Resonance varieties, Chen ranks and formality properties of finitely generated groups

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Overview

1. Techniques
   - 1. Formality properties
   - 2. Chen ranks
   - 3. Resonance varieties
2. Applications
   - Pure braid groups and their relatives
   - Fundamental groups of Seifert manifolds
3. Future work

Background

The basic goal in algebraic topology:

"Find algebraic invariants that classify topological spaces up to homeomorphism, or up to homotopy equivalence."

Let X be a connected CW-complex.

- Fundamental group $\pi_1(X)$.
- Homology groups $H_*(X;\mathbb{Z}) \Longrightarrow$ Betti numbers $b_i(X)$.
- Cohomology ring $H^*(X;\mathbb{Z})$.
- Higher Massey products.
- Cohomology operations.
- Higher homotopy groups $\pi_*(X)$.
- Homotopy Lie algebra $\pi_*(\Omega X)$.
- ......

- X: a connected CW-complex with finitely many 1-cells.
- $G := \pi_1(X)$, a finitely generated group.
- $K(G,1)$: the Eilenberg-MacLane space.

$$G \xleftrightarrow{1-1} K(G,1)$$

Recall our goal: *"Find algebraic invariants ......"*

- The *cohomology algebra* of G, $H^*(G;\mathbb{C}) := H^*(K(G,1);\mathbb{C})$.
- The *associated graded Lie algebra* of G is defined to be

$$\mathrm{gr}(G;\mathbb{C}) := \bigoplus_{k\geq 1}(\Gamma_k(G)/\Gamma_{k+1}(G)) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Here, $\Gamma_k(G)$, $k \geq 1$, are the *lower central series* of G: $\Gamma_1 G = G$ and $\Gamma_{k+1}G = [\Gamma_k G, G]$, $k \geq 1$.

Example (The free group F_n with n generators)

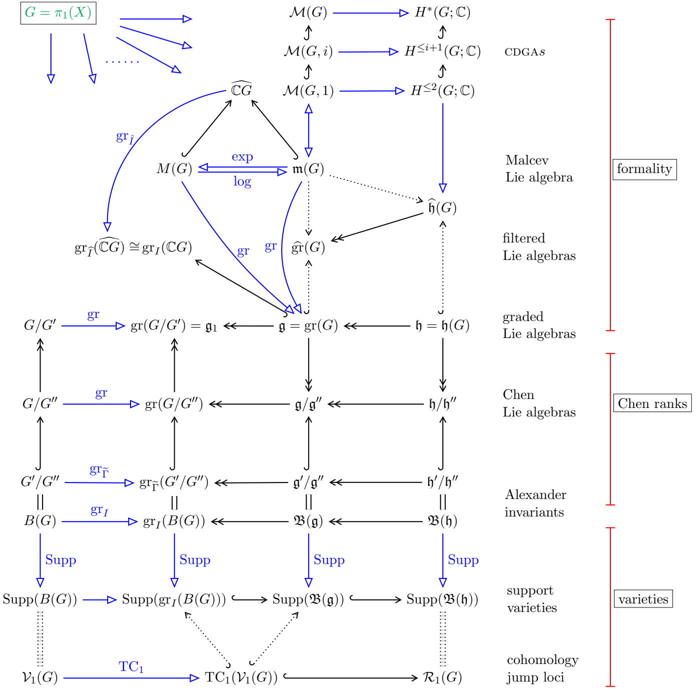
- $K(F_n, 1) = \bigvee_n S^1$.
- $A_n := H^*(F_n; \mathbb{C}) = \bigwedge(u_1, \dots, u_n)/\langle I \rangle$, for $I = \{u_i \wedge u_j \mid 1 \leq i < j \leq n\}$.
- $\mathrm{Hilb}(A_n, t) = 1 + nt$.
- $\mathfrak{g}_n := \mathrm{gr}(F_n; \mathbb{C})$ is isomorphic to the free Lie algebra with n generators.
- $(A_n)^! \cong U(\mathfrak{g}_n) \cong \mathbb{C}\langle x_1, \dots, x_n \rangle$.
- The lower central series (LCS) ranks $\phi_k(F_n) := \dim \mathrm{gr}_k(F_n; \mathbb{C})$ are given by:

$$\phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(k/d) n^d,$$

where μ is the Möbius function.

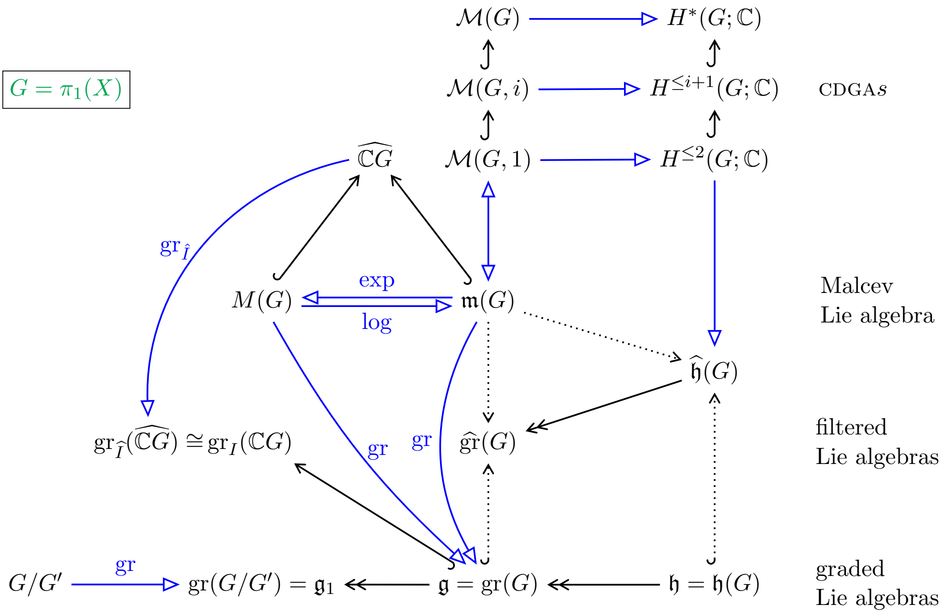
"Find algebraic invariants ......"

