edge (i,j) for each pair of components (L_i, L_j) with non-zero linking number. Suppose the graph Γ is connected. Then, as conjectured by Murasugi [114] and proved by Massey–Traldi [108] and Labute [85], the link group G has the same LCS ranks ϕ_k and the same Chen ranks θ_k as the free group F_{n-1}, for all $k > 1$. Furthermore, G has the same Chen Lie algebra as F_{n-1} (see [117]). The next theorem is a combination of results of [2], Berceanu–Papadima [14], and Papadima–Suciu [117].

Theorem 9.2.14. *Let L be an n-component link in S^3 with connected linking graph Γ, and*

let G be the link group. Then

1. *The group G is graded-formal.*

2. *If L is a pure braid link, then G admits a mild presentation.*

3. *There exists a graded Lie algebra isomorphism* $\operatorname{gr}(G/G'';\mathbb{Q}) \cong \mathfrak{h}(G;\mathbb{Q})/\mathfrak{h}(G;\mathbb{Q})''$.

Proof. Part (1) follows from Lemma 4.1 and Theorems 3.2 and 4.2 in [14]. Part (2) is Theorem 3.7 from [2], while Part (2) is proved in Theorem 10.1 from [117]. □

In general, though, a link group (even a pure braid link group) is not 1-formal. This phenomenon was first detected by W.S. Massey by means of his higher-order products [107], but the graded and especially filtered formality can be even harder to detect.

Example 9.2.15. Let L be the Borromean rings. This is the 3-component link obtained by closing up the pure braid $[A_{1,2}, A_{2,3}] \in P_3'$, where $A_{i,j}$ denote the standard generators of the pure braid group. All the linking numbers are 0, and so the graph Γ is disconnected. It is readily seen that link group G passes Anick's mildness test; thus $\operatorname{gr}(G;\mathbb{Q}) = \mathfrak{lie}(x,y,z)/\operatorname{ideal}([x,[y,z]],[z,[y,x]])$, thereby recovering a computation of Hain [64]. It follows that G is not graded-formal, and thus not 1-formal. Alternatively, the non-1-formality of G can be detected by the triple Massey products $\langle u,v,w\rangle$ and $\langle w,v,u\rangle$.

Example 9.2.16. Let L be the Whitehead link. This is a 2-component link with linking number 0. Its link group is the 1-relator group $G = \langle x, y \mid r\rangle$, where

$$r = x^{-1}y^{-1}xyx^{-1}yxy^{-1}xyx^{-1}y^{-1}xy^{-1}x^{-1}y.$$

By Theorem 9.2.2, this presentation is mild. Direct computation shows that $\operatorname{in}(r) = [x,[y,[x,y]]]$, and so $\operatorname{gr}(G;\mathbb{Q}) = \mathfrak{lie}(x,y)/\operatorname{ideal}([x,[y,[x,y]]])$, again verifying a computation from [64]. In particular, G is not graded-formal. The non-1-formality of G can also be detected by suitable fourth-order Massey products.

We do not know whether the two link groups from above are filtered-formal. Nevertheless, we give an example of a link group which is graded-formal, yet not filtered-formal.

Example 9.2.17. Let L be the link of great circles in S^3 corresponding to the arrangement of transverse planes through the origin of $\mathbb{R}^4$ denoted as $\mathcal{A}(31425)$ in Matei–Suciu [111]. Then L is a pure braid link of 5 components, with linking graph the complete graph K_5; moreover, the link group G is isomorphic to the semidirect product $F_4 \rtimes_\alpha F_1$, where $\alpha = A_{1,3}A_{2,3}A_{2,4} \in P_4$. By Theorem 9.2.14, the group G is graded-formal. On the other hand, as noted by Dimca et al. in [42, Example 8.2], the Tangent Cone theorem does not hold for this group, and thus G is not 1-formal. Consequently, G is not filtered-formal.

9.3 Seifert fibered manifolds

We now use our techniques to study the fundamental groups of orientable Seifert manifolds from a rational homotopy viewpoint. We start our analysis with the fundamental groups of Riemann surfaces.

9.3.1 Riemann surfaces

Let Σ_g be the closed, orientable surface of genus g. The fundamental group $\Pi_g = \pi_1(\Sigma_g)$ is a 1-relator group, with generators $x_1, y_1, \dots, x_g, y_g$ and a single relation, $[x_1, y_1] \cdots [x_g, y_g] = 1$. Since this group is trivial for $g = 0$, we will assume for the rest of this subsection that $g > 0$.

