Northeastern University, Department of Mathematics

MATH G5110: Applied Linear Algebra and Matrix Analysis. (Fall 2020)

• Instructor: **He Wang** Email: **he.wang@northeastern.edu**

§7 Diagonalization; Eigenvalues; Eigenvectors;

Contents

1. **Diagonalization** 1
2. **Eigenvalues** 3
3. **Review of Polynomials** 5
4. **More on Characteristic Polynomials** 7
5. **Eigenspaces** 9
6. **Complex Eigenvalues and Eigenvectors** 11
7. **Examples** 12

1. Diagonalization

Let D be an diagonal matrix. The power D^k is easy to calculate. For example,

$$D^k = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}^k = \begin{bmatrix} (d_1)^k & 0 & 0 & 0 \\ 0 & (d_2)^k & 0 & 0 \\ 0 & 0 & (d_3)^k & 0 \\ 0 & 0 & 0 & (d_4)^k \end{bmatrix}$$

Definition 1. An $n \times n$ matrix A is said to be **diagonalizable** if it is similar to a diagonal matrix D, that is, if there exists an invertible matrix P such that $A = PDP^{-1}$.

Powers of a diagonalizable matrix A are also easy to calculate:

$$A^k = PD^kP^{-1}$$

We see that A^k is similar to the diagonal matrix D^k, and hence also **diagonalizable**.

Question:

1. Are all $n \times n$ matrices A diagonalizable?

2. If a matrix A is diagonalizable, how to find the invertible matrix P and the diagonal matrix D? The answer for this question is called **diagonalize** matrix A.

Solve $A = PDP^{-1}$. That is $AP = PD$. More explicitly (when $n = 3$)

$$A[\vec{b}_1\ \vec{b}_2\ \vec{b}_3] = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3] \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

That is $[A\vec{b}_1\ A\vec{b}_2\ A\vec{b}_3] = [d_1\vec{b}_1\ d_2\vec{b}_2\ d_3\vec{b}_3]$

So, equivalently, we need to find numbers d_1, d_2, d_3 and $\vec{b}_1, \vec{b}_2, \vec{b}_3$ satisfy $A\vec{b}_1 = d_1\vec{b}_1$, $A\vec{b}_2 = d_2\vec{b}_2$, $A\vec{b}_3 = d_3\vec{b}_3$. They are the same equation:

$$A\vec{x} = d\vec{x}$$

Remark: From the point of view of change of coordinates. Recall from §4 the meaning of similar matrices $A = PDP^{-1}$:

Let A be the matrix of a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$ and denote $P = [\vec{b}_1 \ldots \vec{b}_n]$ the change of coordinate matrix. The *matrix of T respect to basis $\mathscr{B}$* is

$$D = \left[[T(\vec{b}_1)]_{\mathscr{B}}\ [T(\vec{b}_2)]_{\mathscr{B}}\ \cdots\ [T(\vec{b}_n)]_{\mathscr{B}} \right]$$

Then, $A = PDP^{-1}$.

Example 2. Let T be the projection transformation onto a line $L = \text{Span}\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}\ \mathbb{R}^3$.

Find a basis $\mathscr{B} = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]$ for $\mathbb{R}^3$ such that the $\mathscr{B}$-matrix of the T is the diagonal matrix $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$.

Method of Solution:

Step 1. Compare the columns of D. It is equivalent to find independent vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ and numbers d_1, d_2, d_3 such that

$$T(\vec{b}_1) = d_1(\vec{b}_1), \quad T(\vec{b}_2) = d_2(\vec{b}_2), \quad T(\vec{b}_2) = d_2(\vec{b}_2)$$

Step 2. Use the geometric properties of the transformation to find those vectors and numbers.

We need to find vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ such that the projection $\text{proj}_L \vec{b}_i$ is the scalar product of $\vec{b}_i$.

Let $\vec{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Then, $A\vec{b}_1 = 1\vec{b}_1$. So, $d_1 = 1$.

Let $\vec{b}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. Then, $A\vec{b}_2 = \vec{0} = 0\vec{b}_2$. So, $d_2 = 0$.

Let $\vec{b}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$. Then, $A\vec{b}_3 = \vec{0} = 0\vec{b}_3$. So, $d_3 = 0$.

