RATIONAL HOMOTOPY THEORY

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Abstract. A mini-course on rational homotopy theory.

Contents

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1. INTRODUCTION

One of the goals of topology is to classify the topological spaces up to some equivalence relations, e.g., homeomorphic equivalence and homotopy equivalence (for algebraic topology).

In algebraic topology, most of the time we will restrict to spaces which are homotopy equivalent to CW complexes.

We have learned several algebraic invariants such as fundamental groups, homology groups, cohomology groups and cohomology rings. Using these algebraic invariants, we can seperate two non-homotopy equivalent spaces.

Another powerful algebraic invariants are the higher homotopy groups. Whitehead theorem shows that the functor of homotopy theory are power enough to determine when two CW complex are homotopy equivalent.

A rational coefficient version of the homotopy theory has its own techniques and advantages: 1. fruitful algebraic structures. 2. easy to calculate.

2. Elementary homotopy theory

2.1. **Higher homotopy groups.** Let X be a connected CW-complex with a base point x_0. Recall that the fundamental group

$$\pi_1(X, x_0) = [(I, \partial I), (X, x_0)]$$

is the set of homotopy classes of maps from pair $(I, \partial I)$ to (X, x_0) with the product defined by composition of paths.

Similarly, for each $n \geq 2$, the *higher homotopy group*

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$$

is the set of homotopy classes of maps from pair $(I^n, \partial I^n)$ to (X, x_0) with the product defined by composition.

$$(f + g)(t_1, t_2, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & t_1 \in [0, 1/2] \\ g(2t_1 - 1, t_2, \dots, t_n) & t_1 \in [1/2, 1] \end{cases}$$

Pictures

Equivalently, the homotopy group $\pi_n(X, x_0)$ can be defined as $\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)]$, the set of homotopy classes of maps from (S^n, s_0) to (X, x_0). To look at the group multiplication, we need to identify S^n with the suspension $\Sigma S^{n-1} = S^{n-1} \times I / S^{n-1} \times \{0, 1\} \cup \{s_0\} \times I$. If there is no confusion, we omit the base point and denote the homotopy group by $\pi_n(X)$.

The homotopy groups of the product $X \times Y$ has an easy formula:

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y).$$

For $n \geq 2$, the homotopy group $\pi_n(X)$ is an abelian group.

(Picture proof)

Higher homotopy groups are homotopy invariants.

Example. $\pi_n(*) = \pi_n(\mathbb{R}^m) = 0$ for $n \geq 1$.

Proposition. If $\tilde{X} \to X$ is a connected covering space, then $\pi_n(\tilde{X}) \xrightarrow{\cong} \pi_n(X)$ for $n \geq 2$.

Example. $\pi_1(S^1) = \mathbb{Z}$ and $\pi_n(S^1) = 0$ for $n \geq 2$.

Example. (HW.) Calculate the higher homotopy groups of Riemann surface S_g of positive genus $g > 0$.

Furthermore,

$$\pi_k(S^n) = \begin{cases} 0 & \text{if } k < n \\ \mathbb{Z} & \text{if } k = n \\ ? & \text{if } k > n. \end{cases}$$

Cellular approximation theorem: Every map $f\colon X \to Y$ of CW complexes is homotopic to a cellular map, i.e., $f(X^{(n)}) \subset Y^{(n)}$ for all n.

Serre showed that the homotopy groups $\pi_n(X)$ of a simply connected finite complex X are finitely generated abelian groups. (Simply-connectness is necessary. For example, $X = S^1 \vee S^2$. (HW: calculate $\pi_2(X)$))

Serre also proved that

$$\pi_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } n = 2m \text{ and } k = 4m-1. \\ \text{finite abelian group} & \text{otherwise} \end{cases}$$

2.2. **The Whitehead Theorem.**

Theorem 2.1 (Whitehead Theorem)**.** *A map $f\colon X \to Y$ between connected CW complexes induces isomorphisms $f_*\colon \pi_n(X) \to \pi_n(Y)$ for all n if and only if f is a homotopy equivalence.*

The Whitehead Theorem does **not** say that two CW complexes X and Y with isomorphic homotopy groups are homotopy equivalent. Homology and cohomology do not have similar properties.

Example.(HW) If $X = S^2 \times \mathbb{R}P^3$ and $Y = \mathbb{R}P^2 \times S^3$. Then X and Y have the same fundamental group $\mathbb{Z}/2$, and the same universal cover $S^2 \times S^3$, hence the same homotopy groups. But $H_2(X) = \mathbb{Z}$ and $H_2(Y) = 0$ by Künneth formula.

2.3. **Hurewicz homomorphism.** A map $f\colon S^n \to X$ induces $f_*\colon H_n(S^n) \to H_n(X)$. We identify $H_n(S^n)$ with $\mathbb{Z}$, then there is a natural transformation

$$h\colon \pi_n(X) \to H_n(X)$$

by sending $[f] \in \pi_n(X)$ to $f_*(1) \in H_n(X)$. This h is a group homomorphism called *Hurewicz homomorphism*.

In particular, if $X = S^n$, then the Hurewicz homomorphism is $h\colon \pi_n(S^n) \to H_n(S^n) = \mathbb{Z}$. In fact, it is an isomorphism from the next theorem.

2.4. **The Hurewicz Theorem.**

Theorem 2.2 (Hurewicz Theorem)**.** *If X is a $(n-1)$-connected CW-complex, i.e., $\pi_k(X) = 0$ for $k \leq n-1$, then,*

(1) $\widetilde{H}_k(X) = 0$ for $k \leq n-1$,
(2) $h\colon \pi_n(X) \to H_n(X)$ is an isomorphism (provided $n \geq 2$).

There is a relative form of Hurewicz theorem.

Holonomy Whitehead theorem: A map $f\colon X \to Y$ between two simply-connected CW complexes is a homotopy equivalence if $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism for each n.

HW: Let X be an n-dimensional CW complex. If $\pi_i(X) = 0$ for all $i \leq n$, then X is contractible.

HW: For $n > 1$, the map $h\colon \pi_n(\vee_i S^n) \to H_n(\vee_i S^n) \cong \bigoplus_i \mathbb{Z}$ is an isomorphism.

HW: Every simply-connected, closed 3-manifold is homotopy equivalent to the 3-sphere S^3.

2.5. **Long exact sequence.** Let $p\colon E \to B$ be a fibration (i.e., has the homotopy lifting property) with fiber $F = \pi^{-1}(b_0)$. There is an exact sequence

$$\cdots \to \pi_{n+1}(B, b_0) \xrightarrow{\partial} \pi_n(F, f_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \cdots .$$

HW: Apply the long exact sequence to the following examples:

Example. Connected covering space $\tilde{X} \to X$.

Example. Hopf fibration: $S^1 \to S^3 \to S^2$.

Example. Hopf fibration: $S^1 \to S^{2n+1} \to \mathbb{C}P^n$.

Example. Let $P(X, x_0)$ be the space of paths in X starting at x_0. Then $p : P(X, x_0) \to X$ defined as $p(f) = f(1)$ is a fibration with fiber the loop space $\Omega(X) = L(X, x_0)$ based at x_0. The path space $P(X, x_0)$ is contractible.

$$\pi_k(X) = \pi_{k-1}(\Omega(X)) \text{ for } k \geq 1.$$

2.6. **Eilenberg–MacLane space.** Let G be a group, or an abelian group if $n > 1$. The *Eilenberg–MacLane space* $K(G, n)$ is a connected CW-complex such that

$$\pi_k(K(G,n)) = \begin{cases} G & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

It is exists and uniquely determined by n and G up to homotopy equivalence.

(Proof)

For an abelian group A and any CW-complex X, there exists a natural isomorphism

$$[X, K(A,n)] \cong H^n(X; A).$$

2.7. **Obstruction theory.** Given two CW complexes X and Y. Is there a map from X to Y? If there are two maps f_1 and f_2 from X to X, is there a homotopy F_t of f_1 and f_2.

In this section, we suppose Y is simply-connected. Obstruction theory deals with these two questions by skeleta from $X^{(n)}$ to $X^{(n+1)}$.

Theorem 2.3. *(1) Given $f_n\colon X^{(n)} \to Y$, there is a cohomology class $\mathcal{O}(f_n) \in H^{n+1}(X; \pi_n(Y))$. This class vanishes if and only if $f_n|_{X^{(n-1)}}$ can be extended to a map $f : X^{(n+1)} \to Y$.*
(2) Given maps f and g from X to Y and let $H\colon (X^{(n)}) \times I \to Y$ be a homotopy between $f|_{X^{(n)}}$ and $g|_{X^{(n)}}$. The obstruction of constructing a homotopy f and g lies in $H^n(X; \pi_n(Y))$.

2.8. **Whitehead products.** Let $\alpha = [f] \in \pi_k(X)$ and $\beta = [g] \in \pi_l(X)$. Then $f\colon S^k \to X$ and $g\colon S^l \to X$, and there is a map $f \vee g\colon S^k \vee S^l \to X$.

The product $S^k \times S^l$ is obtained by attaching a $(k+l)$-cell to the wedge $S^k \vee S^l$ via a map $\phi_{k,l}\colon S^{k+l-1} \to S^k \vee S^l$.

The *Whitehead product* of α and β is a homotopy class $[\alpha, \beta] \in \pi_{k+l-1}(X)$ represented by the composition map

$$S^{k+l-1} \xrightarrow{\phi_{k,l}} S^k \vee S^l \xrightarrow{f \vee g} X.$$

Reassign grading on $\pi_k(X)$ to be $(k-1)$.