The cohomology algebra $A = H^*(\Sigma_g; \mathbb{Q})$ is the quotient of the exterior algebra on generators $a_1, b_1, \dots, a_g, b_g$, in degree 1 by the ideal I generated by $a_ib_i - a_jb_j$, for $1 \leq i < j \leq g$, together with a_ia_j, b_ib_j, a_ib_j, a_jb_i, for $1 \leq i < j \leq g$. It is readily seen that the generators of I form a quadratic Gröbner basis for this ideal; therefore, A is a Koszul algebra.

The Riemann surface Σ_g is a compact Kähler manifold, and thus, a formal space. It follows from Theorem 2.3.20 that the minimal model of Σ_g is generated in degree one,

i.e., $\mathcal{M}(\Sigma_g) = \mathcal{M}(\Sigma_g, 1)$. The formality of Σ_g also implies the 1-formality of Π_g. As a consequence, the associated graded Lie algebra $\mathrm{gr}(\Pi_g; \mathbb{Q})$ is isomorphic to the holonomy Lie algebra $\mathfrak{h}(\Pi_g; \mathbb{Q})$. Using the above presentation for A, we find that

$$\mathfrak{h}(\Pi_g; \mathbb{Q}) = \mathfrak{lie}(2g) \Big/ \Big\langle \sum_{i=1}^{g} [x_i, y_i] = 0 \Big\rangle, \tag{9.14}$$

where $\mathfrak{lie}(2g) := \mathfrak{lie}(x_1, y_1, \dots, x_g, y_g)$. Using again the fact that A is a Koszul algebra, we deduce from Corollary 3.1.9 that $\prod_{k\geq 1}(1-t^k)^{\phi_k(\Pi_g)} = 1 - 2gt + t^2$. In fact, it follows from formula (9.9) that the lcs ranks of the 1-relator group Π_g are given by

$$\phi_k(\Pi_g) = \frac{1}{k} \sum_{d|k} \mu(k/d) \left[\sum_{i=0}^{[d/2]} (-1)^i \frac{d}{d-i} \binom{d-i}{i} (2g)^{d-2i} \right]. \tag{9.15}$$

Using now Theorem 4.3.5, we see that the Chen Lie algebra of Π_g has presentation

$$\mathrm{gr}\left(\Pi_g/\Pi_g''; \mathbb{Q}\right) = \mathfrak{lie}(2g) \Big/ \Big(\Big\langle \sum_{i=1}^{g} [x_i, y_i] \Big\rangle + \mathfrak{lie}''(2g) \Big). \tag{9.16}$$

Furthermore, Proposition 9.2.13 shows that the Chen ranks of our surface group are given by $\theta_1(\Pi_g) = 2g$, $\theta_2(\Pi_g) = 2g^2 - g - 1$, and

$$\theta_k(\Pi_g) = (k-1)\binom{2g+k-2}{k} - \binom{2g+k-3}{k-2}, \quad \text{for } k \geq 3. \tag{9.17}$$

9.3.2 Seifert fibered spaces

We will consider here only orientable, closed Seifert manifolds with orientable base. Every such manifold M admits an effective circle action, with orbit space an orientable surface of genus g, and finitely many exceptional orbits, encoded in pairs of coprime integers $(\alpha_1, \beta_1), \dots, (\alpha_s, \beta_s)$ with $\alpha_j \geq 2$. The obstruction to trivializing the bundle $\eta\colon M \to \Sigma_g$ outside tubular neighborhoods of the exceptional orbits is given by an integer $b = b(\eta)$. A standard presentation for the fundamental group of M in terms of the Seifert invariants is given by

$$\begin{aligned} \pi_\eta := \pi_1(M) = \big\langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_s, h \mid h \text{ central}, \\ [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_s = h^b,\ z_i^{\alpha_i} h^{\beta_i} = 1\ (i = 1, \dots, s) \big\rangle. \end{aligned} \tag{9.18}$$

For instance, if $s = 0$, the corresponding manifold, $M_{g,b}$, is the S^1-bundle over Σ_g with Euler number b. Let $\pi_{g,b} := \pi_1(M_{g,b})$ be the fundamental group of this manifold. If $b = 0$, then $\pi_{g,0} = \Pi_g \times \mathbb{Z}$, whereas if $b = 1$, then

$$\pi_{g,1} = \langle x_1, y_1, \dots, x_g, y_g, h \mid [x_1, y_1] \cdots [x_g, y_g] = h, h \text{ central} \rangle. \tag{9.19}$$

In particular, $M_{1,1}$ is the Heisenberg 3-dimensional nilmanifold and $\pi_{1,1}$ is the group from Example 3.3.6.