The key is to solve $T(\vec{x}) = \lambda\vec{x}$ or equivalently $A\vec{x} = \lambda\vec{x}$.

2. Eigenvalues and Eigenvectors.

Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by matrix $T\vec{x} = A\vec{x}$.

Definition 3.
- An **eigenvector** of A is a nonzero n-dimensional vector $\vec{x}$ such that
$$A\vec{x} = \lambda\vec{x}$$
for some (possibly complex) scalar λ.
- An **eigenvalue** of A is a (possibly complex) scalar λ for which there exists a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. We say that $\vec{x}$ is **an eigenvector corresponding to** λ.
- A basis $\vec{b}_1, \ldots, \vec{b}_n$ of $\mathbb{R}^n$ is called an **eigenbasis** for A if the vectors $\vec{b}_1, \ldots, \vec{b}_n$ are eigenvectors of A.

Example 4. $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Example 5. If $\vec{v}$ is an eigenvector of A corresponding to λ, is $\vec{v}$ an eigenvector of A^k?

Theorem 6. *A is diagonalizable if and only if it has n linearly independent eigenvectors $\vec{b}_1, \ldots, \vec{b}_n$ (eigenbasis).*
In this case $A = PDP^{-1}$ where the columns of P are eigenvectors of A; the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors given by the columns of P.

Proof. We already verified that system of equations $A\vec{b}_1 = \lambda_1\vec{b}_1$, $A\vec{b}_2 = \lambda_2\vec{b}_2$, $\ldots$, $A\vec{b}_n = \lambda_n\vec{b}_n$. is equivalent to matrix equation
$$AP = PD$$
where $P = [\vec{b}_1 \ \ldots \ \vec{b}_n]$ and $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$.
P is invertible if and only if $\{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis of $\mathbb{R}^n$. In this case, $A = PDP^{-1}$ and A is diagonalizable.

□

Example 7. Write down all matrices A, P and D in Example 1.

Example 8. Let T be the rotation through an angle of $\pi/2$ in the counterclock direction. So the matrix of T is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find all eigenvalues and eigenvectors of A. Is A diagonalizable?

Example 9. Find all possible real eigenvalues of an $n \times n$ orthogonal matrix.

Example 10. Which matrix has 0 as an eigenvalue?

2. Eigenvalues

Let A be an $n \times n$ matrix.

- Recall that a (possibly comlex) scalar λ is an eigenvalue of A if $A\vec{x} = \lambda\vec{x}$ has a nonzero solution.
- Equivalently, $(A - \lambda I_n)\vec{x} = \vec{0}$ has a nonzero solution.
- Equivalently, $A - \lambda I_n$ is not invertible.
- Equivalently, the determinant of $A - \lambda I_n$ equals zero.

Hence, we have proved the following theorem.

Theorem 11 (The Characteristic Equation)**.** *Let A be an $n \times n$ matrix. A (possibly comlex) scalar λ is an eigenvalue of A if and only if*

$$\det(A - \lambda I_n) = 0$$

This last equation is called the ***characteristic equation*** *of A.*

Example 12. Finding Eigenvalues for the following matrices:

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

Theorem 13. *The eigenvalues of a (upper or lower) triangular $n \times n$ matrix A equal the diagonal entries of A.*

Proof. Suppose A is an upper triangular matrix.
$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda) = 0$$
Hence, the eigenvalues of A are a_{ii} for $i = 1, ..., n$. $\square$

Example 14. Finding Eigenvalues for the following matrices:

$$A = \begin{bmatrix} 2 & 5 & \sqrt{2} \\ 3 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 0 \\ 3 & 5 & 7 \end{bmatrix}$$

In general for a $n \times n$ matrix A,

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda) + \sum(\text{ terms of degree} \leq (n-2)) \\ &= (-\lambda)^n - (a_{11} + a_{22} + \cdots + a_{nn})(-\lambda)^{n-1} + \sum(\text{ terms of degree} \leq (n-2)) \end{aligned}$$

Definition 15 (Characteristic Polynomial)**.** If A is an $n \times n$ matrix, the degree n polynomial

$$f_A(\lambda) = \det(A - \lambda I_n)$$

is called the **characteristic polynomial** of A.

Example 16. Find the characteristic polynomial for a 2×2 arbitrary matrix.