As shown by Massey, the Whitehead product satisfies the graded Jacobi identity:

$$(-1)^{ik}[\alpha, [\beta, \gamma]] + (-1)^{ij}[\beta, [\gamma, \alpha]] + (-1)^{jk}[\gamma, [\alpha, \beta]] = 0.$$

The Whitehead product is bilinear and graded-symmetric, which make $\pi_k(X)$ to be a graded Lie algebra.

3. Spectral sequences

Consider a fibration $E \to B$ with fiber F. We know that there is a long exact sequence corresponding to this fibration. How to computae the cohomologies of a fibration is a hard question. The method to solve this is Leray-Serre spectral sequence.

Example. For trivial fibration $E = F \times B$, we have the Künneth formula.

We will focus on what is a spectral sequence and how to use it. We will postpone how to construct a spectral sequence.

3.1. Definition of a spectral sequence.

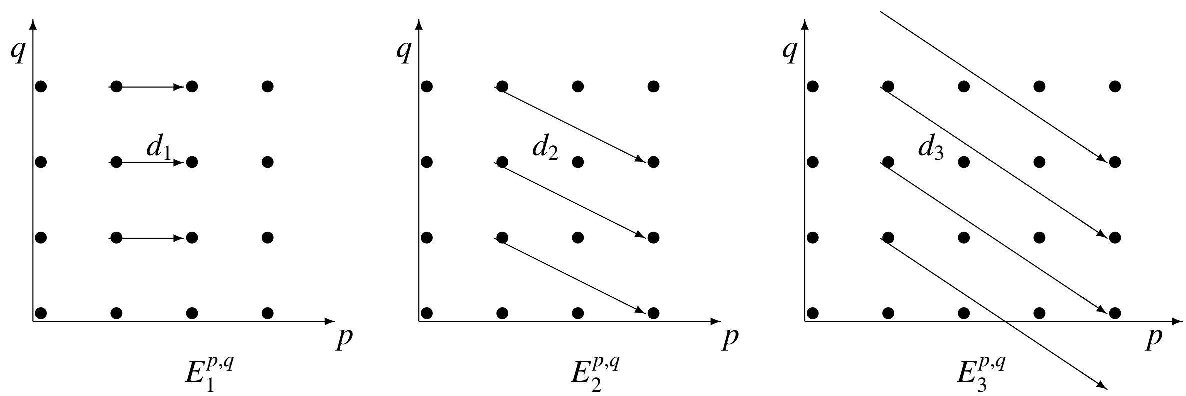
Definition 3.1. A **differential bigraded module** over a ring R (e.g., $\mathbb{Z}$ or $\mathbb{Q}$), is a collection of R-modules, $\{E^{p,q}\}$, where $p, q \in \mathbb{Z}$, together with a R-linear mapping, $d : E^{*,*} \to E^{*+s,*+t}$, satisfying $d \circ d = 0$. Here d is called the differential of bidegree (s, t).

Definition 3.2. A **spectral sequence** is a collection of differential bigraded R-modules $\{E_r^{p,q}, d_r\}$, where $r = 1, 2, \cdots$ and

$$E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}) \cong \ker(d_r : E_r^{p,q} \to E_r^{*,*}) / \operatorname{im}(d_r : E_r^{*,*} \to E_r^{p,q}).$$

The differential d_r has bidegree $(r, 1-r)$ (for a spectral sequence of cohomology type).

Let's see what the (first quadrant) spectral sequence of cohomology type looks like.



Next consider the $E_\infty^{*,*}$-term. (HW: For the first quadrant cohomological spectral sequence, consider $E_r^{p,q}$, when $r > max(p, q+1)$, the differentials $d_r = 0$.)

Thus, $E_{r+1}^{p,q} = E_r^{p,q}$, also $E_{r+k}^{p,q} = E_r^{p,q}$, so, we can define $E_\infty^{p,q} = E_{r+1}^{p,q}$.

3.2. Filtration and convergence.

Definition 3.3. A (decreasing) **filtration** $\mathscr{F}^*$ on a graded module H is a family of submodules $\{\mathscr{F}^p H\}$, such that

$$H \supset \cdots \supset \mathscr{F}^{p-1}H \supset \mathscr{F}^p H \supset \mathscr{F}^{p+1}H \supset \cdots$$

Definition 3.4. If there is a (decreasing) filtration $\mathscr{F}^*$ on H^* such that

$$E_\infty^{p,q} \cong \mathscr{F}^p H^{p+q} / \mathscr{F}^{p+1} H^{p+q}, \tag{1}$$

the spectral sequence $\{E_r^{*,*}, d_r\}$ ***converges*** H^*, denoted as

$$E_r^{p,q} \Rightarrow H^{p+q}$$

A filtration $\mathscr{F}^*$ on graded R-module H^* means that a filtration on each H^n,

$$\mathscr{F}^p H^n = \mathscr{F}^p H^* \cap H^n.$$

How to get information from a spectral sequence converging to H^*? We can see that in the following example.

Example 3.5. Suppose the filtration of H^* is bounded above and below. That is

$$H^* = \mathscr{F}^{-1}H^* \supset \mathscr{F}^0 H^* \supset \mathscr{F}^1 H^* \supset \cdots \supset \mathscr{F}^{n-1}H^* \supset \mathscr{F}^n H^* \supset \mathscr{F}^{n+1}H^* = 0$$

By formula (1), we have a series of short exact sequence

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathscr{F}^n H^{p+q} \arrow[r] \arrow[dl, equal] & E_\infty^{n,p+q-n} \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{F}^n H^{p+q} \arrow[r] \arrow[d, dotted, no head] & \mathscr{F}^{n-1} H^{p+q} \arrow[r] \arrow[d, dotted, no head] & E_\infty^{n-1,p+q-n+1} \arrow[r] \arrow[d, dotted, no head] & 0 \\
0 \arrow[r] & \mathscr{F}^k H^{p+q} \arrow[r] \arrow[d, dotted, no head] & \mathscr{F}^{k-1} H^{p+q} \arrow[r] \arrow[d, dotted, no head] & E_\infty^{k-1,p+q-k+1} \arrow[r] \arrow[d, dotted, no head] & 0 \\
0 \arrow[r] & \mathscr{F}^1 H^{p+q} \arrow[r] & \mathscr{F}^0 H^{p+q} \arrow[r] \arrow[dl, equal] & E_\infty^{0,p+q} \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{F}^0 H^{p+q} \arrow[r] & H^{p+q} \arrow[r] & E_\infty^{-1,p+q+1} \arrow[r] & 0
\end{tikzcd}$$

If H^* is vector space (over $\mathbb{Q}$), then

$$\boxed{H^{p+q} \cong \bigoplus_{i+j=p+q} E_\infty^{i,j}}$$

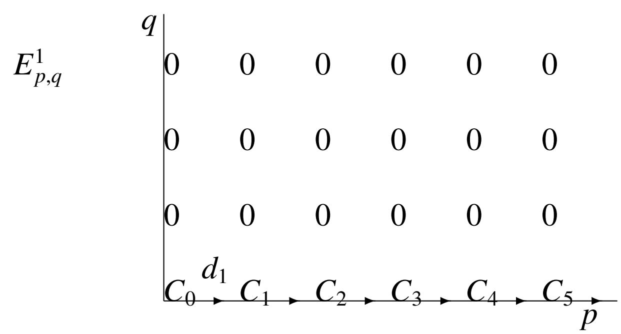
Example 3.6 (Cellular homology)**.** Let X be a CW-complex. Then X has a natural filtration by the p-skeleton X^p. There is a spectral sequence, $\{E^r_{*,*}, d_r\}$ converging to $H_*(X, d)$, with

$$E^1_{p,q} \cong H_{p+q}(X^p, X^{p-1}).$$

Recall that

$$H_{p+q}(X^p, X^{p-1}) \cong \begin{cases} C^{cell}_p(X), & q = 0, \\ 0, & q \neq 0. \end{cases}$$

So, the $E^1_{*,*}$ is

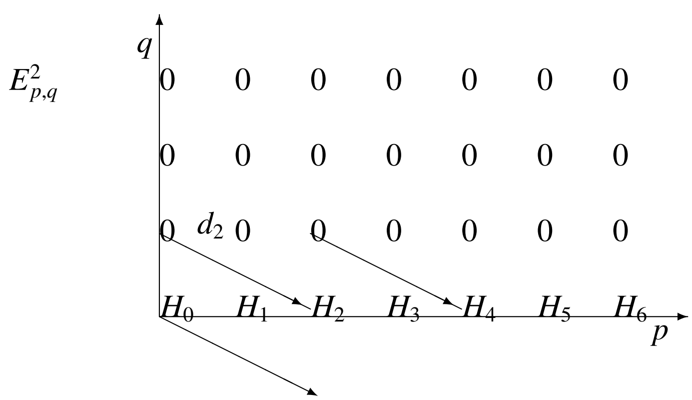


And the first differential $d_1 : E_{p,0} \to E^1_{p-1,0}$ is the same as the cellular differential

$$d : C^{cell}_p(X) = H_p(X^p, X^{p-1}) \to C^{cell}_{p-1}(X) = H_{p-1}(X^{p-1}, X^{p-2})$$

Therefore, E^2 is given in terms of the cellular homology by

$$E^2_{p,q} \cong \begin{cases} H^{cell}_p(X), & q = 0, \\ 0, & q \neq 0. \end{cases}$$



So, $d_r = 0$ for $r \geq 2$. Then $E^r_{p,q} = E^2_{p,q}$ for $r \geq 2$. So, $H_p(X) = H^{cell}_p(X)$.

3.3. **Leray-Serre spectral sequence.** Suppose $F \hookrightarrow X \xrightarrow{\pi} B$ is a fibration, where B is a path-connected CW-complex.

HW: All fibers $F_b = \pi^{-1}(b)$ are homotopy equivalent to a fixed fiber.