9.3.3 Malcev Lie algebra

As shown by Scott in [135], the Euler number $e(\eta)$ of the Seifert bundle $\eta\colon M \to \Sigma_g$ satisfies

$$e(\eta) = -b(\eta) - \sum_{i=1}^{s} \beta_i/\alpha_i. \tag{9.20}$$

If the base of the Seifert bundle has genus 0, the group π_η has first Betti number 0 or 1, according to whether $e(\eta)$ is non-zero or 0. Thus, π_η is 1-formal, and the Malcev Lie algebra $\mathfrak{m}(\pi_\eta; \mathbb{Q})$ is either 0, or the completed free Lie algebra of rank 1. To analyze the case when $g > 0$, we will employ the minimal model of M, as constructed by Putinar in [130].

Theorem 9.3.1 ([130]). *Let $\eta\colon M \to \Sigma_g$ be an orientable Seifert fibered space with $g > 0$. The minimal model $\mathcal{M}(M)$ is the Hirsch extension $\mathcal{M}(\Sigma_g) \otimes (\bigwedge(c), d)$, where the differential is given by $d(c) = 0$ if $e(\eta) = 0$, and $d(c) \in \mathcal{M}^2(\Sigma_g)$ represents a generator of $H^2(\Sigma_g; \mathbb{Q})$ if $e(\eta) \neq 0$.*

More precisely, recall that Σ_g is formal, and so there is a quasi-isomorphism $f\colon \mathcal{M}(\Sigma_g) \to (H^*(\Sigma_g; \mathbb{Q}), d = 0)$. Thus, there is an element $a \in \mathcal{M}^2(M)$ such that $d(a) = 0$ and $f^*([a]) \neq 0$ in $H^2(\Sigma_g; \mathbb{Q}) = \mathbb{Q}$. We then set $d(c) = a$ in the second case.

To each Seifert fibration $\eta\colon M \to \Sigma_g$ as above, let us associate the S^1-bundle $\bar{\eta}\colon M_{g,\epsilon(\eta)} \to \Sigma_g$, where $\epsilon(\eta) = 0$ if $e(\eta) = 0$, and $\epsilon(\eta) = 1$ if $e(\eta) \neq 0$. For instance, $M_{0,0} = S^2 \times S^1$ and $M_{0,1} = S^3$. The above theorem implies that

$$\mathcal{M}(M) \cong \mathcal{M}(M_{g,\epsilon(\eta)}). \tag{9.21}$$

Corollary 9.3.2. *Let $\eta\colon M \to \Sigma_g$ be an orientable Seifert fibered space. The Malcev Lie algebra of the fundamental group $\pi_\eta = \pi_1(M)$ is given by $\mathfrak{m}(\pi_\eta;\mathbb{Q}) \cong \mathfrak{m}(\pi_{g,e(\eta)};\mathbb{Q})$.*

Proof. The case $g = 0$ follows from the above discussion, while the case $g > 0$ follows from (9.21). □

Corollary 9.3.3. *Let $\eta\colon M \to \Sigma_g$ be an orientable Seifert fibered space with $g > 0$. Then M admits a minimal model with positive Hirsch weights.*

Proof. We know from §9.3.1 that the minimal model $\mathcal{M}(\Sigma_g)$ is formal, and generated in degree one (since $g > 0$). By Theorem 3.2.5, $\mathcal{M}(\Sigma_g)$ is isomorphic to a minimal model of Σ_g with positive Hirsch weights; denote this model by $\mathcal{H}(\Sigma_g)$.