Definition 17. Th sum of the diagonal entries of a square matrix is called the **trace** of A, denoted by $\operatorname{tr} A$.

Summarize Example 2: the characteristic polynomial for a 2×2 A:

$$\det(A - \lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + \det(A)$$

More generally,

Theorem 18. *Let A be an $n \times n$ matrix. Then the characteristic polynomial of A is*

$$f_A(\lambda) = \det(A - \lambda I) = (-\lambda)^n + (\operatorname{tr} A)(-\lambda)^{n-1} + \cdots + \det(A).$$

Proof. The constant of the polynomial $= f_A(0) = \det(A)$ □

Remark: We have no formulas for the middle terms.

3. Review of Polynomials

Definition 19.

- A **polynomial** with coefficients in field $\mathbb{F}$ is a function $p : \mathbb{F} \to \mathbb{F}$ of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n.$$

- If $a_m \neq 0$, we say that the polynomial $p(t)$ has **degree** n
- A number λ is called a **root** of the polynomial $p(t)$ if $p(\lambda) = 0$.

Proposition 20. *λ is root of a degree n polynomial $p(t)$ if and only if there is a degree $n-1$ polynomial $q(t)$ such that*

$$p(t) = (t - \lambda)q(t)$$

Proof. Backward direction "$\Leftarrow$" is obvious. Let's show forward direction "$\Rightarrow$"
Since λ is root, we have $a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n = 0$.
So,

$$\begin{aligned} p(t) &= p(t) - a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n \\ &= a_1(t - \lambda) + a_2(t^2 - \lambda^2) + \cdots + a_n(t^n - \lambda^n) \\ &= (t - \lambda)\big[a_1 + a_2(t + \lambda) + \cdots + a_n(t^{n-1} + t^{n-2}\lambda + \cdots + \lambda^{n-1})\big] \\ &= (t - \lambda)q(t) \end{aligned}$$

Here $q(t)$ has degree $n-1$ since $a_n \neq 0$. □

Proposition 21. *A degree n polynomial has at most n (distinct) roots in $\mathbb{F}$.*

Proof. From the above theorem by induction. $\square$

Proposition 22. *If $a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n = 0$ for all $t \in \mathbb{F}$, then $a_0 = a_1 = \cdots = a_n = 0$.*

Proof. Only zero polynomial $p = 0$ has infinitely many solutions. $\square$

This means that $\{1, t, t^2, \ldots, t^n\}$ is independent in polynomial vector space P.

Proposition 23 (Division Algorithm)**.** *Suppose $p(t)$ and $q(t)$ are non-zero polynomials. There exists polynomials $r(t)$ and $s(t)$ such that*

$$p(t) = s(t)q(t) + r(t)$$

and $\deg(r) < \deg(q)$.

Similar as integers, we can think this as divide $p(t)$ by $q(t)$ and the remainder is $r(t)$.

Theorem 24 (Fundamental Theorem of Algebra)**.** *Every polynomial $p(t)$ of degree $n \geq 1$ with complex coefficient has n roots. That is*

$$p(t) = a_n(t - z_1)(t - z_2)\cdots(t - z_n)$$

The above factorization is unique if we do not count the order.

Proposition 25. *Suppose $p(t)$ is a polynomial with real coefficients. If $z \in C$ is a root of $p(t)$, then the conjugate of z is also a root.*

Proof. If $p(z) = 0$, then take the conjugate of both sides, we have $\overline{p(z)} = 0$ and hence $p(\bar{z}) = 0$ by properties of conjugate. $\square$

Theorem 26 (Real roots)**.** *Every polynomial $p(t)$ of degree $n \geq 1$ with real coefficient can be factorized as*

$$p(t) = a_n(t - c_1)(t - c_2)\cdots(t - c_p)(t^2 + a_1 t + b_1)(t^2 + a_2 t + b_2)\cdots(t^2 + a_m t + b_m)$$

where all numbers in the factorization are real numbers and $a_i^2 < 4b_i$ for $i = 1, 2, ..., m$

Proof. First $p(t) = a_n(t - z_1)(t - z_2)\cdots(t - z_n)$ has been factored as complex roots. Since complex roots come in pairs for real polynomials. Suppose $z = a + bi$ is a root, then $p(t)$ contains a real polynomial factor $(t - z)(t - \bar{z}) = t^2 - 2at + |z|^2$. $\square$