Algebraic invariants



1. Formality properties

$G = \pi_1(X)$



Rational homotopy theory

- $A := (A^*, d)$ a commutative differential graded algebra (CDGA) over $\mathbb{Q}$.
- A CDGA morphism $f\colon A \to B$ is an *i-quasi-isomorphism* if $f^*\colon H^j(A) \to H^j(B)$ is an isomorphism for each $j \le i$ and monomorphism for $j = i+1$.
- Two CDGAs A and B are *i-weakly equivalent* ($A \simeq_i B$), if there exist i-quasi-isomorphisms: $A \to C_1 \leftarrow C_2 \to \cdots \leftarrow C_m \to B$.
- If B is a 'minimal' CDGA generated by elements of degree $\le i$, and there exists an i-quasi-isomorphism $f\colon B \to A$, then we say that B is an *i-minimal model* for A.

$$B_1^* \subset B_2^* \subset \cdots \subset B_j^* \subset \cdots .$$

- Each connected CDGA A has an i-minimal model $\mathcal{M}(A, i)$, (a minimal model $\mathcal{M}(A)$) unique up to isomorphism. (Sullivan 77, Morgan 78)

Formality Properties

- $A_{PL}(X)$: the rational Sullivan model of a connected space X.

$$X \rightsquigarrow A_{PL}(X) \rightsquigarrow \mathcal{M}(X, i)$$

- $A := (A^*, d)$ is said to be *i-formal* if there exists an i-quasi-isomorphism $\mathcal{M}(A, i) \to (H^*(A), 0)$. Equivalently, $(A^*, d) \simeq_i (H^*(A), 0)$.
- X is said to be (i-)*formal*, if $A_{PL}(X)$ is (i-)formal, i.e.,

$$A_{PL}(X) \xleftarrow{i-\text{quasi-iso.}} \mathcal{M}(X, i) \xrightarrow{i-\text{quasi-iso.}} (H^*(X; \mathbb{Q}), 0)$$

- The 1-formality of a path-connected space X depends only on $\pi_1(X)$.
- A finitely generated group G is called *1-formal* if $X = K(G, 1)$ is 1-formal, i.e., $\mathcal{M}(X, 1)$ is 1-quasi-isomorphic to $(H^*(G; \mathbb{Q}), 0)$.

Malcev Lie algebra

- G: a finitely generated group.
- There exists a tower of nilpotent Lie algebras [Malcev 51]

$$\mathfrak{L}((G/\Gamma_2 G)\otimes\mathbb{Q}) \twoheadleftarrow \mathfrak{L}((G/\Gamma_3 G)\otimes\mathbb{Q}) \twoheadleftarrow \mathfrak{L}((G/\Gamma_4 G)\otimes\mathbb{Q}) \twoheadleftarrow$$

The inverse limit of the tower is called the *Malcev Lie algebra* of G, denoted by $\mathfrak{m}(G;\mathbb{Q})$.

- Let $\mathcal{M}(G,1)$ be the 1-minimal model of $K(G,1)$. Taking the dual of

$$\mathcal{M}(G,1)^1_1 \subset \mathcal{M}(G,1)^1_2 \subset \cdots \subset \mathcal{M}(G,1)^1_j \subset \cdots,$$

we also get a tower of nilpotent Lie algebras

$$\mathfrak{L}_1(G) \twoheadleftarrow \mathfrak{L}_2(G) \twoheadleftarrow \cdots \twoheadleftarrow \mathfrak{L}_j(G) \twoheadleftarrow \cdots.$$

Theorem (Sullivan 77, Cenkl–Porter 81)

These two towers of nilpotent Lie algebras are isomorphic.

- The universal enveloping algebra of $\mathfrak{m}(G;\mathbb{Q})$ is isomorphic to $\widehat{\mathbb{Q}G}$. [Quillen 69]

Example

Let X be the Heisenberg manifold, and $G = \pi_1(X) \cong F_2/\Gamma_3 F_2$.

- The (1-)minimal model $\mathcal{M}(X)$ is $\bigwedge(a,b,c)$ with $d(a) = d(b) = 0$ and $d(c) = a \wedge b$.
- $$\begin{array}{ccc} \mathcal{M}(G)_1^1 & \hookrightarrow & \mathcal{M}(G)_2^1 \\ \| & & \| \\ \mathbb{Q}^2\{a,b\} & & \mathbb{Q}^3\{a,b,c\} \end{array} \qquad \begin{array}{ccc} \mathfrak{L}_1(G) & \twoheadleftarrow & \mathfrak{L}_2(G) \\ \| & & \| \\ \mathbb{Q}^2\{a^*,b^*\} & & \mathbb{Q}^3\{a^*,b^*,c^*\} \end{array}$$
  $d(a) = d(b) = 0$ $\qquad$ $[a^*, b^*] = c^*$

  $d(c) = a \wedge b$ $\qquad$ $[a^*, c^*] = [b^*, c^*] = 0.$
- The Malcev Lie algebra $\mathfrak{m}(G;\mathbb{Q}) \cong \mathrm{Lie}(x,y)/\Gamma_3 \mathrm{Lie}(x,y)$.
- G is not 1-formal. (Non-vanishing Massey triple products)
- $\mathfrak{m}(G;\mathbb{Q}) \cong \mathrm{gr}(G;\mathbb{Q})$.

Holonomy Lie algebra

- The *holonomy Lie algebra* of a finitely generated group G is defined to be

$$\mathfrak{h}(G;\mathbb{C}) := \mathrm{Lie}(H_1(G;\mathbb{C}))/\langle \mathrm{im}(\partial_G)\rangle.$$

Here, ∂_G is the dual of $H^1(G;\mathbb{C}) \wedge H^1(G;\mathbb{C}) \xrightarrow{\cup} H^2(G;\mathbb{C})$.

Theorem (Suciu-W.)

Let G be a group with a finite presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m\rangle$. There is an explicit method to compute the above cup products and find a finite presentation for $\mathfrak{h}(G;\mathbb{C})$ using Magnus expansions.

If G is a commutator-relators group, the above theorem has been obtained by Fenn–Sjerve 87, Matei–Suciu 98, Papadima–Suciu 04.