By Theorem 9.3.1 and Lemma 2.3.3, the Hirsch extension $\mathcal{H}(\Sigma_g) \otimes \bigwedge(c)$ is a minimal model for M, generated in degree one. Moreover, the weight of c equals 1 if $e(\eta) = 0$, and equals 2 if $e(\eta) \neq 0$. Clearly, the differential d is homogeneous with respect to these weights, and this completes the proof. □

Using Theorem 9.3.1 and Lemma 2.3.3 again, we obtain a quadratic model for the Seifert manifold M in the case when the base has positive genus.

Corollary 9.3.4. *Suppose $g > 0$. Then M has a quadratic model of the form $\big(H^*(\Sigma_g;\mathbb{Q}) \otimes \bigwedge(c), d\big)$, where $\deg(c) = 1$ and the differential d is given by $d(a_i) = d(b_i) = 0$ for $1 \leq i \leq g$, $d(c) = 0$ if $e(\eta) = 0$, and $d(c) = a_1 \wedge b_1$ if $e(\eta) \neq 0$.*

We give now an explicit presentation for the Malcev Lie algebra $\mathfrak{m}(\pi_\eta;\mathbb{Q})$ as the degree completion of a certain graded Lie algebra.

Theorem 9.3.5. *The Malcev Lie algebra of π_η is the degree completion of the graded Lie algebra*

$$L(\pi_\eta) = \begin{cases} \mathfrak{lie}(x_1, y_1, \dots, x_g, y_g, z)/\langle \sum_{i=1}^g [x_i, y_i] = 0,\ z \text{ central}\rangle & \text{if } e(\eta) = 0; \\ \mathfrak{lie}(x_1, y_1, \dots, x_g, y_g, w)/\langle \sum_{i=1}^g [x_i, y_i] = w,\ w \text{ central}\rangle & \text{if } e(\eta) \neq 0, \end{cases} \tag{9.22}$$

where $\deg(w) = 2$ *and the other generators have degree* 1. *Moreover,* $\mathrm{gr}(\pi_\eta; \mathbb{Q}) \cong L(\pi_\eta)$.

Proof. The case $g = 0$ was already dealt with, so let us assume $g > 0$. There are two cases to consider.

If $e(\eta) = 0$, Corollary 9.3.2 says that $\mathfrak{m}(\pi_\eta; \mathbb{Q})$ is isomorphic to the Malcev Lie algebra of $\pi_{g,0} = \Pi_g \times \mathbb{Z}$, which is a 1-formal group. Furthermore, we know from (9.14) that $\mathrm{gr}(\Pi_g; \mathbb{Q})$ is the quotient of the free Lie algebra $\mathfrak{lie}(2g)$ by the ideal generated by $\sum_{i=1}^{g}[x_i, y_i]$. Hence, $\mathfrak{m}(\pi_\eta; \mathbb{Q})$ is isomorphic to the degree completion of $\mathrm{gr}(\Pi_g \times \mathbb{Z}) = \mathrm{gr}(\Pi_g; \mathbb{Q}) \times \mathrm{gr}(\mathbb{Z}; \mathbb{Q})$, which is precisely the Lie algebra $L(\pi_\eta)$ from (9.22).

If $e(\eta) \neq 0$, Corollary 9.3.4 provides a quadratic model for our Seifert manifold. Taking the Lie algebra dual to this quadratic model and using [17, Theorem 4.3.6] or [13, Theorem 3.1], we obtain that the Malcev Lie algebra $\mathfrak{m}(\pi_\eta)$ is isomorphic to the degree completion of the graded Lie algebra $L(\pi_\eta)$. Furthermore, by formula (3.21), there is an isomorphism $\mathrm{gr}(\mathfrak{m}(\pi_\eta; \mathbb{Q})) \cong \mathrm{gr}(\pi_\eta; \mathbb{Q})$. This completes the proof. □

Corollary 9.3.6. *Fundamental groups of orientable Seifert manifolds are filtered-formal.*

Proof. The claim follows at once from the above theorem and the definition of filtered-formality. Alternatively, the claim also follows from Theorem 3.2.5 and Corollary 9.3.3. □

9.3.4 Holonomy Lie algebra

We now give a presentation for the holonomy Lie algebra of a Seifert manifold group.