In particular, a loop γ at a basepoint of B gives homotopy equivalence $L_\gamma : F \to F$ for F the fiber over the base point. The association $\gamma \to L_{\gamma *}$ defines an action of $\pi_1(B)$ on $H_*(F)$. We may assume that this action is trivial in the following theorem, meaning that $L_{\gamma *}$ is the identity for all loops γ. Then the fibration is called **orientable**. Consider the cohomology with coefficients in a ring R.

Theorem 3.7 (the cohomology Leray-Serre spectral sequence)**.** *Suppose $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibration, where B is path-connected. If $\pi_1(B)$ acts trivially on $H^*(F; R)$, then there is a spectral sequence $\{E_r^{p,q}, d_r\} \Rightarrow H^*(E)$, with*

$$E_2^{p,q} \cong H^p(B; H^q(F; R)).$$

This spectral is natural with respect to fibre-preserving maps of fibrations.

Proposition: The Leray-Serre spectral sequence for cohomology can be provided with bilinear products $E_r^{p,q} \times E_r^{s,t} \xrightarrow{\star_r} E_r^{p+s,q+t}$ satisfying

(1) $d_r(x \star_r y) = (d_r x) \star_r y + (-1)^{p+q} x \star_r (d_r y)$ for $x \in E_r^{p,q}$ and $y \in E_r^{s,t}$.
(2) $x \star_2 y = (-1)^{qs} x \cup y$, for $x \in E_2^{p,q}$ and $y \in E_2^{s,t}$, where the coefficients are multiplied via the cup product $H^q(F; R) \times H^t(F; R) \to H^{q+t}(F; R)$.
(3) The cup product in $H^*(X; R)$ restricts to maps $F^p H^m \times F^s H^n \to F^{p+s} H^{m+n}$. These induce quotient maps
$$F^p H^m / F^{p+1} H^m \times F^s H^n / F^{s+1} H^n \to F^{p+s} H^{m+n} / F^{p+s+1} H^{m+n}.$$
that coincide with the products $E_\infty^{p,m-p} \times E_\infty^{s,n-s} \to E_\infty^{p+s,m+n-p-s}$.

3.4. **Application of Leray-Serre spectral sequence.** If either the fiber F or the base B is a n-sphere S^n, then we can have some nice results by the Leray-Serre spectral sequence.

Example 3.8. An orientable fibration $F \hookrightarrow E \xrightarrow{\pi} B$ with fiber $F = S^n$ an n-sphere, $n > 0$, is called spherical fibration. Then

$$H^q(F; R) = \begin{cases} R, & \text{if } q = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, in Leray-Serre spectral sequence,

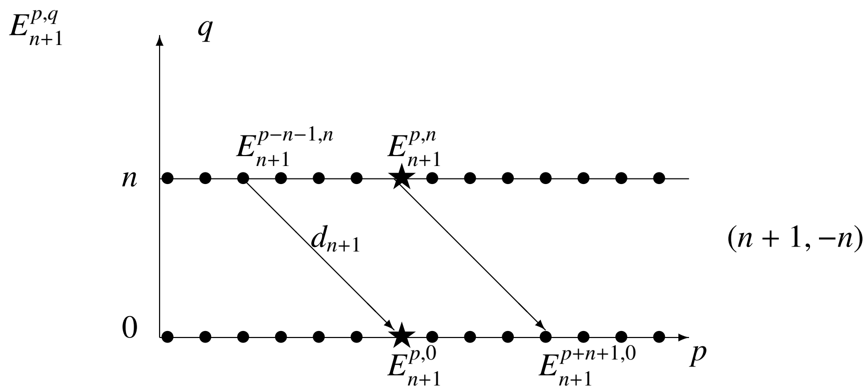
$$E_2^{p,q} = H^p(B; H^q(F; R)) \begin{cases} H^p(B; R), & \text{if } q = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

The bidegree of d_r is $(r, 1-r)$, so the only non-trivial differential is d_{n+1}. Hence,

$$E_2^{p,q} = E_3^{p,q} = \cdots = E_n^{p,q} = E_{n+1}^{p,q}$$

and

$$E_{n+2}^{p,q} = E_{n+3}^{p,q} = \cdots = E_\infty^{p,q}$$



Clearly,

$$E_{n+2}^{p,n} = \ker[d_{n+1} : E_{n+1}^{p,n} \to E_{n+1}^{p+n+1,0}]$$
$$E_{n+2}^{p,0} = \operatorname{coker}[d_{n+1} : E_{n+1}^{p-n-1,n} \to E_{n+1}^{p,0}].$$

So, we have exact sequence

$$0 \to E_{n+2}^{p,n} \to E_{n+1}^{p,n} \xrightarrow{d_{n+1}} E_{n+1}^{p+n+1,0} \to E_{n+2}^{p+n+1,0} \to 0.$$

That is

$$0 \to E_\infty^{p,n} \to E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} \to E_\infty^{p+n+1,0} \to 0. \tag{2}$$

The only nontrivial E_∞-term are $E_\infty^{p,0}$ and $E_\infty^{p,n}$ for $p \geq 0$. And,

$$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}.$$

Consider $H^m(E; R)$, we have the only two nontrivial E_∞-term

$$\begin{cases} E_\infty^{m,0} = F^m H^m / F^{m+1} H^m \\ E_\infty^{m-n,n} = F^{m-n} H^m / F^{m-n+1} H^m \end{cases}$$

Then the filtration on $H^m(E; R)$ have the form

$$H^m = \cdots = F^{m-n} H^m \supset F^{m-n+1} H^m = \cdots = F^m H^m \supset F^{m+1} H^m = \cdots = \{0\}.$$

Then we have

$$\begin{cases} E_\infty^{m,0} = F^m H^m \\ E_\infty^{m-n,n} = H^m / F^{m-n+1} H^m \end{cases}$$

This yields a short exact sequence

$$0 \to E_\infty^{m,0} \to H^m(E;R) \to E_\infty^{m-n,n} \to 0. \tag{3}$$

Gluing exact sequence (2) and (3), and recalling that

$$E_2^{p,0} = E_2^{p,n} = H^p(B;R),$$

We get a long exact sequence

$$\boxed{\cdots \to H^m(E;R) \xrightarrow{\phi} H^{m-n}(B;R) \xrightarrow{d_{n+1}} H^{m+1}(B;R) \xrightarrow{\pi^*} H^{m+1}(E;R) \to \cdots,} \tag{4}$$

which is called **Gysin sequence** of a spherical fibration $\pi : E \to B$. (In fact, for $n = 0$, Gysin sequence is also true, but it need another proof.)

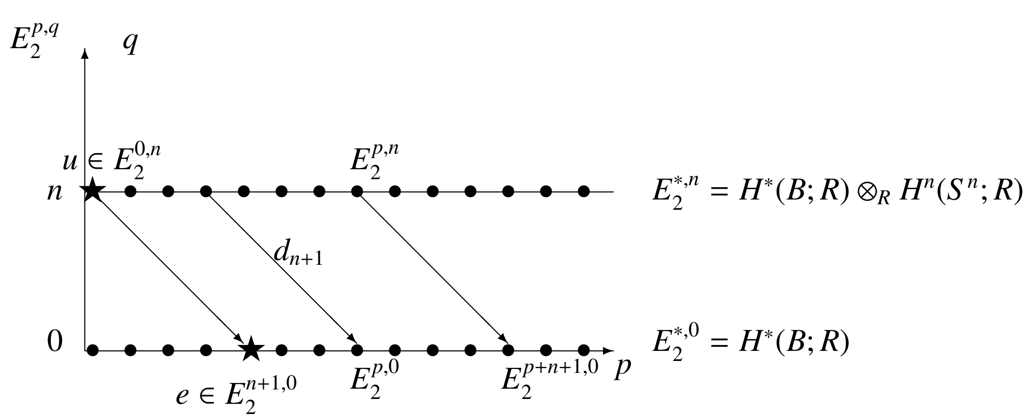
The following formula explain how the gluing works

$$\begin{array}{llllllll} 0 \to E_\infty^{m,0} \to & H^m(E;R) \to & E_\infty^{m-n,n} \to 0 & & & & \\ & & 0 \to & E_\infty^{m-n,n} \to E_2^{m-n,n} \xrightarrow{d_{n+1}} E_2^{m+1,0} \to E_\infty^{m+1,0} \to 0. \end{array}$$

The map π^* need some argument. We are more interested in map d_{n+1}, which can be described in another useful way.

$$E_2^{p,n} = H^p(B;H^0(S^n;R)) \cong H^p(B;R) \otimes_R H^n(S^n;R)$$

$$E_2^{p,0} = H^p(B;H^0(S^n;R)) \cong H^p(B;R)$$



Let $u \in H^n(S^n; R)$ be a generator, we can also regard u as lying in $E_2^{0,n} = H^0(B; H^n(S^n; R)) \cong H^n(S^n; R)$. Let

$$e = d_{n+1}u \in E_2^{n+1,0} = H^{n+1}(B; H^0(S^n; R)) \cong H^{n+1}(B; R).$$

e is called the **Euler class** of the spherical fibration.

Claim:

$$d_{n+1} : H^{m-n}(B; R) \to H^{m+1}(B; R)$$

is the cup-product with the class $x \to e \cup x$.