Partial formality of groups

- There exists an epimorphism $\Phi_G \colon \mathfrak{h}(G;\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G;\mathbb{Q})$. [Lambe 86]
- We say that a group G is *graded-formal*, if $\Phi_G \colon \mathfrak{h}(G;\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G;\mathbb{Q})$ is an isomorphism of graded Lie algebras.
- $\operatorname{gr}(G;\mathbb{Q}) \xrightarrow{\cong} \operatorname{gr}(\mathfrak{m}(G;\mathbb{Q}))$. [Quillen 68]
- A group G is 1-formal iff $\mathfrak{m}(G;\mathbb{Q}) \cong \widehat{\mathfrak{h}}(G;\mathbb{Q})$. [Markl–Papadima 92]
- We say that a group G is *filtered-formal*, if there is a filtered Lie algebra isomorphism $\mathfrak{m}(G;\mathbb{Q}) \cong \widehat{\operatorname{gr}}(G;\mathbb{Q})$.

$$\begin{array}{ccc}
\mathfrak{m}(G;\mathbb{Q}) & \xrightarrow{\text{1-formal}} & \widehat{\mathfrak{h}}(G;\mathbb{Q}) \\
\text{filtered-formal} \Big\downarrow & & \Big\downarrow \text{graded-formal} \\
\widehat{\operatorname{gr}}(\mathfrak{m}(G;\mathbb{Q})) & \xrightarrow[\text{Quillen}]{\cong} & \widehat{\operatorname{gr}}(G;\mathbb{Q}).
\end{array}$$

Remark

- formal $\Longrightarrow$ i-formal $\Longrightarrow$ 1-formal $\Longleftrightarrow$ $\begin{array}{c}\text{graded-formal}\\ + \\ \text{filtered-formal.}\end{array}$
- The filtered formality of finite-dimensional, nilpotent Lie algebras has many different names: 'Carnot', 'naturally graded', 'homogeneous' and 'quasi-cyclic'.
- A finitely generated, torsion-free, 2-step nilpotent group is filtered-formal. [Suciu-W.]
- Recently, Bar-Natan has explored the "Taylor expansion" of $\mathbb{Q}G$.
  $\mathbb{Q}G$ has a Taylor expansion $\Longleftrightarrow$ G is filtered-formal.
  $\mathbb{Q}G$ has a quadratic Taylor expansion $\Longleftrightarrow$ G is 1-formal.

Propagation of partial formality properties

Proposition (Suciu–W.)

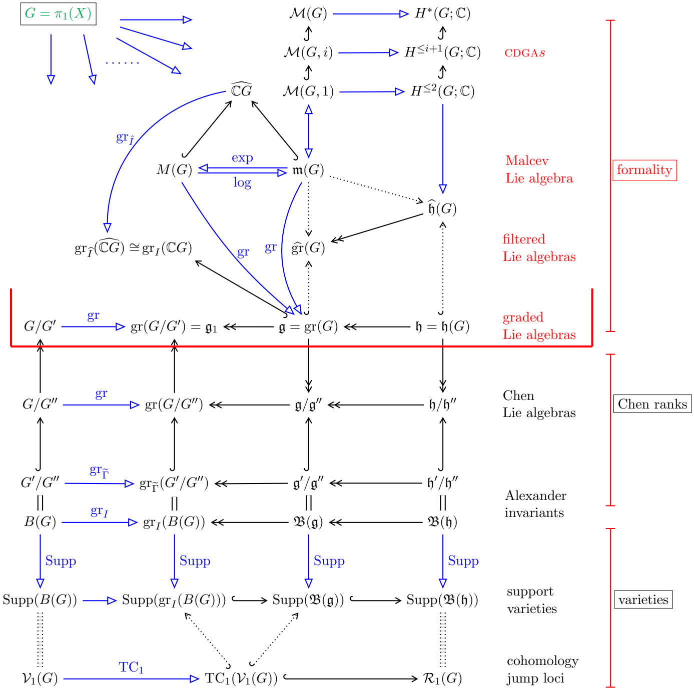
Let G be a finitely generated group, and let $K \leq G$ be a finitely generated subgroup. Suppose there is a split monomorphism $\iota \colon K \to G$. Then:

1. *If G is graded-formal, then K is also graded-formal.*
2. *If G is filtered-formal, then K is also filtered-formal.*
3. *If G is 1-formal, then K is also 1-formal.*

Proposition (Suciu–W.)

Let G_1 and G_2 be two finitely generated groups. The following conditions are equivalent.

1. *G_1 and G_2 are graded-formal (respectively, filtered-formal, or 1-formal).*
2. *$G_1 * G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).*
3. *$G_1 \times G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).*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Malcev
Lie algebra
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$\widehat{\mathfrak{h}}(G)$
$\mathrm{gr}_{\hat{I}}(\widehat{\mathbb{C}G}) \cong \mathrm{gr}_I(\mathbb{C}G)$
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$\mathfrak{g}'/\mathfrak{g}''$
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Alexander
invariants
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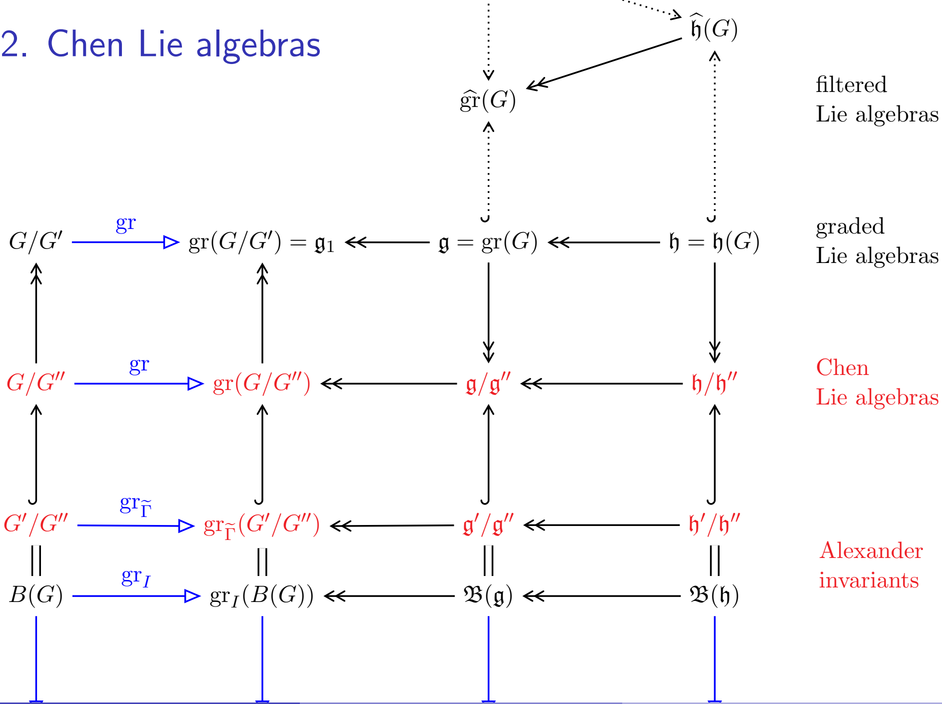
2. Chen Lie algebras