Theorem 9.3.7. *Let* $\eta\colon M \to \Sigma_g$ *be a Seifert fibration. The rational holonomy Lie algebra of the group* $\pi_\eta = \pi_1(M)$ *is given by*

$$\mathfrak{h}(\pi_\eta; \mathbb{Q}) = \begin{cases} \mathfrak{lie}(x_1, y_1, \dots, x_g, y_g, h)/\langle \sum_{i=1}^{s}[x_i, y_i] = 0, h \text{ central} \rangle & \text{if } e(\eta) = 0; \\ \mathfrak{lie}(2g) & \text{if } e(\eta) \neq 0. \end{cases}$$

Proof. First assume $e(\eta) = 0$. In this case, the row-echelon approximation of π_η has presentation

$$\begin{aligned}\widetilde{\pi}_\eta = \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_s, h \mid z_i^{\alpha_i} h^{\beta_i} = 1 \ (i = 1, \dots, s),\\ ([x_1, y_1] \cdots [x_g, y_g])^{\alpha_1 \cdots \alpha_s} = 1, \ h \text{ central} \rangle\end{aligned} \tag{9.23}$$

It is readily seen that the rank of the Jacobian matrix associated to this presentation has rank s. Furthermore, the map $\pi\colon F_{\mathbb{Q}} \to H_{\mathbb{Q}}$ is given by $x_i \mapsto x_i$, $y_i \mapsto y_i$, $z_j \mapsto (-\beta_i/\alpha_i)h$, $h \mapsto h$. Let κ be the Magnus expansion from Definition 4.1.2. A Fox Calculus computation shows that κ takes the following values on the commutator-relators of $\widetilde{\pi}_\eta$:

$$\begin{aligned}
\kappa(r) &= 1 + (\alpha_1 \cdots \alpha_s)(x_1 y_1 - y_1 x_1 + \cdots + x_g y_g - y_g x_g) + \text{terms of degree} \geq 3,\\
\kappa([x_i, h]) &= 1 + x_i h - h x_i + \text{terms of degree} \geq 3,\\
\kappa([y_i, h]) &= 1 + y_i h - h y_i + \text{terms of degree} \geq 3,\\
\kappa([y_i, z]) &= 1 + \text{terms of degree} \geq 3,
\end{aligned}$$

where $r = ([x_1, y_1] \cdots [x_g, y_g])^{\alpha_1 \cdots \alpha_s}$. The first claim now follows from Theorem 4.3.1.

Next, assume $e(\eta) \neq 0$. Then the row-echelon approximation of π_η is given by

$$\begin{aligned}\widetilde{\pi}_\eta = \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_s, h \mid z_i^{\alpha_i} h^{\beta_i} = 1 \ (i = 1, \dots, s),\\ ([x_1, y_1] \cdots [x_g, y_g])^{\alpha_1 \cdots \alpha_s} h^{e(\eta)\alpha_1 \cdots \alpha_s} = 1, \ h \text{ central} \rangle,\end{aligned} \tag{9.24}$$

while the homomorphism $\pi\colon F_{\mathbb{Q}} \to H_{\mathbb{Q}}$ is given by $x_i \mapsto x_i$, $y_i \mapsto y_i$, $z_j \mapsto (-\beta_i/\alpha_i)h$, $h \mapsto 0$. As before, the second claim follows from Theorem 4.3.1, and we are done. □

9.3.5 LCS ranks

We end this section with a computation of the ranks of the various graded Lie algebras attached to the fundamental group of a Seifert manifold. Comparing these ranks, we derive some consequences regarding the non-formality properties of such groups.

We start with the LCS ranks $\phi_k(\pi_\eta) = \dim \operatorname{gr}_k(\pi_\eta; \mathbb{Q})$ and the holonomy ranks are defined as $\bar{\phi}_k(\pi_\eta) = \dim(\mathfrak{h}(\pi_\eta; \mathbb{Q})_k$.

Proposition 9.3.8. *The LCS ranks and the holonomy ranks of a Seifert manifold group π_η are computed as follows.*

1. *If $e(\eta) = 0$, then $\phi_1(\pi_\eta) = \bar{\phi}_1(\pi_\eta) = 2g + 1$, and $\phi_k(\pi_\eta) = \bar{\phi}_k(\pi_\eta) = \phi_k(\Pi_g)$ for $k \geq 2$.*

2. *If $e(\eta) \neq 0$, then $\bar{\phi}_k(\pi_\eta) = \phi_k(F_{2g})$ for $k \geq 1$.*

3. *If $e(\eta) \neq 0$, then $\phi_1(\pi_\eta) = 2g$, $\phi_2(\pi_\eta) = g(2g-1)$, and $\phi_k(\pi_\eta) = \phi_k(\Pi_g)$ for $k \geq 3$.*

Here the LCS ranks $\phi_k(\Pi_g)$ are given by formula (9.15).