Example 3.9. The cohomology ring of $\mathbb{C}P^n$ and $\mathbb{C}P^\infty$ by Gysin sequence. Consider Hopf fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$ for every $n \geq 1$. The fiberation is orientable because $\pi_1(\mathbb{C}P^n, *) = 0$. The Euler class $e \in H^2(\mathbb{C}P^n)$ and for $0 < m < 2n$, Gysin sequence 4 becomes

$$0 \to H^{m-1}(\mathbb{C}P^n; R) \xrightarrow{\cup e} H^{m+1}(\mathbb{C}P^n; R) \to 0.$$

We know that $H^1(\mathbb{C}P^n; R) = 0$, and hence

$$H^{2n}(\mathbb{C}P^n; R) \cong H^{2n-2}(\mathbb{C}P^n; R) \cong \cdots \cong H^2(\mathbb{C}P^n; R) \cong H^0(\mathbb{C}P^n; R) \cong R;$$

$$H^{2n-1}(\mathbb{C}P^n; R) \cong H^{2n-3}(\mathbb{C}P^n; R) \cong \cdots \cong H^3(\mathbb{C}P^n; R) \cong H^1(\mathbb{C}P^n; R) \cong 0.$$

and $H^{2r}(\mathbb{C}P^n; R)$ is generated by e^r. Thus

$$H^*(\mathbb{C}P^n; R) \cong R[e]/(e^{n+1}),$$

a truncated polynomial algebra, where $e \in H^2(\mathbb{C}P^n; R)$.

Similarly $H^*(\mathbb{C}P^\infty; R) \cong R[e]$.

Example 3.10 (H.W.)**.** Compute the cohomology algebra of $\mathbb{C}P^n$ directly using Leray-Serre spectral sequence.

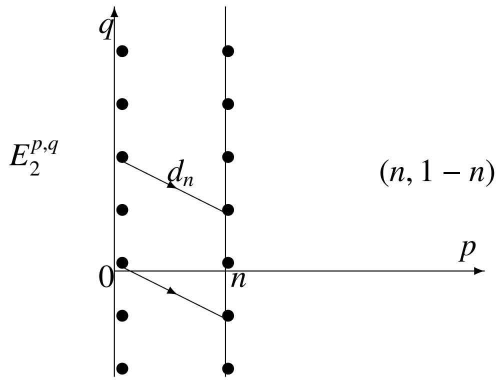
Example 3.11. Compute the cohomology algebra of $\mathbb{R}P^n$ with $\mathbb{Z}_2$ coefficient. (Use Gysin sequence)

Example 3.12. If the base B is a sphere S^n for $n > 0$ in an orientable fibration $F \hookrightarrow E \xrightarrow{\pi} B$, Then

$$H^p(B; R) = \begin{cases} R, & \text{if } p = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, in Leray-Serre spectral sequence,

$$E_2^{p,q} = H^p(B; H^q(F; R)) \begin{cases} H^q(F; R), & \text{if } p = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$



By the similar method (HW: practice) of getting the Gysin sequence, we can get **Wang Sequence**.(Not me.)

$$\boxed{\cdots \to H^m(E) \xrightarrow{i^*} H^m(F) \xrightarrow{d_n} H^{m-n+1}(F) \to H^{m+1}(E) \to \cdots .}$$

where $i : F \to E$ is the inclusion.

Example 3.13. Compute the **rational** cohomology algebra of Eilenberg-Maclane space $K(\mathbb{Z}, n)$.

We already know the results for the integer cohomology algebras of $K(\mathbb{Z}, 1) = S^1$ and $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$.

Consider the path fibration

$$P(K(\mathbb{Z}, 2k)) \to K(\mathbb{Z}, 2k)$$

with fiber the loop space $\Omega(K(\mathbb{Z}, 2k)) \simeq K(\mathbb{Z}, 2k-1)$.

Recall the Leray-Serre spectral sequence $\{E_r^{p,q}, d_r\}$ converges to $H^*(P(K(\mathbb{Z}, 2k)); \mathbb{Q})$.

Since $P(K(\mathbb{Z}, 2k))$ is contractible, we must have $H^i(P(K(\mathbb{Z}, 2k)); \mathbb{Q}) = 0$ if $i > 0$. Hence $E_\infty^{p,q} = 0$ if $(p, q) \neq (0, 0)$. (Here, we used the $\mathbb{Q}$ coefficient.)

The second page of the spectral sequence is

$$E_2^{p,q} \cong H^p(B; H^q(F; \mathbb{Q})) \cong H^p(B; \mathbb{Q}) \otimes H^q(F; \mathbb{Q}).$$

Here we used $\mathbb{Q}$ coefficient again in the universal coefficient theorem.

HW. Compute the **rational** cohomology algebra of Eilenberg-Maclane space $K(\mathbb{Z}, 3)$ and $K(\mathbb{Z}, 4)$.

4. Postnikov towers and rational homotopy theory

4.1. **Postnikov tower.** A *Postnikov tower* for a simply-connected space X is a tower of fibrations

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & X_n \arrow[d] & K(\pi_n(X), n) \arrow[l] \\
 & \vdots \arrow[d] & \\
 & X_4 \arrow[d] & K(\pi_4(X), 4) \arrow[l] \\
 & X_3 \arrow[d] & K(\pi_3(X), 3) \arrow[l] \\
X \arrow[r, "f_2"] \arrow[ur, "f_3"] \arrow[uur, "f_4"] \arrow[uuuur, "f_n"] & X_2 \arrow[r, equal] & K(\pi_2(X), 2)
\end{tikzcd}$$

such that

(1) $\pi_j(X_i) = 0$ for all $j > i$.
(2) The map $X_i \to X_{i-1}$ is a principal fibration with fiber $K(\pi_i(X), i)$ induced by some map $k^{i+1} \colon X_{i-1} \to K(\pi_i(X), i+1)$.
(3) $(f_i)_* \colon \pi_j(X) \to \pi_j(X_i)$ is an isomorphism for $j \leq i$.

A *principal fibration* $K(\pi_i(X), i) \to X_i \to X_{i-1}$ is fibration pulled back from a path loop fibration:

$$\begin{tikzcd}
K(\pi_i(X), i) \arrow[r, "\simeq"] \arrow[d] & K(\pi_i(X), i) \arrow[d] \\
X_i \arrow[r] \arrow[d] & P(K(\pi_i(X), i+1)) \arrow[d] \\
X_{i-1} \arrow[r, "k^{i+1}"] & K(\pi_i(X), i+1)
\end{tikzcd}$$

Proposition: A fibration $K(A, i) \to E \to B$ is a principal fibration if and only if $\pi_1(B)$ acts trivially on the fiber. (We can use Leray-Serre spectral sequence on it.)

Recall that

$$[X_{i-1}, K(\pi_i(X), i+1)] \cong H^{i+1}(X_{i-1}; \pi_i(X)).$$

The map k^{i+1} represents a class

$$[k^{i+1}] \in H^{i+1}(X_{i-1}; \pi_i(X))$$

which is called the $(i+1)$-st *k-invariant.*

One can recover X (up to homotopy equivalence) from a Postnikov tower $\{X_n\}$ as $\varprojlim X_n$, which is a subspace of $\prod X_n$. (In particular, the space X is homotopy equivalent to the product $\prod_{i\geq 2} K(\pi_i(X), i)$ if and only if all k-invariants are trivial.)

Obstruction theory shows that for any simply-connected CW-complex X, there is a Postnikov tower of X, and it is unique up to homotopy.

Let's look at the first obstruction, and the others are done by induction.

First, let $X_2 = K(\pi_2(X), 2)$. Consider the pair of CW complexes (X_2, X).

$$0 \to \pi_4(X_2, X) \to \pi_3(X) \xrightarrow{i_*} \pi_3(X_2) = 0 \to \pi_3(X_2, X) \xrightarrow{\partial} \pi_2(X) \xrightarrow{\cong} \pi_2(X_2) \xrightarrow{p_*} \pi_2(X_2, X) = 0 \to 0$$

So, $\pi_3(X_2, X) = 0$ and $\pi_4(X_2, X) \cong \pi_3(X)$.

By the (relative) Hurewicz Theorem $H_2(X_2, X) = H_3(X_2, X) = 0$ and

$$H_4(X_2, X) \cong \pi_4(X_2, X) \cong \pi_3(X).$$

By the universal coefficient theorem,

$$H^4(X_2, X; \pi_3(X)) \cong \mathrm{Hom}(\pi_3(X), \pi_3(X))$$

The class corresponding to the identity homomorphism $\mathrm{id} : \pi_3(X) \to \pi_3(X)$ determines a principal fibration $X_3 \to X_2$ and maps $f_3 : X \to X_3$ satisfying the Postnikov tower condition.

Example: In particular, suppose X is a CW complex of **dimension** 2.

$$H_4(X) \to H_4(X_2) \to H_4(X_2, X) \to H_3(X) = 0 \xrightarrow{i_*} H_3(X_2) \to H_3(X_2, X) = 0 \xrightarrow{\partial} H_2(X) \xrightarrow{\cong} H_2(X_2) \xrightarrow{p_*} H_2(X_2, X)$$

Then, $H_4(X_2, X) \cong H_4(X_2)$. Recall that $H_4(X_2, X) \cong \pi_3(X)$. So, $H_4(X_2) \cong \pi_3(X)$.

$$\begin{tikzcd}
K(\pi_3(X),3) \arrow[r, "\simeq"] \arrow[d] & K(\pi_3(X),3) \arrow[d] \\
X_3 \arrow[r] \arrow[d] & P(K(\pi_3(X),4)) \arrow[d] \\
X_2 \arrow[r, "k^4"] & K(\pi_3(X),4)
\end{tikzcd}$$

The k-invariant k^4 is the identity map in $H^4(X_2; \pi_3(X)) \cong \mathrm{Hom}(\pi_3(X), \pi_3(X))$.