Chen Lie algebra

- The *Chen Lie algebra* of a finitely generated group G is defined to be the graded Lie algebra,

$$\mathrm{gr}(G/G'';\mathbb{C}),$$

  of the maximal metabelian quotient of G.
- The quotient map $G \twoheadrightarrow G/G''$ induces $\mathrm{gr}(G;\mathbb{C}) \twoheadrightarrow \mathrm{gr}(G/G'';\mathbb{C})$.
- The *Chen ranks* of G are defined as $\theta_k(G) := \dim(\mathrm{gr}_k(G/G'';\mathbb{C}))$.
- Recall the LCS ranks $\phi_k(G) = \dim(\mathrm{gr}_k(G;\mathbb{C}))$.
- $\phi_k(G) \geq \theta_k(G)$. Equality holds for $k \leq 3$.
- $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$, $k \geq 2$. [K.-T. Chen (51)]

$$0 \to \mathrm{gr}^{\widetilde{\Gamma}}(G'/G'';\mathbb{C}) \to \mathrm{gr}(G/G'';\mathbb{C}) \to \mathrm{gr}(G/G';\mathbb{C}) \to 0.$$

Alexander invariants and Chen ranks

- The *Alexander invariant* of G is the $\mathbb{Z}[G_{\text{ab}}]$-module $B(G) = G'/G''$.
- The $\mathbb{Z}[G_{\text{ab}}]$-module structure on $B(G)$ is determined by the extension

$$0 \to G'/G'' \to G/G'' \to G/G' \to 0.$$

  with G/G' acting on the cosets of G'' via conjugation:
  $gG' \cdot hG'' = ghg^{-1}G''$, for $g \in G$, $h \in G'$.

- $I := \ker \epsilon \colon \mathbb{Z}[G_{\text{ab}}] \to \mathbb{Z}$.
- The module $B(G)$ has an I-adic filtration $\{I^k B(G)\}_{k \geq 0}$.
- $\text{gr}(B(G)) := \bigoplus_{k \geq 0} I^k B(G)/I^{k+1}B(G)$ is a graded $\text{gr}(\mathbb{Z}[G_{\text{ab}}])$-module.
- $\text{gr}_k(G/G'') \cong \text{gr}_{k-2}(B(G))$. [W. Massey (80)]
- This isomorphism gives $\text{Hilb}(\text{gr}(B(G)) \otimes \mathbb{C}, t) = \sum_{k \geq 0} \theta_{k+2}(G) t^k$.
- $\text{gr}^{\widetilde{\Gamma}}(G'/G''; \mathbb{C}) \cong \text{gr}(B(G) \otimes \mathbb{C})$. [Suciu–W.]

Infinitesimal Alexander invariants

- $\mathfrak{g}$: a finitely generated, graded Lie algebra.
- The *infinitesimal Alexander invariant* of $\mathfrak{g}$ is the graded S-module $\mathfrak{B}(\mathfrak{g}) := \mathfrak{g}'/\mathfrak{g}''$. Here, S is the symmetric algebra on $\mathfrak{g}_1$.
- The exact sequence of graded Lie algebras

$$0 \longrightarrow \mathfrak{g}'/\mathfrak{g}'' \longrightarrow \mathfrak{g}/\mathfrak{g}'' \longrightarrow \mathfrak{g}/\mathfrak{g}' \longrightarrow 0$$

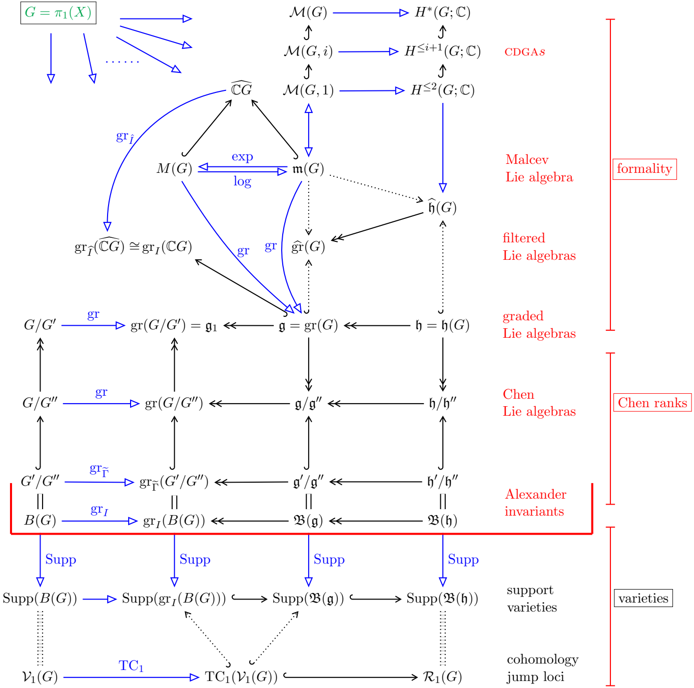
gives the graded S-module structure on $\mathfrak{B}(\mathfrak{g})$.

Theorem (Suciu-W.)

Let G be a finitely generated group. There exist surjective morphisms of graded S-modules,

$$\mathfrak{B}(\mathfrak{h}(G;\mathbb{C})) \overset{\psi}{\twoheadrightarrow} \mathfrak{B}(\mathrm{gr}(G;\mathbb{C})) \overset{\phi}{\twoheadrightarrow} \mathrm{gr}(B(G)\otimes\mathbb{C}) \cong \mathrm{gr}^{\widetilde{\Gamma}}(G'/G'';\mathbb{C}).$$

Moreover, if G is graded-formal, then ψ is an isomorphism, and if G is filtered-formal, then ϕ is an isomorphism.



