# Lie algebras of finitely generated groups and their formality properties

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Auslander Distinguished Lectures and International Conference
Woods Hole, MA

May 1, 2014

# References

Alexander I. Suciu and He Wang,
*Cup products, Magnus expansions, and central series of finitely generated groups and Lie algebras*, preprint, 2014.

# Overview


1. Lie algebras of finitely generated groups
   - Associated graded Lie algebra
   - Holonomy Lie algebra
   - Malcev Lie algebra
2. Formality properties
3. Magnus expansion and cup products
4. A presentation for the holonomy Lie algebra
5. Examples
   - The pure flat braid groups
   - The pure braid groups on Riemann surfaces

# Associated graded Lie algebra

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**Example (Chen-Fox-Lyndon)**

If $F$ is the free group of rank $n$, then $\mathrm{gr}(F;\Bbbk)$ is the free graded Lie algebra $\mathrm{Lie}(\Bbbk^n)$.

# Holonomy Lie algebra

## Definition

The *holonomy Lie algebra* of a group $G$ is defined to be

$$\mathfrak{h}(G;\Bbbk) := \mathrm{Lie}(H_1(G;\Bbbk))/\langle \mathrm{im}(\partial_G)\rangle.$$

Here, $\partial_G$ is the composition map:

$$H_2(G;\Bbbk) \xrightarrow{\cup^*} H_1(G;\Bbbk) \wedge H_1(G;\Bbbk) \to \mathrm{Lie}^2(H_1(G;\Bbbk)).$$

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## Lemma (Sullivan-Lambe)

*There exists an epimorphism of graded $\mathbb{k}$-Lie algebras*

$$\Phi_G : \mathfrak{h}(G;\mathbb{k}) \twoheadrightarrow \mathrm{gr}(G;\mathbb{k}).$$

# Malcev Lie algebra

## Definition (Malcev)

The tower

$$\cdots \longrightarrow (G/\Gamma_4 G) \otimes \Bbbk \longrightarrow (G/\Gamma_3 G) \otimes \Bbbk \longrightarrow G/(\Gamma_2 G) \otimes \Bbbk$$

is an inverse limit system. The prounipotent group is defined by

$$\mathcal{P}(G;\Bbbk) = \varprojlim_k ((G/\Gamma_k G) \otimes \Bbbk).$$

The corresponding pronilpotent Lie algebra $\mathfrak{p}(G;\Bbbk)$ gives the *Malcev Lie algebra* (over $\Bbbk$) of $G$.

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## Example

The Malcev Lie algebra of the free group $F$ is $\widehat{\mathrm{Lie}(\Bbbk^n)}$.

# Quillen's construction

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- The *Malcev group* $M(G) := \{$ all group-like elements of $\hat{A}$ $\}$.
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## Theorem (Quillen)

1. *There is a filtered group isomorphism* $\mathcal{P}(G;\mathbb{Q}) \to M(G)$.
2. *There is a filtered Lie algebra isomorphism* $\mathfrak{p}(G;\mathbb{Q}) \to \mathfrak{m}(G)$
3. *There is a graded Lie algebra isomorphism* $\mathrm{gr}(G;\mathbb{Q}) \to \mathrm{gr}(\mathfrak{m}(G))$.

# Formality properties

The next definition is from the rational homotopy theory.

**Definition (roughly)**

- A topological space $X$ is called *1-formal* if there exists a CDGA homomorphism from the Sullivan 1-minimal model $\mathcal{M}(X)$ to the CDGA $(H^*(X;\Bbbk), d = 0)$ inducing isomorphism in cohomology of degree 1 and monomorphism in degree 2.
- A group $G$ is *1-formal* if the associated Eilenberg-MacLane space $K(G,1)$ is 1-formal.

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## Example

Examples of 1-formal groups include free groups, Artin groups, the pure braid groups, Kähler groups, etc.

# Formality properties

## Definition

- A group $G$ is *graded-formal* (over $\Bbbk$), if the canonical projection $\Phi_G \colon \mathfrak{h}(G;\Bbbk) \twoheadrightarrow \operatorname{gr}(G;\Bbbk)$ is an isomorphism.
- A group $G$ is *filtered-formal*, if there is a filtered Lie algebras isomorphism $\mathfrak{m}(G) \cong \widehat{\operatorname{gr}(\mathfrak{m}(G))}$.

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## Lemma (Sullivan-Papadima-Suciu)

*The group $G$ is 1-formal $\Longleftrightarrow$ the Malcev Lie algebra of $G$ is isomorphic to the degree completion of a rational quadratic Lie algebra.*

# Formality properties

**Corollary**

*$G$ is 1-formal $\Longleftrightarrow$ $G$ is graded-formal (over $\mathbb{Q}$) and filtered-formal.*

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## Corollary

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$$\begin{array}{ccc}
\mathfrak{m}(G) & \xrightarrow{\quad 1-\textit{formal} \quad} & \widehat{\mathfrak{h}(G;\mathbb{Q})} \\
& \searrow_{\textit{filtered}} \qquad\qquad {}_{\textit{graded}}\swarrow & \\
& \widehat{\mathrm{gr}(\mathfrak{m}(G))} \cong \widehat{\mathrm{gr}(G;\mathbb{Q})} &
\end{array}$$

# Torsion-free nilpotent groups

In the lower central series $\{\Gamma_k G\}_{k\geq 1}$, if $\Gamma_t G \neq 1$ and $\Gamma_{t+1}(G) = 1$, then $G$ is called *t-step nilpotent group*.