More generally,

$$H_{i+2}(X_i) \cong \pi_{i+1}(X).$$

Example. Consider the case $X = S^2$ then $X_2 = K(Z, 2) = \mathbb{C}P^\infty$. Hence,

$$\pi_3(S^2) = H_4(X_2) = H_4(\mathbb{C}P^\infty) = \mathbb{Z}.$$

Apply the Leray-Serre spectral sequence and universal coefficient theorem for

$$K(\mathbb{Z}, 3) \to X_3 \to K(\mathbb{Z}, 2).$$

One has $H_5(X_3) = \mathbb{Z}/2$ (H.W.). Hence, $\pi_4(S^2) = \mathbb{Z}/2$.

Example. (H.W.) Suppose X is a CW complex of **dimension** n. Do the same computation as above.

4.2. **Rational homotopy theory (for simply connected spaces).**

Definition. (Thm) A map $f\colon X \to Y$ between simply-connected spaces is called a *rational homotopy equivalence*,(denoted by $X \xrightarrow{\simeq_{\mathbb{Q}}} Y$), if it satisfies the following equivalent conditions

(1) The induced map $f_*\colon \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism.
(2) The induced map $f_*\colon H_*(X;\mathbb{Q}) \to H_*(Y;\mathbb{Q})$ is an isomorphism.

Definition. Two simply connected spaces X and Y have the same *rational homotopy type*, (or are rationally homotopy equivalent), (denoted by $X \simeq_{\mathbb{Q}} Y$), if there is a zig-zag of rational homotopy equivalences connecting X and Y.

Definition.(Thm) A simply-connected space X is called a $\mathbb{Q}$-*space* if it satisfies:

(1) X is homotopy equivalent to a CW-complex.
(2) $\pi_i(X)$ is a $\mathbb{Q}$ vector space for all $i \geq 2$. Or equivalently
(2') $H_i(X)$ is a $\mathbb{Q}$ vector space for all $i \geq 2$.

Definition. A *rationalization*, (or $\mathbb{Q}$-localization, or 0-localization) of a simply connected space X is a $\mathbb{Q}$-space $X_{(0)}$ together with a rational homotopy equivalence $r : X \to X_{(0)}$.

Theorem A rationalization $r : X \to X_{(0)}$ is *universal* for maps of X into $\mathbb{Q}$-spaces.

Every rational homotopy equivalence between $\mathbb{Q}$- CW complexes is a homotopy equivalence.

The construction of the localization of a space goes by induction on the Postnikov tower of the space. The idea is to tensor both the groups and the k-invariants with $\mathbb{Q}$.

$K(\mathbb{Q}, n)$ is the 0-localization for $K(\mathbb{Z}, n)$.

Example: Localization for S^{2n-1}.

Since

$$H^*(K(\mathbb{Q}, 2n-1)) = \begin{cases} \mathbb{Q} & \text{if } * = 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

Hence the map $S^{2n-1} \to K(\mathbb{Q}, 2n-1)$ is a 0-localization for S^{2n-1}. Hence

$$\pi_k(S^{2n-1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

Example: Localization for S^{2n}.

Since $H^*(K(\mathbb{Q}, 2n)) = \mathbb{Q}[\alpha]$ with $|\alpha| = 2n$, the map $S^{2n} \to K(\mathbb{Q}, 2n)$ induces isomorphisms on rational cohomology up to degree $4n-1$.

Consider the principle fibration $K(\mathbb{Q}, 4n-1) \to E \to K(\mathbb{Q}, 2n)$ with k-invariant α^2. Using Leray-Serre spectral sequence, one can show that $S^{2n} \to E$ is a 0-localization for S^{2n}. (This shows that this fibration gives the rational Postnikov tower of S^{2n}.) Hence

$$\pi_k(S^{2n}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2n \text{ or } 4n-1 \\ 0 & \text{otherwise} \end{cases}$$

5. Commutative differential graded algebras

Does not have to be simply-connected.

5.1. **Definition.** A *commutative differential graded algebra* (for short, a CDGA) over $\mathbb{Q}$ is a graded $\mathbb{k}$-algebra $A^\bullet = \bigoplus_{n\geq 0} A^n$ equipped with a differential $d\colon A \to A$ of degree 1 satisfying

(1) $ab = (-1)^{mn} ba$,
(2) $d(ab) = d(a)\cdot b + (-1)^{|a|} a \cdot d(b)$ for any $a \in A^m$ and $b \in A^n$.

A CDGA A is said to be connected if $H^0(A) = \mathbb{Q}$. A CDGA A is said to be simply-connected if $H^1(A) = 0$.

Examples.

(1) Let M be a connected smooth manifold. The de Rham algebra of forms $A_{DR}(M)$ a CDGA.
(2) A commutative graded algebra (e.g., cohomology algebra) with zero differential is a CDGA.
(3) $\wedge(a, b, c)$ with $d(a) = d(b) = 0$ and $dc = ab$.
(4) Singular cohain algebra $C^*(X; \mathbb{Q})$ is NOT commutative, hence not a CDGA.

A morphism $f\colon A^\bullet \to B^\bullet$ between two CDGAS is a degree zero algebra map which commutes with the differentials, i.e., $fd_A = d_B f$.

A CDGA map $f\colon A \to B$ is said to be a *quasi-isomorphism* if all the induced maps in cohomology, $H^j(f)\colon H^j(A) \to H^j(B)$, are isomorphisms.

Two CDGAS A and B are *weakly equivalent* (written $A \simeq B$) if there is a zig-zag of quasi-isomorphisms connecting them.

$$A \to C_1 \leftarrow C_2 \to \cdots \leftarrow C_s \to B.$$

A CDGA $(A^\bullet, d)$ over $\mathbb{Q}$ is said to be *formal* if $(A^\bullet, d)$ is weakly equivalent to it cohomology $(H^\bullet(A), d = 0)$.

Example. $A = \wedge(x, y)$, $|x| = 2$, $|y| = 3$, $dx = 0$ and $dy = x^2$. Show that it is formal.

5.2. **From spaces to CDGAS: construction of $A_{PL}(X)$.** The standard n-simplex Δ^n is the convex hull of the standard basis $e_0, e_1, \cdots, e_n$ in $\mathbb{R}^{n+1}$:

$$\Delta^n = \left\{ (t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \middle| \sum_{i=0}^{n} t_i = 1, t_i \geq 0, j = 0, \cdots, n \right\}.$$

Denote the set of singular n-simplices on X(i.e. continuous maps from $\Delta^n \to X$) by $S_n(X)$. The sets $S_n(X)$ constitute a simplicial set whose boundary(face) operator ∂_i and degeneracy operators s_j are defined by:

$$\begin{aligned} \partial_i : \quad & S_n(X) \to S_{n-1}(X) \\ & \partial_i(\sigma)(t_0, \cdots, t_{n-1}) = \sigma(t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}) \\ s_j : \quad & S_n(X) \to S_{n+1}(X) \\ & s_j(\sigma)(t_0, \cdots, t_{n+1}) = \sigma(t_0, \cdots, t_j + t_{j+1}, \cdots, t_{n+1}) \end{aligned}$$

The simplicial CDGA A_{PL} is defined by

$$(A_{PL})_n = \frac{\wedge(t_0, \cdots, t_n, dt_0, \cdots, dt_n)}{(\sum t_i - 1, \sum dt_i)} = \frac{\mathbb{Q}[t_0, \cdots, t_n] \otimes \wedge(dt_0, \cdots, dt_n)}{(\sum t_i - 1, \sum dt_i)},$$

where the element t_i are in degree 0, the dt_i are in degree 1, and the differential d is defined by $d(t_i) = dt_i$.

This is an acyclic CDGA. that can be viewed as an algebra of polynomial $\mathbb{Q}$-forms on Δ^n. The face and degeneracy operator of the simplicial CDGA A_{PL} are the morphisms of CDGA's defined by

$$\begin{aligned} \partial_i : \quad & (A_{PL})_n \to (A_{PL})_{n-1} \\ & \partial_i(t_k) = \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases} \\ s_j : \quad & (A_{PL})_n \to (A_{PL})_{n+1} \\ & s_j(t_k) = \begin{cases} t_k, & k < j \\ t_k + t_{k+1}, & k = j \\ t_{k+1}, & k > j. \end{cases} \end{aligned}$$

The CDGA $A_{PL}(X)$ is then defined as a set of simplicial maps

$$A_{PL}(X) = \mathrm{Hom}_{Simplicial}(S_*(X), (A_{PL})_*).$$

$A_{PL}(-)$ is a contravariant functor from the category of topological spaces to the category of CDGAS.

The graded algebras $H^*(X; \mathbb{Q})$ and $H^*(A_{PL}(X))$ are naturally isomorphic.

Definition. A CDGA (A, d) is called a **model** for X if (A, d) is weak-equivalent to $A_{PL}(X)$.

A space X is called **formal** if $A_{PL}(X)$ is formal.

5.3. Triple Massey Products.

Let $(A, d, \mu = \cdot)$ be a dg-algebra over a field $\mathbb{k}$ of characteristic zero.

Let $a_i \in A^{r_i}$ for $i = 1, 2, 3$ such that

$$[a_1] \cdot [a_2] = 0 \quad \text{and} \quad [a_2] \cdot [a_3] = 0.$$

Thus, there are $a_{12} \in A^{r_1+r_2-1}$ and $a_{23} \in A^{r_2+r_3-1}$ such that

$$da_{12} = \bar{a}_1 \cdot a_2 \quad \text{and} \quad da_{23} = \bar{a}_2 \cdot a_3.$$

Here we denote $\overline{x} := (-1)^{1+i}x$ for $x \in A^i$.

Define an element $\omega \in A^{r_1+r_2+r_3-1}$ by

$$\omega = \overline{a}_1 \cdot a_{23} + \overline{a}_{12} \cdot a_3.$$

Claim: ω is a cocycle, i.e., $d(\omega) = 0$.