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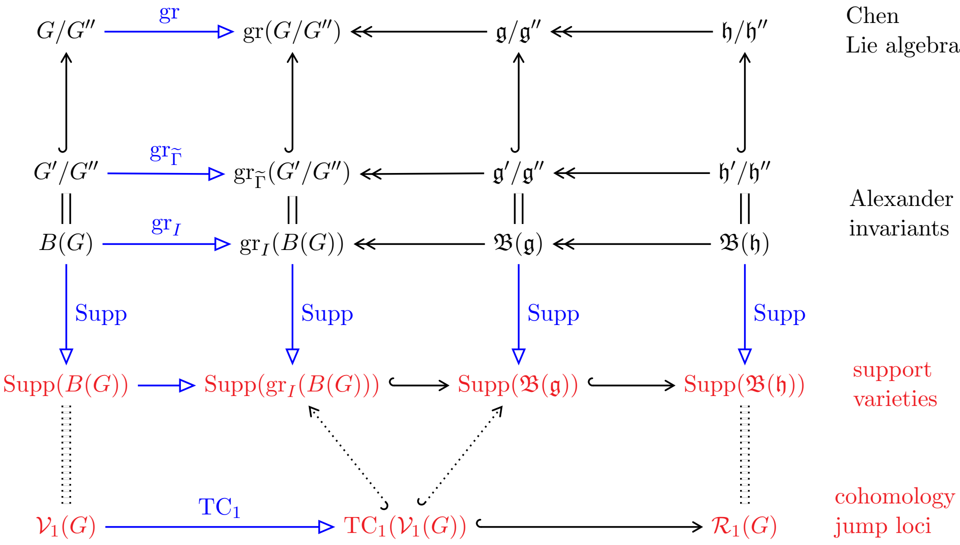
3. Resonance varieties



Resonance varieties

- Suppose $A^* := H^*(G, \mathbb{C})$ has finite dimension in each degree.
- For each $a \in A^1$, define a cochain complex of finite-dimensional $\mathbb{C}$-vector spaces,

$$(A, a) : A^0 \xrightarrow{a\cup -} A^1 \xrightarrow{a\cup -} A^2 \xrightarrow{a\cup -} \cdots ,$$

  with differentials given by left-multiplication by a.
- The *resonance varieties* of G are the homogeneous subvarieties of A^1

$$\mathcal{R}_k(G, \mathbb{C}) = \{a \in A^1 \mid \dim_{\mathbb{C}} H^1(A^*; a) \geq k\}.$$

- $\mathcal{R}_1(\mathbb{Z}^n, \mathbb{C}) = \{0\}$; $\mathcal{R}_1(F_g, \mathbb{C}) = \mathbb{C}^g$; $\mathcal{R}_1(\pi_1(\Sigma_g), \mathbb{C}) = \mathbb{C}^{2g},\ g \geq 2$.
- $\mathcal{R}_1(G; \mathbb{C}) \cong \text{Supp}(\mathfrak{B}(\mathfrak{h}(G; \mathbb{C})))$, (away from the origin). [Matei–Suciu 98]

Characteristic varieties

- X: a connected CW-complex of finitely many 1-cells, with $G = \pi_1(X)$.
- The *rank 1 local system* on X is a 1-dimensional $\mathbb{C}$-vector space $\mathbb{C}_\rho$ with a right $\mathbb{C}G$-module structure $\mathbb{C}_\rho \times G \to \mathbb{C}_\rho$ given by $\rho(g) \cdot a$ for $a \in \mathbb{C}_\rho$ and $g \in G$ for $\rho \in \mathrm{Hom}(G, \mathbb{C}^*)$.
- The *characteristic varieties* of X over $\mathbb{C}$ are the Zariski closed subsets

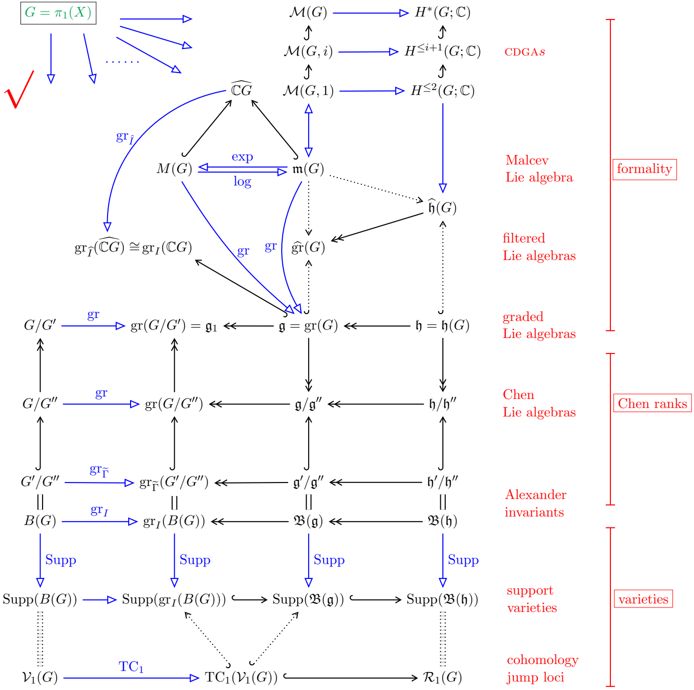
$$\mathcal{V}_k(X, \mathbb{C}) = \{\rho \in \mathrm{Hom}(G, \mathbb{C}^*) \mid \dim_{\mathbb{C}} H_1(X, \mathbb{C}_\rho) \geq k\}.$$

- $\mathcal{V}_1(T^n, \mathbb{C}) = \{1\}$; $\mathcal{V}_1(F_n, \mathbb{C}) = (\mathbb{C}^*)^g$; $\mathcal{V}_1(\Sigma_g, \mathbb{C}) = (\mathbb{C}^*)^{2g}$ for $g \geq 2$.
- $\mathcal{V}_1(G) \cong \mathrm{Supp}(B(G))$, (away from the origin). [E. Hironaka 97]

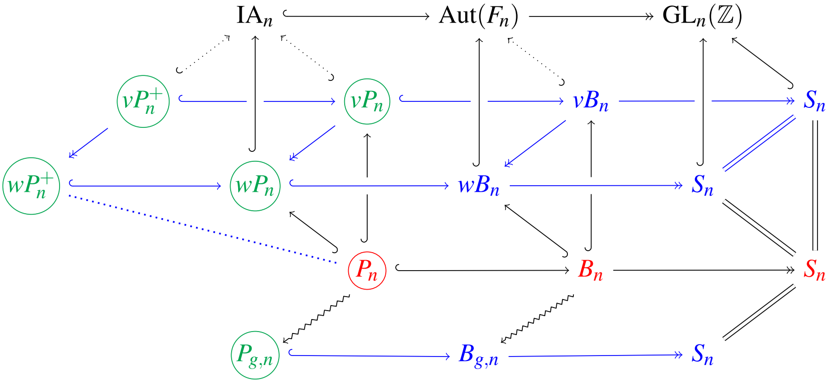
Theorem (Tangent Cone Theorem) (Dimca–Papadima–Suciu 09)

If G is 1-formal, then the tangent cone $\mathrm{TC}_1(\mathcal{V}_k(G, \mathbb{C}))$ equals $\mathcal{R}_k(G, \mathbb{C})$. Moreover, $\mathcal{R}_k(G, \mathbb{C})$ is a union of rationally defined linear subspaces of $H^1(G, \mathbb{C})$.