## Theorem (Suciu-Wang)

*The torsion-free, 2-step nilpotent group $G$ is filtered-formal. That is* $\mathfrak{m}(G) \cong \operatorname{gr}(G; \mathbb{Q})$.

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*The torsion-free, 2-step nilpotent group $G$ is filtered-formal. That is $\mathfrak{m}(G) \cong \mathrm{gr}(G;\mathbb{Q})$.*

By Igusa and Orr's lemma, the torsion-free, 2-step nilpotent group $G$ has a presentation

$$\left\langle x_1,\dots,x_n,y_1,\dots,y_m \,\middle|\, [x_i,x_j]=\prod_{k=1}^{m} y_k^{c_{ijk}},\ [y_i,y_j]=1, \text{for } i<j; [x_i,y_j]=1 \right\rangle.$$

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**Example**

The class of filtered-formal groups includes the unipotent groups $U_n(\mathbb{Z})$ (Lambe-Priddy) and the $n$-step free nilpotent groups $F/\Gamma_n F$ (Massuyeau).

## Remark

In general, torsion-free nilpotent groups need NOT be filtered-formal.

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This phenomenon is illustrated by an example adapted from Lambe and Priddy.

## Example

The Malcev Lie algebra $\mathfrak{m}(G)$ of a group $G$ is given by

$$\left[\begin{array}{c|ccccccc} [-,-] & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline e_1 & 0 & e_3 & e_4 & e_5 & e_6 & e_7 & 0 \\ e_2 & -e_3 & 0 & e_6 & e_7 & -e_7 & 0 & 0 \\ e_3 & -e_4 & -e_6 & 0 & e_7 & 0 & 0 & 0 \\ e_4 & -e_5 & -e_7 & -e_7 & 0 & 0 & 0 & 0 \\ e_5 & -e_6 & e_7 & 0 & 0 & 0 & 0 & 0 \\ e_6 & -e_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ e_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The Lie algebra $\mathfrak{m}(G)$ is not isomorphic to $\mathrm{gr}(G;\mathbb{Q})$.

# Magnus expansion

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## Definition

The *quasi-Magnus expansions* of the group $G$, denoted by $\kappa$, is the composite

$$\Bbbk F \xrightarrow{M} \Bbbk\langle\langle x_1, \ldots, x_n\rangle\rangle \xrightarrow{\widehat{\pi}} \Bbbk\langle\langle y_1, \ldots, y_b\rangle\rangle,$$

where $\widehat{\pi}$ can be identified by $\widehat{T}(\pi)\colon \widehat{T}(H_1(F;\Bbbk)) \twoheadrightarrow \widehat{T}(H_1(G;\Bbbk))$ which is induced by $\pi\colon H_1(F;\Bbbk) \to H_1(G;\Bbbk)$.

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In particular, if $G$ is a commutator-relators group, then $\widehat{\pi}$ is identity.

# Basis for (co)homology groups

- $G$ has a presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$.
- $K_G$ is the 2-complex associated to this presentation of $G$.

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- There exists a matrix $C = (c_{lk})_{m \times m} \in \mathrm{GL}(m; \mathbb{Z})$ such that $C \cdot (d_2^G)^*$ is in row echelon form, where $d_2^G \colon C_2(K_G; \Bbbk) \to C_1(K_G; \Bbbk)$

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- $H_1(K_G;\Bbbk)$ has basis $\{y_1, \ldots, y_b\}$. The dual basis is $\{u_1, \ldots, u_b\}$.
- $H_2(K_G;\Bbbk)$ has basis $\{\gamma_{d+1}, \ldots, \gamma_m\}$. Here, $\gamma_k := \sum_{l=1}^m c_{lk} r_l$, $d = n - b$. Then $H^2(K_G;\Bbbk)$ has dual basis $\{\beta_{d+1}, \ldots, \beta_m\}$.