Proof.

$$\begin{aligned} d(\omega) &= (-1)^{r_1}\bar{a}_1 \cdot da_{23} + (-1)^{r_1+r_2} da_{12} \cdot a_3 \\ &= (-1)^{r_1+r_2+1}\bar{a}_1 \cdot a_2 \cdot a_3 + (-1)^{r_1+r_2}\bar{a}_1 \cdot a_2 \cdot a_3 \\ &= 0. \end{aligned}$$

□

Definition: The ***triple Massey product*** of $[a_1], [a_2]$ and $[a_3]$ is defined by

$$\langle [a_1], [a_2], [a_3] \rangle := \{\ [\omega] \text{ obtained by the above procedure }\}$$

This is well-defined because of the following proposition.

Proposition 5.1. *The triple Massey product $\langle u_1, u_2, u_3 \rangle$ depends only on the cohomology classes of the cocycles a_1, a_2, a_3.*

Proof. We need to check that $\langle [a_1], [a_2], [a_3] \rangle = \langle [a_1 + d(x)], [a_2], [a_3] \rangle$ for $x \in A^{r_1-1}$. Define $a'_{12} = a_{12} - \bar{x}a_2 \in A^{r_1+r_2-1}$ and $a_{23} \in A^{r_2+r_3-1}$ such that

$$da'_{12} = \bar{a}_1 \cdot a_2 + (d\bar{x})a_2 = (\overline{a_1 + dx}) \cdot a_2 \quad \text{and} \quad da_{23} = \bar{a}_2 \cdot a_3.$$

Hence, $\omega' = \overline{a_1 + dx} \cdot a_{23} + \overline{a'_{12}} \cdot a_3 = ... = \omega - d(\bar{x} \cdot a_{23})$, which gives $[\omega'] = [\omega]$. □

Definition.: The triple Massey product $\langle u_1, u_2, u_3\rangle$ is called ***vanishing***(or trivial) if $0 \in \langle u_1, u_2, u_3\rangle$.

Theorem. If X is a formal space, then all triple Massey products of X are vanishing.

The subset

$$In\langle u_1, u_2, u_3\rangle := \{a - b \mid a, b \in \langle u_1, u_2, u_3\rangle\} \subseteq H^{r_1+r_2+r_3-1}(A)$$

is called the ***indeterminacy*** of triple Massey product $\langle u_1, u_2, u_3\rangle$.

For triple Massey product, the indeterminacy can be interpreted as

$$In\langle u_1, u_2, u_3\rangle = u_1 H^{r_2+r_3-1}(A) + H^{r_1+r_2-1}(A)u_3$$

Example 5.2. Let $A = \bigwedge V$ is an exterior algebra over $\Bbbk$-vector space V with basis $\{a, b, c\}$. The differential is defined by $d(a) = d(b) = 0$ and $d(c) = a \wedge b$.

$$\begin{cases} A^1 = \Bbbk^3 \text{ with basis } \{a, b, c\}; \\ A^2 = \Bbbk^3 \text{ with basis } \{a \wedge b, a \wedge c, b \wedge c\}; \\ A^3 = \Bbbk \;\text{ with basis } \{a \wedge b \wedge c\}. \end{cases}$$

The cohomology $H^*(A)$ is given by

$$\begin{cases} H^1(A) = \Bbbk^2 \text{ with basis } \{u_1 := a, u_2 := b\}; \\ H^2(A) = \Bbbk^2 \text{ with basis } \{v_1 := a \wedge c, v_2 := b \wedge c\}; \\ H^3(A) = \Bbbk \;\text{ with basis } \{w := a \wedge b \wedge c\}. \end{cases}$$

The nontrivial Massey products are given by

$$\begin{cases} \langle u_1, u_1, u_2\rangle = v_1 \\ \langle u_1, u_2, u_1\rangle = -2v_1 \\ \langle u_2, u_1, u_1\rangle = v_1 \\ \langle u_2, u_1, u_2\rangle = 2v_2 \\ \langle u_2, u_2, u_1\rangle = -v_2 \\ \langle u_1, u_2, u_2\rangle = -v_2 \end{cases}$$

Example 5.3 (Borromean ring)**.**

$H^1(X) = \Bbbk^3$ with basis $\{[a], [b], [c]\}$.
$H^2(X) = \Bbbk^2$ with basis $\{[u], [v]\}$.
All cup products are zero.
The triple Massey products $\langle [a], [b], [c]\rangle = [u]$ and $\langle [c], [a], [b]\rangle = [v]$.

6. Minimal models

6.1. Minimal algebras.

Definition. A *Hirsch extension* (of degree i) is a CDGA inclusion

$$\alpha\colon (A^\bullet, d_A) \hookrightarrow (A^\bullet \otimes \bigwedge(V^i), d),$$

where V^i is a $\mathbb{Q}$-vector space concentrated in degree i, while $\bigwedge(V)$ is the free graded-commutative algebra generated by V^i, and d sends V^i into A^{i+1}.

We say this is a *finite* Hirsch extension if $\dim V^i < \infty$.

Definition. A CDGA $(M^\bullet, d)$ is called *minimal* if $M^0 = \mathbb{Q}$, and the following conditions are satisfied:

(1) $M^\bullet = \bigcup_{j\geq 0} M_j^\bullet$, where $M_0 = \mathbb{Q}$, and M_j is a Hirsch extension of M_{j-1}, for all $j \geq 0$.
(2) The differential is *decomposable*, i.e., $dM^\bullet \subset M^+ \wedge M^+$, where $M^+ = \bigoplus_{i\geq 1} A^i$.

The first condition implies that $M^\bullet$ has an increasing, exhausting filtration by the sub-CDGAS $M_j^\bullet$;

$$\mathbb{Q} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_j \subset \ldots$$

In particular, $M^\bullet$ is free as a graded-commutative algebra on generators of degree ≥ 1.

$$M \cong \mathrm{Symmetric}(V^{even}) \otimes \mathrm{Exterior}(V^{odd})$$

(A CDGA is called *Sullivan algebra* if it only satisfies the first condition.)
(Note that we use the lower-index for the filtration, and the upper-index for the grading.)

Proposition. Suppose $(M^\bullet, d)$ is simply-connected, $M^1 = 0$. Let M_i be the subalgebra of M generated in degree $\leq n$. The first condition is equivalent to the condition that $M^\bullet$ is free as a graded-commutative algebra on generators of degree ≥ 2 and the differential d is decomposable.

Sketch of proof

If $(M^\bullet, d)$ is not simply-connected, we don't have the above theorem. For example $\wedge(x, y, z)$ with $|x| = |y| = |z| = 1$ and $d(x) = yz$, $d(y) = zx$ and $d(z) = yx$ is a free CDGA with decomposable differential. But It is not a minimal CDGA.

6.2. **Existence.**

A *minimal model* of a connected CDGA A is a minimal CDGA M together with a quasi-morphism $M \to A$.

A *n-minimal model* of CDGA A is a minimal CDGA M_n together with a quasi-morphism $\rho_n : M_n \to A$ such that

(1) M_n is minimal and generated by elements in degrees $\leq n$.
(2) ρ_n^* is an isomorphism on cohomology in degrees $\leq n$.
(3) ρ_n^* is injective on cohomology in degree $n+1$.

Theorem. Any simply-connected CDGA A has a minimal model $\wedge V \to A$. If $H^*(A)$ is of finite type, then V is of finite type.

Sketch of proof for simply-connected cases, i.e., $H^1(A) = 0$.

We need to construct a sequence of Hirsch extensions:

$$\mathbb{Q} = M_0 = M_1 \subset M_2 \subset \cdots \subset M_j \subset \ldots$$

together with maps

$$\rho_n \colon M_n \to A$$

such that

$$\rho_n|_{M_k} = \rho_k \text{ for } k \leq n$$

and $\rho_n \colon M_n \to A$ is a n-minimal model of A.

Then, we can define $M = \bigcup M_n$ and define map $\rho : M \to A$ by $\rho|_{M_n} = \rho_n$. This map is a quasi-isomorphism, since cohomology commutes with direct limits. Then $\rho : M \to A$ gives a minimal model of A.

Before the construction of the sequence of Hirsch extensions, we need to review the relative cohomology $H(C, D)$ of a CDGA $f : C \to D$. Define the mapping cylinder M_f as

$$M_f^n = C^n \oplus D^{n-1}$$

and the differential $d_M : M_f^n \to M_f^{n+1}$ if given by

$$\begin{bmatrix} d_C & 0 \\ f & -d_D \end{bmatrix}$$

Check: $d_M d_M = 0$.

The relative cohomology $H^*(C, D)$ is defined to be $H^*(C, D) := H^*(M_f)$. There is a long exact sequence:

$$\ldots \to H^i(C, D) \to H^i(C) \xrightarrow{f^*} H^i(D) \to H^{i+1}(C, D) \to H^{i+1}(C) \to \ldots$$

Let's go back to the constructions of the sequence of Hirsch extensions.

Step 1. Let $M_1 = \mathbb{Q}$ and $\rho_1 : M_1 \to A$ is the map sending 1 to 1. Clearly, this is a 1-minimal model of A.

Step 2. Let's see how to construct M_2. The others are similarly by induction.

By long exact sequence,

$$\cdots \to H^i(M_1, A) \to H^i(M_1) \xrightarrow{\rho_1^*} H^i(A) \to H^{i+1}(M_1, A) \to H^{i+1}(M_1) \to \ldots$$

the relative cohomology $H^i(M_1, A) = 0$ for $i \leq 2$. Let $V = H^3(M_1, A)$ and give degree of V to be 2 (we may denoted it as V^2). Now define M_2 as algebra

$$M_2 = M_1 \otimes \wedge(V^2).$$

We need to define the differential on $v \in V$ such that $dv \in M_1^3$ (and then extended it to M_2 by Leibnitz rule). We also need to extend ρ_1 to $\rho_2 : M_2 \to A$, such that $\rho_1(dv) = d\rho_2(v)$.