Algebraic invariants √



Applications: Pure braid groups and their relatives



- The Artin braid groups B_n are subgroups of $\mathrm{Aut}(F_n)$.
  $B_n = \{\beta \in \mathrm{Aut}(F_n) \mid \beta(x_i) = wx_{\tau(i)}w^{-1}, \beta(x_1 \cdots x_n) = x_1 \cdots x_n\}$.
- The welded braid groups $wB_n = \{\beta \in \mathrm{Aut}(F_n) \mid \beta(x_i) = wx_{\tau(i)}w^{-1}\}$.
- The pure braid groups $P_n = B_n \cap IA_n$ and the pure welded braid groups $wP_n = wB_n \cap IA_n$.
- The pure welded braid group wP_n has presentation [McCool 86]

$$\left\langle x_{ij}, (1 \le i \ne j \le n) \;\middle|\; \begin{array}{l} x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}, \\ [x_{ij}, x_{kl}] = 1, \\ [x_{ij}, x_{kj}] = 1, \text{ for } i,j,k,l \text{ distinct} \end{array} \right\rangle.$$

- wP_n^+ is the subgroup of wP_n generated by $\{x_{ij} \mid 1 \le i < j \le n\}$.
  wP_n (wP_n^+) are also called (upper) McCool groups.
- A classifying space for P_n is the configuration space

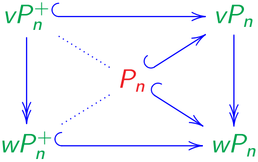
$$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \ne z_j \text{ for } i \ne j\}.$$

- wP_n^+ and wP_n are the fundamental groups of the untwisted flying rings spaces given by Brendle and Hatcher (13).

- The pure virtual braid groups vP_n come from the virtual knot theory introduced by Kauffman (99).
- The following presentation of vP_n was given by Bardakov (04):

$$\left\langle x_{ij}, (1 \le i \ne j \le n) \;\middle|\; \begin{array}{l} x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}, \\ [x_{ij}, x_{kl}] = 1, \text{ for } i,j,k,l \text{ distinct} \end{array} \right\rangle.$$

- vP_n^+ is the subgroup of vP_n generated by $\{x_{ij} \mid 1 \le i < j \le n\}$.
- Bartholdi et al. (06) constructed classifying spaces for vP_n and vP_n^+ by taking quotients of permutahedra by suitable actions of the symmetric groups.



<table>
<tr><th>G_n</th><th>P_n</th><th>wP_n</th><th>wP_n^+</th><th>vP_n, vP_n^+</th></tr>
<tr><td>$H^*(G_n)$</td><td>Arnol'd (69)</td><td rowspan="2">Jensen, McCammond, and Meier (06) Conjectured by Brownstein and R. Lee (93).</td><td rowspan="2">F. Cohen, Pakhianathan, Vershinin, and Wu (07)</td><td rowspan="3">Bartholdi, Enriquez, Etingof, and Rains (06) P. Lee(13)</td></tr>
<tr><td>$\mathrm{gr}(G_n)$</td><td>Kohno(85) Falk–Randell (85)</td></tr>
<tr><td>Koszul</td><td>Arnol'd (69), Kohno(85)</td><td>D. Cohen–Pruidze (08)</td><td>(No for $n \geq 4$) Conner–Goetz (08)</td></tr>
<tr><td>1-formal</td><td>Kohno(83)</td><td colspan="2">Berceanu–Papadima (09)</td><td>?</td></tr>
<tr><td>Resonance $\mathcal{R}_1(G_n)$</td><td>D. Cohen–Suciu (99)</td><td>D. Cohen (09)</td><td>?</td><td>?</td></tr>
<tr><td>Chen Ranks $\theta_k(G_n)$</td><td>D. Cohen–Suciu (93)</td><td>D. Cohen–Schenck (15) for $k \gg 1$</td><td>?</td><td>?</td></tr>
</table>

- F. Cohen, et al. (07) asked a question: Are wP_n^+ and P_n isomorphic for $n \geq 4$?
- $vP_n^+ \hookrightarrow vP_n$ is split [Bartholdi et al.]   Bellingeri asked: Is $wP_n^+ \hookrightarrow wP_n$ split ?

Upper pure welded braid groups (Upper McCool groups)

Theorem (Suciu–W.)

The *Chen ranks* θ_k *of* wP_n^+ *are given by* $\theta_1 = \binom{n}{2}, \theta_2 = \binom{n}{3}, \theta_3 = 2\binom{n+1}{4}$,

$$\theta_k = \binom{n+k-2}{k+1} + \theta_{k-1} = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}, \; k \geq 4.$$

Theorem (Suciu–W.)

The *first resonance variety* *of the upper McCool group* wP_n^+ *has irreducible decomposition:*

$$\mathcal{R}_1(wP_n^+, \mathbb{C}) = \bigcup_{n \geq i > j \geq 2} L_{i,j},$$

where $L_{i,j} = \mathbb{C}^j$ *is the linear subspace defined by the equations*

$$\begin{cases} x_{i,l} + x_{j,l} = 0 & \text{for } 1 \leq l \leq j-1, \\ x_{i,l} = 0 & \text{for } j+1 \leq l \leq i-1, \\ x_{s,t} = 0 & \text{for } s \neq i,\ s \neq j, \text{ and } 1 \leq t < s. \end{cases}$$

Outline of the proofs:

1. Find a minimal presentation for the infinitesimal Alexander invariant $\mathfrak{B}(wP_n^+) := \mathfrak{B}(\mathfrak{h}(wP_n^+;\mathbb{C}))$ using the cohomology algebra $H^*(wP_n^+;\mathbb{C})$.
2. Find a Gröbner basis for the above presentation of $\mathfrak{B}(wP_n^+)$.

For Chen ranks theorem:

3. Compute the Hilbert series of $\mathfrak{B}(wP_n^+)$ using the Gröbner basis.
4. Using formula $\mathrm{Hilb}(\mathfrak{B}(G)\otimes\mathbb{C},t)=\sum_{k\geq 0}\theta_{k+2}(G)t^k$, find the Chen ranks.