Proof. If $e(\eta) = 0$, then $\pi_\eta \cong \Pi_g \times \mathbb{Z}$, and claim (1) readily follows. So suppose that $e(\eta) \neq 0$. In this case, we know from Theorem 9.3.7 that $\mathfrak{h}(\pi_\eta; \mathbb{Q}) = \mathfrak{h}(F_{2g}; \mathbb{Q})$, and thus claim (2) follows.

By Theorem 9.3.5, the associated graded Lie algebra $\mathrm{gr}(\pi_\eta; \mathbb{Q})$ is isomorphic to the quotient of the free Lie algebra $\mathfrak{lie}(x_1, y_1, \dots, x_g, y_g, w)$ by the ideal generated by the elements $\sum_{i=1}^{g}[x_i, y_i] - w$, $[w, x_i]$, and $[w, y_i]$. Define a morphism $\chi\colon \mathrm{gr}(\pi_\eta; \mathbb{Q}) \to \mathrm{gr}(\Pi_g; \mathbb{Q})$ by sending $x_i \mapsto x_i$, $y_i \mapsto y_i$, and $w \mapsto 0$. It is readily seen that the kernel of χ is the Lie ideal of $\mathrm{gr}(\pi_\eta; \mathbb{Q})$ generated by w, and this ideal is isomorphic to the free Lie algebra on w. Thus, we get a short exact sequence of graded Lie algebras,

$$0 \longrightarrow \mathfrak{lie}(w) \longrightarrow \mathrm{gr}(\pi_\eta; \mathbb{Q}) \xrightarrow{\chi} \mathrm{gr}(\Pi_g; \mathbb{Q}) \longrightarrow 0\,. \tag{9.25}$$

Comparing Hilbert series in this sequence establishes claim (3) and completes the proof. □

Corollary 9.3.9. *If $g = 0$, the group π_η is always 1-formal, while if $g > 0$, the group π_η is graded-formal if and only if $e(\eta) = 0$.*

Proof. First suppose $e(\eta) = 0$. In this case, we know from Theorem 9.3.5 that $\mathrm{gr}(\pi_\eta; \mathbb{Q}) \cong \mathrm{gr}(\Pi_g; \mathbb{Q}) \times \mathrm{gr}(\mathbb{Z}; \mathbb{Q})$. It easily follows that $\mathrm{gr}(\pi_\eta; \mathbb{Q}) \cong \mathfrak{h}(\pi_\eta; \mathbb{Q})$ by comparing the presentations of these two Lie algebras. Hence, π_η is graded-formal, and thus 1-formal, by Corollary 9.3.6.

It is enough to assume that $g > 0$ and $e(\eta) \neq 0$, since the other claims are clear. By Proposition 9.3.8, we have that $\bar{\phi}_3(\pi_\eta) = (8g^3 - 2g)/3$, whereas $\phi_3(\pi_\eta) = (8g^3 - 8g)/3$. Hence, $\mathfrak{h}(\pi_\eta; \mathbb{Q})$ is not isomorphic to $\operatorname{gr}(\pi_\eta; \mathbb{Q})$, proving that π_η is not graded-formal. □

9.3.6 Chen ranks

Recall that the Chen ranks are defined as $\theta_k(\pi_\eta) = \dim \operatorname{gr}_k(\pi_\eta/\pi_\eta''; \mathbb{Q})$, while the holonomy Chen ranks are defined as $\bar{\theta}_k(\pi_\eta) = \dim(\mathfrak{h}/\mathfrak{h}'')_k$, where $\mathfrak{h} = \mathfrak{h}(\pi_\eta; \mathbb{Q})$.