Consider the projection from cocycles to cohomology classes:

$$Z^3(M_1, A) \twoheadrightarrow H^3(M_1, A) = V$$

Choose a split for this projection as

$$s : V \to Z^3(M_1, A) \subset M_1^3 \oplus A^2.$$

Suppose $s(v) = (m_v, a_v)$, then we can define

$$d(v) = m_v \text{ and } \rho_2(v) = a_v.$$

Check $dd(v) = 0$ and $\rho_1(dv) = d\rho_2(v)$. Hence $\rho : M_2 \to A$ is a CDGA morphism.

Claim: $H^i(M_2, A) = 0$ for $i \leq 3$.

By long exact sequence and five lemma, we can show that $\rho_2 : M_2 \to A$ is a 2-minimal model.

$$\cdots \to H^i(M_2, A) \to H^i(M_2) \xrightarrow{\rho_2^*} H^i(A) \to H^{i+1}(M_2, A) \to H^{i+1}(A) \to \ldots$$

By induction, we can construct the desired sequence of Hirsch extensions.

6.3. **Uniqueness.**

The minimal model of a CDGA is unique up to isomorphism.

Definition. Two CDGA morphisms f and g from A to B are **homotopic** via a homotopy

$$H : A \to B \otimes \wedge(t, dt)$$

satisfying

$$H|_{t=0} = f \text{ and } H|_{t=1} = g.$$

The minimal algebra $\wedge(t, dt)$ is acyclic. It is isomorphic to the Sullivan algebra $A_{PL}(I)$ of the interval $I = [0, 1]$. Define the maps $p_i : \wedge(t, dt) \to \mathbb{Q}$ as $p_i(t) = i$ and $p_i(dt) = 0$. $H|_i := p_i \circ H$.

Lifting Lemma. Let $(\wedge V, d)$ be a minimal CDGA, $f : A \to B$ be a quasi-isomorphism of CDGA's, and $\rho : \wedge V \to B$ be a morphism of CDGAS. Then there is a morphism of CDGA's $\phi : \wedge V \to B$ and a homotopy H from $f\phi$ to ρ.

$$\begin{tikzcd} & A \arrow[d, "f"] \\ \wedge V \arrow[ur, dotted, "\phi"] \arrow[r, "\rho"'] & B \end{tikzcd}$$

Theorem. Given a CDGA A and two minimal models $\rho : M \to A$ and $\rho' : M' \to A$. Then, there is an isomorphism $I : M \to M'$ and a homotopy H from ρ to $\rho' I$.

$$\begin{tikzcd} M \arrow[dd, dotted, "\cong"', "I"] \arrow[dr, "\rho"] & \\ & A \\ M' \arrow[ur, "\rho'"'] & \end{tikzcd}$$

A quasi-isomorphism between minimal Sullivan algebras is an isomorphism.

On object level, a minimal model is uniquely determined. On map level, it is not unique.

Any model A of X admits the following quasi-isomorphisms:

$$A \xleftarrow{\simeq} M(X) \xrightarrow{\simeq} A_{PL}(X)$$

Theorem. Two simply-connected spaces X and Y of finite $\mathbb{Q}$ homotopy type are rationally homotopy equivalent if and only if their minimal models M_X and M_Y are isomorphic as CDGA's.

$$\left\{ \text{ rational homotopy types } \right\} \xleftrightarrow{\text{bijection}} \left\{ \text{ minimal algebras } \right\} / \cong$$

Theorem. Two simply-connected CDGA's A and B are weak-equivalent if and only if their minimal models M_A and M_B are isomorphic as CDGA's.

$$\left\{ \text{ CDGA's } \right\} /_{\simeq} \xleftrightarrow{\text{bijection}} \left\{ \text{ minimal algebras } \right\} /_{\cong}$$

6.4. **Duality between homotopy groups and minimal models.**

This section deal with **simply-connected** space (CDGA) with of **finite type**.

Let π be a finitely generated abelian group. Let $E \to B$ be a principal $K(\pi, n)$ fibration corresponding to the k-invariant $k^{n+1} \in H^{n+1}(B; \pi)$.

Apply the Sullivan's contravariant functor $A_{PL}(-)$, we have

$$\begin{array}{ccc} K(\pi,n) & & A^*_{PL}(B) \\ \downarrow & & \downarrow \\ E & \rightsquigarrow & A^*_{PL}(E) \\ \downarrow & & \downarrow \\ B & & A^*_{PL}(K(\pi,n)) \end{array}$$

In rational homotopy theory, let $V = \pi \otimes \mathbb{Q}$ be a vector space of finite dimension. We have the corresponding cohomology class

$$k^{n+1} \otimes 1 \in H^{n+1}(B; \pi) \otimes \mathbb{Q} = H^{n+1}(B; V)$$

Let $\rho : M(B) \to A_{PL}(B)$ be the minimal model of B. As in the construction, $M(B) = \bigcup M_i(B)$ where M_i is the i-minimal model of B.

Let $[u]$ be the corresponding element in

$$H^{n+1}(M(B); V) = \mathrm{Hom}(H_{n+1}(M(B)), V) \cong \mathrm{Hom}(V^*, H^{n+1}(M(B))).$$

Hence $[u]$ determines a Hrisch extension [Lemma 10.1 in [1]]

$$M' := M(B) \otimes \wedge((V^*)^n)$$

There is a one to one corresponding between principle $K(V, n)$-fibration over B and Hirsch extensions $M(B) \otimes \wedge((V^*)^n)$. Moreover,

Theorem.(Hirsch extension $\leftrightarrow$ Principle $K(\pi, n)$ fibration)
There is a map $\rho' : M' \to A_{PL}(E)$ giving a minimal model for E and the following diagram commutes:

$$\begin{tikzcd} M(B) \arrow[r, "\rho"] \arrow[d] & A^*_{PL}(B) \arrow[d] \\ M' \arrow[r, "\rho'"] & A^*_{PL}(E) \end{tikzcd}$$

Apply this theorem to Postnikov tower of a simply-connected space X

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_3 \to X_2 = K(\pi_2, 2),$$

we have a sequence of Hirsch extensions

$$M(X_2) \to M(X_3) \to \cdots \to M(X_{n-1}) \to M(X_n) \to \cdots$$

such that $M := \bigcup M(X_n)$ is the minimal model for X.

In particular, $M(X_2) = \bigwedge((\pi_2 \otimes \mathbb{Q})^*)$ with zero differential, and

$$M(X_n) = M(X_{n-1}) \otimes \bigwedge((\pi_n \otimes \mathbb{Q})^*).$$

In particular, we have the dual relation between minimal model and rational homotopy groups.

Theorem.(minimal model and homotopy groups)
Let X be a simply-connected space with finite Betti numbers, and let $(\wedge V, d)$ be its minimal model. Then, for $n \geq 2$, we have a natural isomorphism

$$V^n \cong \mathrm{Hom}(\pi_n(X) \otimes \mathbb{Q}, \mathbb{Q}) = \mathrm{Hom}(\pi_n(X), \mathbb{Q}).$$

6.5. **Minimal model and Whitehead products.** We start with the general (non-simply-connected) case.

Let $M = \bigcup M_i$ be a union of sequence of Hirsch extensions, where $M_0 = \mathbb{Q}$ and $M_i(X)$ is the subalgebra of $M(X)$ generated in degree $\leq i$. Hence M is a free graded commutative algebra $M = \bigwedge(V)$ with $V = \oplus_{j \geq 1} V^j$.

$M = \bigwedge(V)$ can be decomposed by arity as

$$\bigwedge(V) = \wedge^0 V \oplus \wedge^1 V \oplus \wedge^2 V \oplus \wedge^3 V \oplus \cdots$$

The differential $d(\wedge^k V) \subseteq \wedge^{\geq k} V$ by Leibniz rule. The differential d can decomposed as

$$d = d_0 + d_1 + d_2 + \cdots$$

where $d_r(\wedge^k V) \subseteq \wedge^{k+r} V$. The relation $dd = 0$ is equivalent to the relations

$$\begin{aligned} d_0 d_0 &= 0 \\ d_0 d_1 + d_1 d_0 &= 0 \\ d_0 d_2 + d_1 d_1 + d_2 d_0 &= 0 \\ d_0 d_3 + d_2 d_1 + d_1 d_2 + d_3 d_0 &= 0 \\ &\vdots \end{aligned}$$

Claim: $M = \bigwedge(V)$ is minimal if and only if $d_0 = 0$, i.e., $d(V) \subseteq \wedge^{\geq 2} V$.

Hence, if M is a minimal algebra, then $d_1 d_1 = 0$.

Simply-connected case. Recall the minimal model of a simply-connected space X is given by $M(X) = \bigcup M_i(X)$, where $M_i(X)$ is the subalgebra of $M(X)$ generated in degree $\leq i$.

Let $L_n = \pi_{n+1}(X) \otimes \mathbb{Q}$. Recall that $L = \oplus_{n \geq 1} L_n$ is a graded Lie algebra with the Lie bracket induced by the Whitehead product

$$\pi_k(X) \otimes \pi_l(X) \xrightarrow{[-,-]} \pi_{k+l-1}(X)$$

Define a vector space D such that $sD = \mathrm{Hom}(V, \mathbb{Q})$, here $(sD)_k = D_{k-1}$. The dual of d_1 corresponds to the Lie bracket

$$\langle v; s[x, y] \rangle = (-1)^{|y+1|} \langle d_1 v; sx, sy \rangle$$

The relation $d_1 d_1 = 0$ corresponds to the Jacobi identity of the Lie brackets.