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**Theorem (Suciu-Wang)**

*The cup-product map* $\cup\colon H^1(K_G;\Bbbk) \wedge H^1(K_G;\Bbbk) \to H^2(K_G;\Bbbk)$ *is given by the formula*

$$u_i \cup u_j = \sum_{k=d+1}^{m} \kappa(r_k')_{i,j}\beta_k,$$

*where* $\kappa$ *is the quasi-Magnus expansion and* $r_k' = r_1^{c_{1k}} r_2^{c_{2k}} \cdots r_m^{c_{mk}}$ *for* $d+1 \le k \le m$ *with* $c_{lk}$ *defined above.*

# Cup products

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*where* $\kappa$ *is the quasi-Magnus expansion and* $r'_k = r_1^{c_{1k}} r_2^{c_{2k}} \cdots r_m^{c_{mk}}$ *for* $d+1 \leq k \leq m$ *with* $c_{lk}$ *defined above.*

The idea of the proof:

- Define a new group $\tilde{G}$ by the same generators of $G$ with relations $r'_k$, $1 \leq k \leq m$.
- Finding a chain transformation $T\colon C_*(\widetilde{K_{\tilde{G}}};\mathbb{Z}) \to B_*(\tilde{G})$.
- Find the cup product formula for $\tilde{G}$.
- Transfer the formula to group $G$.

# A presentation for the holonomy Lie algebra

## Corollary (Suciu-Wang)

*There exists an isomorphism of graded Lie algebras*

$$\mathfrak{h}(G;\Bbbk) \xrightarrow{\cong} \mathsf{Lie}_{\Bbbk}[y_1,\dots,y_b]/\mathrm{ideal}(\mathrm{I})\ .$$

*Here* $\mathsf{Lie}_{\Bbbk}[y_1,\dots,y_b]$ *is the free Lie algebra over* $\Bbbk$ *generated by elements* $y_1,\dots,y_b$ *in degree* $1$*, and* $I$ *is the set*

$$I := \left\{ \sum_{1\le i<j\le b} \sum_{l=1}^{m} c_{lk}\kappa(r_l)_{i,j}[y_i,y_j], \quad d+1\le k\le m \right\}$$

*where* $b$ *is the first Betti number of* $G$*,* $d = n - b$*.*

If $G$ is a commutator-relators group, the result can be simplified.

# The pure flat braid groups (The triangular groups $Tr_n$)

$Tr_n$ has a presentation

$$\left\langle x_{ij}, (1 \leq i < j \leq n) \;\middle|\; \begin{array}{c} x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij} \text{ for } i < j < k; \\ x_{ij}x_{kl} = x_{kl}x_{ij} \text{ for } i \neq j \neq k \neq l, \\ \qquad\qquad i < j \text{ and } k < l \end{array} \right\rangle.$$

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**Proposition (Bartholdi-Enriquez-Etingof-Rains (06)-Lee(13))**

*The Lie algebra* $\mathrm{gr}(Tr_n; \mathbb{Q})$ *is generated by* $a_{i,j}$ $(1 \leq i < j \leq n)$ *with defining relations given by the Yang-Baxter equation,*
$[a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}] = 0$ *for* $i < j < k$ *and*
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The triangular group $Tr_n$ is graded-formal, that is $\mathrm{gr}(Tr_n;\mathbb{Q}) \cong \mathfrak{h}(Tr_n;\mathbb{Q})$.

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The triangular group $Tr_n$ is graded-formal, that is $\mathrm{gr}(Tr_n; \mathbb{Q}) \cong \mathfrak{h}(Tr_n; \mathbb{Q})$.

**Proposition (Suciu-Wang)**

*The triangular group $Tr_n$ is not filtered-formal for $n \geq 4$.*

# The pure braid group on Riemann surfaces

- $P_{g,n}$ is the pure braid group on $n$ strings of the underlying Riemann surface of genus $g$.
- $P_{g,n} = \pi_1(F(C_g, n))$, where $F(C_g, n)$ is the configuration of $C_g$, which is a smooth compact complex curve of genus $g$ ($g \geq 1$).

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**Proposition (Bezrukavnikov 94)**

*The Malcev Lie algebra of $P_{g,n}$ is given by the completion of Lie algebra generated by $\{a_i^l, b_i^l, s_{ij}\}$ ($1 \leq i \leq n, 1 \leq l \leq g$) with relations*

$$
\begin{aligned}
&[a_i^l, b_j^k] = 0 (i \neq j, l \neq k);\\
&[a_i^l, a_j^k] = [b_i^l, b_j^k] = 0 (i \neq j);\\
&[a_i^l, b_j^l] = [a_j^k, b_i^k] = s_{ij} (i \neq j);\\
&\textstyle\sum_{l=1}^{g} [a_i^l, b_i^l] = -\sum_{i \neq j} s_{ij};\\
&[a_i^l, s_{jk}] = [b_i^l, s_{jk}] = 0 (i \neq j \neq k).
\end{aligned}
$$

# The pure braid groups on Riemann surfaces

## Proposition

- $P_{n,g}$ *is* 1*-formal for* $g \geq 2$. *(Bezrukavnikov 94)*
- $P_{n,1}$ *is not* 1*-formal for* $n \geq 3$. *(Dimca-Papadima-Suciu 09)*
- $P_{n,1}$ *is filtered formal for* $n \geq 3$. *(Bezrukavnikov94-Calaque-Enriquez-Etingof 09)*

## Corollary

$P_{n,1}$ *is not graded-formal for* $n \geq 3$.

*Thank You!*