Proposition 9.3.10. *The Chen ranks and the holonomy Chen ranks of a Seifert manifold group π_η are computed as follows.*

1. *If $e(\eta) = 0$, then $\theta_1(\pi_\eta) = \bar{\theta}_1(\pi_\eta) = 2g + 1$, and $\theta_k(\pi_\eta) = \bar{\theta}_k(\pi_\eta) = \theta_k(\Pi_g)$ for $k \geq 2$.*
2. *If $e(\eta) \neq 0$, then $\bar{\theta}_k(\pi_\eta) = \theta_k(F_{2g})$ for $k \geq 1$.*
3. *If $e(\eta) \neq 0$, then $\theta_1(\pi_\eta) = 2g$, $\theta_2(\pi_\eta) = g(2g-1)$, and $\theta_k(\pi_\eta) = \theta_k(\Pi_g)$ for $k \geq 3$.*

Here the Chen ranks $\theta_k(F_{2g})$ and $\theta_k(\Pi_g)$ are given by formulas (6.5) and (9.17), respectively.

Proof. Claims (1) and (2) are easily proved, as in Proposition 9.3.8. To prove claim (3), start by recalling from Corollary 9.3.6 that the group π_η is filtered-formal. Hence, by Theorem 6.1.5, the Chen Lie algebra $\operatorname{gr}(\pi_\eta/\pi_\eta''; \mathbb{Q})$ is isomorphic to $\operatorname{gr}(\pi_\eta; \mathbb{Q})/\operatorname{gr}(\pi_\eta; \mathbb{Q})''$. As before, we get a short exact sequence of graded Lie algebras,

$$0 \longrightarrow \mathfrak{lie}(w) \longrightarrow \operatorname{gr}(\pi_\eta/\pi_\eta''; \mathbb{Q}) \longrightarrow \operatorname{gr}(\Pi_g/\Pi_g''; \mathbb{Q}) \longrightarrow 0\,. \tag{9.26}$$

Comparing Hilbert series in this sequence completes the proof. □

Remark 9.3.11. The above result can be used to give another proof of Corollary 9.3.9. Indeed, suppose $e(\eta) \neq 0$. Then, by Proposition 9.3.10, we have that $\bar{\theta}_3(\pi_\eta) - \theta_3(\pi_\eta) = 2g$. Consequently, by Corollary 6.1.7, the group π_η is not 1-formal. Hence, by Corollary 9.3.6, π_η is not graded-formal.

9.4 Pure braid groups on Riemann surfaces

Recall that the *configuration space* of n ordered points in a connected manifold M is defined to be $\mathrm{Conf}_n(M) := \{(x_1, \cdots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$. Algebraic models of $\mathrm{Conf}_n(M)$ were studied by Cohen and Taylor [35] for $M = \mathbb{R}^l$, and by Fulton and MacPherson [59], Kříž [82], and Totaro [149] for M a smooth projective variety.

The configuration space $\mathrm{Conf}_n(\Sigma_g)$ is a classifying space for $P_{g,n}$. which is another important class of braid-like groups are the pure braid groups on compact Riemann surfaces Σ_g of genus g.

Much work has been done for this class of groups, see [16, 13, 23, 65], but still there are several unsolved problems. In our recent and future work, we compute the resonance varieties of $P_{g,n}$, the resonance varieties of the CDGA model of $P_{g,n}$, and the Chen ranks of $P_{g,n}$ and explore their relationship.

9.5 Picture groups from quiver representations

For every quiver of finite type, there is a finitely presented group called a *picture group*, which was introduced recently by Igusa, Orr, Todorov, and Weyman [73]. They proved that the integral cohomology groups of the picture group $G(A_n)$ of type A_n with straight orientation are free abelian with ranks given by the 'ballot numbers'. As shown by Igusa in [74], the classifying space of the category of non-crossing partitions is a $K(G(A_n), 1)$.

We noticed that for each picture group $G(A_n)$ there is a right-angled Artin group $R(A_n)$ such that these two groups have the same resonance varieties. Hence, the resonance varieties for these groups can be determined from the results of Papadima and Suciu [120].

In our future work, we will construct a finite CDGA model for $G(A_n)$ and compute the Malcev Lie algebra of $G(A_n)$. We we also investigate the relationship between the LCS ranks, the Chen ranks of $G(A_n)$ and the resonance varieties of the CDGA model. Explore

properties of picture groups from these algebraic invariants. We conjectured a finite CDGA model for $G(A_n)$. Using this model, we conjectured that $G(A_n)$ is filtered-formal, but not 1-formal by computing non-trivial Massey products.

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