For Resonance varieties theorem:

5. Using the presentation of $\mathfrak{B}(wP_n^+)$, show that the right side is a lower bound for the first resonance variety.
6. Using the Gröbner basis, show that the right side is a upper bound.

Corollary

*The pure braid group P_n, the upper McCool groups wP_n^+, and the product group $\Pi_n := \prod_{i=1}^{n-1} F_i$ are **not** isomorphic for $n \geq 4$.*

Proof: $\theta_4(P\Sigma_n^+) = 2\binom{n+1}{4} + \binom{n+2}{5}, \theta_4(P_n) = 3\binom{n+1}{4}, \theta_4(\Pi_n) = 3\binom{n+2}{5}$. $\square$

Corollary

*There is **no** epimorphism from wP_n to wP_n^+ for $n \geq 4$. In particular, the inclusion $\iota \colon wP_n^+ \to wP_n$ admits no splitting for $n \geq 4$.*

The proof is based on the first resonance varieties of wP_n and wP_n^+ together with the following lemma.

Lemma [Papadima–Suciu 06]
If $\alpha \colon G_1 \to G_2$ is an epimorphism, then the induced monomorphism, $\alpha^* \colon H^1(G_2; \mathbb{C}) \to H^1(G_1; \mathbb{C})$, takes $\mathcal{R}_1(G_2, \mathbb{C})$ to $\mathcal{R}_1(G_1, \mathbb{C})$.

Pure virtual braid groups

Theorem (Suciu, W.)

The pure virtual braid groups vP_n and vP_n^+ are 1-formal if and only if $n \leq 3$.

Sketch of proof:

Lemma

There are split monomorphisms

$$\begin{array}{ccccccccccc}
vP_2^+ & \hookrightarrow & vP_3^+ & \hookrightarrow & vP_4^+ & \hookrightarrow & vP_5^+ & \hookrightarrow & vP_6^+ & \hookrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
vP_2 & \hookrightarrow & vP_3 & \hookrightarrow & vP_4 & \hookrightarrow & vP_5 & \hookrightarrow & vP_6 & \hookrightarrow & \dots
\end{array}$$

To show: vP_3 is 1-formal and vP_4^+ is not 1-formal.

Lemma

The group vP_3 is 1-formal.

Proof: $vP_3 \cong N * \mathbb{Z}$, and $P_4 \cong N \times \mathbb{Z}$.

Lemma

The group vP_4^+ is not 1-formal.

Proof: The first resonance variety $\mathcal{R}_1(vP_4^+, \mathbb{C})$ is the subvariety of $\mathbb{C}^6$ given by the equations

$$\begin{aligned}
&x_{12}x_{24}(x_{13}+x_{23}) + x_{13}x_{34}(x_{12}-x_{23}) - x_{24}x_{34}(x_{12}+x_{13}) = 0,\\
&x_{12}x_{23}(x_{14}+x_{24}) + x_{12}x_{34}(x_{23}-x_{14}) + x_{14}x_{34}(x_{23}+x_{24}) = 0,\\
&x_{13}x_{23}(x_{14}+x_{24}) + x_{14}x_{24}(x_{13}+x_{23}) + x_{34}(x_{13}x_{23}-x_{14}x_{24}) = 0,\\
&x_{12}(x_{13}x_{14}-x_{23}x_{24}) + x_{34}(x_{13}x_{23}-x_{14}x_{24}) = 0.
\end{aligned}$$

The group vP_4^+ is not 1-formal, by the Tangent Cone Theorem.

Application: Seifert fibered manifolds

Let $\eta\colon M \to \Sigma_g$ be an orientable Seifert fibered manifold with Seifert invariants $(g, b, (\alpha_i, \beta_i), i = 1, \ldots, s)$. Let $e(\eta)$ be its Euler number.

Theorem (Putinar 98)

If $g > 0$, the minimal model $\mathcal{M}(M)$ is the Hirsch extension $\mathcal{M}(\Sigma_g) \otimes (\bigwedge(c), d)$, with differential $d(c) = 0$ if $e(\eta) = 0$, and $d(c) \in \mathcal{M}^2(\Sigma_g)$ represents a generator of $H^2(\Sigma_g; \mathbb{Q})$ if $e(\eta) \neq 0$.

Proposition

The Malcev Lie algebra of $\pi_\eta := \pi_1(M)$ is the degree completion of the graded Lie algebra

$$L(\pi_\eta) = \begin{cases} \mathrm{Lie}(x_1, y_1, \ldots, x_g, y_g, z)/\langle \sum_{i=1}^{g} [x_i, y_i] = 0,\ z \text{ central}\rangle & \text{if } e(\eta) = 0; \\ \mathrm{Lie}(x_1, y_1, \ldots, x_g, y_g, w)/\langle \sum_{i=1}^{g} [x_i, y_i] = w,\ w \text{ central}\rangle & \text{if } e(\eta) \neq 0, \end{cases}$$

where $\deg(w) = 2$ and the other generators have degree 1.
Moreover, $\mathrm{gr}(\pi_\eta; \mathbb{Q}) \cong L(\pi_\eta)$.

Proposition (Suciu–W. 15)

Let $\eta\colon M \to \Sigma_g$ be a Seifert fibration. The holonomy Lie algebra of the group $\pi_\eta = \pi_1(M)$ is given by

$$\mathfrak{h}(\pi_\eta;\mathbb{Q}) = \begin{cases} \mathrm{Lie}(x_1, y_1, \dots, x_g, y_g, h)/\langle \sum_{i=1}^{s}[x_i, y_i] = 0, h \text{ central}\rangle & \text{if } e(\eta) = 0; \\ \mathrm{Lie}(2g) & \text{if } e(\eta) \neq 0. \end{cases}$$

Corollary

Fundamental groups of orientable Seifert manifolds are filtered-formal.

Corollary

If $g = 0$, the group π_η is always 1-formal.
If $g > 0$, the group π_η is graded-formal if and only if $e(\eta) = 0$.

Future work: Techniques

1. Generalized Tangent Cone theorem

- Tangent Cone Theorem [Dimca–Papadima–Suciu 09]
  If G is 1-formal, then the tangent cone $\mathrm{TC}_1(\mathcal{V}_k(G,\mathbb{C}))$ equals $\mathcal{R}_k(G,\mathbb{C})$.
  Moreover, $\mathcal{R}_k(G,\mathbb{C})$ is a union of $\mathbb{Q}$-defined linear subspaces of $H^1(G,\mathbb{C})$.