The graded Lie algebras L and D are isomorphic.

6.6. **Examples of minimal models of spaces.**

Example 6.1 ($K(\pi, n)$)**.**

For $n \geq 2$, the minimal model of $K(\pi, n)$ is $\wedge((\pi \otimes \mathbb{Q})^*)$ with zero differential.

Example 6.2 (The sphere S^n)**.**

The rational cohomology of the sphere S^n is an exterior algebra on one generator in degree n. Denote by ω a cocycle in degree n in $A_{PL}(S^n)$ representing the fundamental class. Then we get a morphism of CDGA's

$$\varphi : (\wedge(x), 0) \to A_{PL}(S^n)$$

defined by $\varphi(x) = \omega$.

When n is odd, $\wedge(x)$ is an exterior algebra on one generator and φ is a quasi-isomorphism.

$$\pi_k(S^{2m-1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2m-1 \\ 0 & \text{otherwise} \end{cases}$$

When n is even, $\wedge(x)$ is the polynomial algebra $\mathbb{Q}[x]$ and $H^*(\varphi) : \mathbb{Q}[x] \to \mathbb{Q}[w]/w^2$ is not an isomorphism. For degree reason ω^2 is then a coboundary, $d\alpha = \omega^2$. Then we add a new generator y to $\wedge(x)$ of degree $2n-1$ with $dy = x^2$, and define

$$\phi : (\wedge(x, y), d) \to A_{PL}(S^n)$$

by putting $\phi(x) = \omega$ and $\phi(y) = \alpha$. Since $H^*(\wedge(x, y), d) \cong \mathbb{Q}[x]/x^2$, then ϕ is a quasi-isomorphism. Hence

$$\pi_k(S^{2m}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2m \text{ or } 4m-1 \\ 0 & \text{otherwise} \end{cases}$$

Example 6.3 (The complex projective space $\mathbb{C}P(n)$)**.**

$H^*(\mathbb{C}P(n); \mathbb{Q}) \cong \mathbb{Q}[x]/x^{n+1}$ with $|x| = 2$. We can choose, in $A_{PL}(\mathbb{C}P(n))$, elements α and β of respective degree 2 and $2n+1$ such that the class of α is x and $d\beta = \alpha^{n+1}$. We can construct a morphism of CDGA's

$$\varphi : (\wedge(x, y), d) \to A_{PL}(\mathbb{C}P(n))$$

defined by $|x| = 2, |y| = 2n+1, dx = 0, dy = x^{n+1}, \varphi(x) = \alpha$ and $\varphi(y) = \beta$. This morphism is a quasi-isomorphism.

Example 6.4 (Product of manifolds)**.**

If X and Y are path connected spaces, then there is a quasi-isomorphism between $A_{PL}(X \times Y)$ and $A_{PL}(X) \otimes A_{PL}(Y)$. The minimal model of $X \times Y$ is the tensor product of the minimal models of X and Y.

Example 6.5 (The torus T^n)**.**

The minimal model of the torus T^n is the CDGA $(\wedge(x_1, \cdots, x_n), d = 0)$, where all the x_i have degree 1.

Example 6.6 (Wedge of two simply-connected spaces)**.** Let X and Y be spaces with minimal models $(\wedge V, d)$ and $(\wedge W, d)$. Then, a minimal model for $X \vee Y$ is obtained by taking a minimal model of $(\wedge V, d) \oplus_{\mathbb{Q}} (\wedge W, d)$.

HW. [Wedge of two circles $S^1 \wedge S^1$]

Example 6.7 (Formal spaces)**.** The minimal model of $(H, d = 0)$.

7. Fundamental groups

Example. $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$. It's rational cohomology algebra is $\tilde{H}^*(\mathbb{R}P^2;\mathbb{Q}) = 0$. It is a formal space. So, $\tilde{H}^*(\mathbb{R}P^2;\mathbb{Q}) = 0$ with zero differential is a cdga model for $\mathbb{R}P^2$.

The sphere S^2 is a covering space of $\mathbb{R}P^2$. Hence $\pi_n(\mathbb{R}P^2) = \pi_n(S^2)$ for $n \geq 2$. So, its rational homotopy type is not a point.

The minimal model of non-simply-connected spaces can not determine the rational homotopy type. However, it relates the important properties of the fundamental group.

7.1. **1-minimal model.** Recall that a 1-*minimal model* of cdga A is a minimal cdga M_1 together with a quasi-morphism $\rho : M_1 \to A$ such that

(1) M_1 is minimal and generated by elements in degree 1.
(2) ρ^* is an isomorphism on cohomology in degree 1.
(3) ρ^* is injective on cohomology in degree 2.

Theorem. Any connected cdga A has a 1-minimal model $M_1 \to A$. M_1 is unique up to isomorphism.

Consider the filtration

$$\Bbbk = M_{1,0} \subset M_{1,1} \subset M_{1,2} \subset \cdots \subset M = \bigcup_i M_{1,i},$$

where $M_{1,1}$ is the subalgebra of M_1 generated by $x \in M_1^1$ such that $dx = 0$, and $M_{1,i}$ is the subalgebra of M_1 generated by $x \in M_1^1$ such that $dx \in M_{1,i-1}$ for $i > 1$. Each inclusion $M_{1,i-1} \subset M_{1,i}$ is a Hirsch extension of the form $M_{1,i} = M_{1,i-1} \otimes \bigwedge(V_i)$. Taking the degree 1 part of the filtration (7.1), we obtain the filtration

$$\Bbbk = M_{1,0}^1 \subset M_{1,1}^1 \subset \cdots \subset M_1^1.$$

Now assume each of the above Hirsch extensions is finite, i.e., $\dim(V_i) < \infty$ for all i. Using the fact that $d(V_i) \subset M_{1,i-1}$, we see that each dual vector space $L_i = (M_{1,i}^1)^*$ acquires the structure of a $\Bbbk$-Lie algebra by setting

$$\langle [u^*, v^*], w\rangle = \langle u^* \wedge v^*, dw\rangle$$

for $u, v, w \in M_{1,i}^1$.

Using the vector space decompositions $M_{1,i}^1 = M_{1,i-1}^1 \oplus V_i$ and $M_{1,i}^2 = M_{1,i-1}^2 \oplus (M_{1,i-1}^1 \otimes V_i) \oplus \bigwedge^2(V_i)$ we easily see that the canonical projection $L_i \twoheadrightarrow L_{i-1}$ (i.e., the dual of the inclusion map $M_{1,i-1} \hookrightarrow M_i$) has kernel V_i^*, and this kernel is central inside L_i. Therefore, we obtain a tower of finite-dimensional nilpotent $\Bbbk$-Lie algebras,

$$0 \twoheadleftarrow L_1 \twoheadleftarrow L_2 \twoheadleftarrow \cdots \twoheadleftarrow L_i \twoheadleftarrow \cdots .$$

The inverse limit of this tower, $L = L(M_1)$, endowed with the inverse limit filtration, is a complete, filtered Lie algebra such that $L/\widehat{\Gamma}_{i+1}L = L_i$, for each $i \geq 1$. Conversely, from a tower as in (7.1), we can construct a sequence of finite Hirsch extensions $M_{1,i}$ as in (7.1).

Furthermore, the DGA $M_{1,i}$, with differential defined by (7.1), coincides with the Chevalley–Eilenberg complex $(\bigwedge(L_i^*), d)$ associated to the finite-dimensional Lie algebra $L_i = L(M_{1,i})$. In particular,

$$H^{\bullet}(M_{1,i}) \cong H^{\bullet}(L_i; \Bbbk) .$$

7.2. Malcev Lie algebra. $\cdots \to G/\Gamma_4 G \to G/\Gamma_3 G \to G/\Gamma_2 G = G_{\mathrm{ab}}$.

Let $\mathfrak{Lie}((G/\Gamma_k G) \otimes \Bbbk)$ be the Lie algebra of the nilpotent Lie group $(G/\Gamma_k G) \otimes \Bbbk$. The pronilpotent Lie algebra

$$\mathfrak{m}(G; \Bbbk) := \varprojlim_k \mathfrak{Lie}((G/\Gamma_k G) \otimes \Bbbk), \tag{5}$$

Sullivan and Cenkl–Porter proved that $\mathfrak{m}(G; \Bbbk) \cong L(G)$

Quillen proved that $\mathfrak{m}(G; \Bbbk) \cong \mathrm{Prim}\,(\widehat{\Bbbk G})$.

7.3. graded Lie algebra.

Let G be a finitely generated group, and let $\{\Gamma_k G\}_{k \geq 1}$ be its lower central series (LCS). The LCS quotients of G are finitely generated abelian groups. Taking the direct sum of these groups, we obtain a graded Lie ring over $\mathbb{Z}$,

$$\mathrm{gr}(G; \mathbb{Z}) = \bigoplus_{k \geq 1} \Gamma_k G/\Gamma_{k+1} G. \tag{6}$$

The Lie bracket $[x, y]$ on $\mathrm{gr}(G; \mathbb{Z})$ is induced from the group commutator, $[x, y] = xyx^{-1}y^{-1}$. More precisely, if $x \in \Gamma_r G$ and $y \in \Gamma_s G$, then $[x + \Gamma_{r+1} G, y + \Gamma_{s+1} G] = xyx^{-1}y^{-1} + \Gamma_{r+s+1} G$. The Lie algebra

$$\mathrm{gr}(G; \mathbb{Q}) = \mathrm{gr}(G; \mathbb{Z}) \otimes \mathbb{Q}$$

is called the *associated graded Lie algebra* (over $\mathbb{Q}$) of the group G.

Quillen proved that $\mathrm{gr}(G; \mathbb{Q}) \cong \mathrm{gr}(\mathfrak{m}(G; \mathbb{Q}))$

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