- The resonance varieties of a CDGA (A,d) with finite dimension A^1, recently studied by Dimca, Papadima, Suciu, et al., are defined to be

  $$\mathcal{R}^i_k(A,d) := \{a \in A^1 \mid \dim(H^i(A;\delta_a)) \geq k\},$$

  where $\delta_a(u) = d(u) + a \cdot u$ for $u \in A^i$.

- Generalized Tangent Cone theorem (conjecture):

  If G is filtered-formal, then the tangent cone $\mathrm{TC}_1(\mathcal{V}_k(G,\mathbb{C}))$ equals $\mathcal{R}_k(A(G))$, for a finite type CDGA model of G.

Future work: Techniques

2. Generalized Chen ranks conjecture

- Chen ranks conjecture [Suciu (01)] Let G be an arrangement group. Let h_n be the number of n-dimensional irreducible components of $\mathcal{R}_1(G)$.

$$\theta_k(G) = \sum_{n \geq 2} h_n \cdot \theta_k(F_n), \quad \text{for} \quad k \gg 1.$$

- D. Cohen and Schenck (15) proved this conjecture in a wider range: The above formula holds if G is a 1-formal, commutator-relators group, such that the resonance variety $\mathcal{R}_1(G)$ is 0-isotropic, projectively disjoint, and reduced as a scheme.

  They also show that the McCool group wP_n satisfies these conditions.

- However, Chen ranks formula does not hold for wP_n^+.

- Generalized Chen ranks conjecture

  1-formal $\longrightarrow$ filtered-formal

  resonance varieties $\longrightarrow$ resonance varieties of a finite type CDGA model.

Future work: Applications

1. Picture groups

- For every quiver of finite type there is a finitely presented group called a picture group (Igusa, Orr, Todorov and Weyman (14)).
- They computed the cohomology ring of $G(A_n)$, the picture group of type A_n with straight orientation.
- Classifying spaces of $G(A_n)$ were given by Igusa (14).
- Future work: finite type CDGA models, resonance varieties, resonance varieties of the models, Malcev Lie algebras, Chen ranks, formality properties?

Future work: Applications

2. Pure braid group on Riemann surfaces

Let $P_{g,n}$ be the pure braid group on compact Riemann surface Σ_g of genus $g \geq 1$.

- The configuration space $M_{g,n} := F(\Sigma_g, n)$ gives a classifying space of $P_{g,n}$.
- Bezrukavnikov (94) computed the Malcev Lie algebra of $P_{g,n}$ using the CDGA model given by Kříž and Totaro.
- $P_{g,n}$ is 1-formal for $g \geq 2$.
- $P_{1,n}$ is filtered-formal for $n \geq 3$. (Bezrukavnikov 94, Calaque-Enriquez-Etingof 09)
- $P_{1,n}$ is not 1-formal for $n \geq 3$ by computing $\mathcal{R}_1(P_{1,n})$. (Dimca-Papadima-Suciu 09)
- Future work: resonance varieties, resonance varieties of the model, Chen ranks?

Continued?

Holonomy Lie algebra (continued)

- Suppose G has a finite presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$.
- The *Magnus expansion* $M\colon \mathbb{Q}F \to \mathbb{Q}\langle\langle x_1, \ldots, x_n \rangle\rangle$, is a ring homomorphism defined by $M(x_i) = 1 + x_i$ and $M(x_i^{-1}) = 1 - x_i + x_i^2 - x_i^3 + \cdots$.

Theorem (Fenn-Sjerve 87, Matei-Suciu 98)

Suppose G is a commutator-relators group, such that $H_2(G)$ is free abelian. The cup-product $\cup\colon H^1(G) \wedge H^1(G) \to H^2(G)$ is given by

$$u_i \cup u_j = \sum_{k=1}^{m} M(r_k)_{i,j} \beta_k.$$

Proposition (Papadima-Suciu 04)

If G is a commutator-relators group, then

$$\mathfrak{h}(G; \mathbb{Q}) = \mathrm{Lie}(x_1, \ldots, x_n) / \langle M_2(r_1), \ldots, M_2(r_m) \rangle.$$

Holonomy Lie algebra (continued)

- The *Magnus expansion* of a group G is the composition

$$\kappa : \quad \mathbb{Q}F \xrightarrow{M} \mathbb{Q}\langle\langle x_1, \dots, x_n \rangle\rangle \xrightarrow{\widehat{\pi}} \mathbb{Q}\langle\langle y_1, \dots, y_b \rangle\rangle,$$

  where $b = \dim H_1(G; \mathbb{Q})$ and $\widehat{\pi}$ is induced by $\pi \colon H_1(F; \mathbb{Q}) \to H_1(G; \mathbb{Q})$.
- In particular, if G is a commutator-relators group, then $\widehat{\pi}$ is identity.
- A group G has an *echelon presentation* $\langle x_1, \dots, x_n \mid w_1, \dots, w_m \rangle$, if the augmented Jocobian matrix of Fox derivative $(M(w_k)_i)$ is in row-echelon form.
- $\kappa_2(r)$: the degree 2 homogeneous part of $\kappa(r)$.
- $\kappa(r)_{i,j}$: the coefficient of $y_i y_j$ in $\kappa(r)$.

Holonomy Lie algebra (continued)

Theorem (Suciu-W. 15)

Let G be a group with a presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$.
Let K_G be the 2-complex associated to the presentation.

1. *There exists a group $\widetilde{G}$ with echelon presentation $\langle x_1, \ldots, x_n \mid w_1, \ldots, w_m \rangle$ such that*

$$\mathfrak{h}(G;\mathbb{Q}) \cong \mathfrak{h}(\widetilde{G};\mathbb{Q}) \text{ and } H^{\leq 2}(K_G;\mathbb{Q}) \cong H^{\leq 2}(K_{\widetilde{G}};\mathbb{Q}).$$

2. *The cup-product map $\cup\colon H^1(K_G;\mathbb{Q}) \wedge H^1(K_G;\mathbb{Q}) \to H^2(K_G;\mathbb{Q})$ is given by*

$$u_i \cup u_j = \sum_{k=n-b+1}^{m} \kappa(w_k)_{i,j}\beta_k.$$

3. *There exists an isomorphism of graded Lie algebras*

$$\mathfrak{h}(G;\mathbb{Q}) \xrightarrow{\cong} \mathrm{Lie}(y_1, \ldots, y_b)/\mathrm{ideal}(\kappa_2(w_{n-b+1}), \ldots, \kappa_2(w_m)) .$$