Proposition 27 (Rational roots)**.** *Let $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ be a polynomial of degree $n \geq 1$ with integer coefficient. Suppose rational number $\frac{p}{q}$ is a root of $p(t)$ such that $(p, q) = 1$, then $p | a_0$ and $q | a_n$.*

4. More on Characteristic Polynomials

Definition 28 (Algebraic Multiplicity)**.** An eigenvalue λ_0 of A is said to have **algebraic multiplicity** k if it has multiplicity k as a root of the characteristic polynomial $f_A(t)$. Equivalently,

$$f_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda)$$

such that $g(\lambda_0) \neq 0$.

Theorem 29. *An $n \times n$ matrix has at most n eigenvalues, even counted with algebraic multiplicities.*

Example 30. Find all eigenvalues and their algebraic multiplicities of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Example 31. Find the characteristic polynomial of $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. Which of the following numbers 1, -1, 4 are eigenvalues of A?

Example 32. Find the characteristic polynomial of $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$. Verify that 3 and 5 are eigenvalues.

Theorem 33. *Let A be an $n \times n$ matrix. Suppose A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, (listed with algebraic multiplicities.) Then*

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

and

$$\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

This theorem comes from

$$f_A(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

Theorem 34 (On Eigenvalues of Similar Matrices)**.** *If A and B are similar, i.e., $A = PBP^{-1}$, then they have the same characteristic polynomial, i.e. $f_A(\lambda) = f_B(\lambda)$, and hence the same eigenvalues with the same multiplicities.*

Proof. $f_A(\lambda) = \det(A - \lambda I) = \det(PBP^{-1} - \lambda I) = \det(P(B - \lambda I)P^{-1}) = \det(P)\det(B - \lambda I)\det(P^{-1}) = \det(B - \lambda I) = f_B(\lambda)$. So, A and B have the same characteristic polynomial. $\square$

If A and B are similar, we also have
$\text{rank}(A) = \text{rank}(B)$, $\det(A) = \det(B)$, $\text{tr}(A) = \text{tr}(B)$.

The converse of the preceding theorem is not generally true. There do exist matrices with the same characteristic polynomial that are not similar matrices. See example later.

Proposition 35. *If A and B are similar, we also have* $\det(A) = \det(B)$, $\text{tr}(A) = \text{tr}(B)$.

Proof. Proof. Since determinant and trace are determined by characteristic polynomial, so we get the result by the above theorem. □

Proposition 36. *If A and B are similar, then* $\text{rank}(A) = \text{rank}(B)$.

Proof. $A = PBP^{-1}$. Multiplying an invertible matrix does not change the rank. So, $\text{rank}(A) = \text{rank}(B)$. □

Example 37. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are not similar, but they have the same rank, the same determinant, the same trace, the same eigenvalues.

If B similar to $A = I$, then $B = PAP^{-1} = I$ which is a contradiction.

Example 38. Are the following two matrices similar to each other? $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$

$\text{tr}(A) = 5$ but $\text{tr}(B) = 6$
$|A| = 2$ but $|B| = -1$

Warning: Similar matrices may have **different** eigenvectors.

Think about Example 1 in §7.1. The projection matrix $A = \dfrac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ is similar to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of D but it is not an eigenvector of A.

Theorem 39. *Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue for A if and only if the matrix equation*

$$(A - \lambda I_n)\vec{x} = \vec{0}$$

has a nontrivial *solution $\vec{x}$.*
Said differently, λ is an eigenvalue for A if and only if

$$\mathrm{Nul}(A - \lambda I_n) \neq \{\vec{0}\}.$$

Definition 40. Let A be an $n \times n$ matrix and λ be a eigenvalue of A. The set of all eigenvectors of A corresponding to λ together with the zero vector, is called the **eigenspace** of A corresponding to λ, and it equals the subspace

$$\mathrm{Nul}(A - \lambda I_n).$$

The dimension of $\mathrm{Nul}(A - \lambda I_n)$ is called the **geometric multiplicity** of λ. ($G.m.(\lambda)$)

Proposition 41. *1$\leq$ Geometric multiplicity of $\lambda \leq$ Algebraic multiplicity of $\lambda \leq n$.*

Proof. There is at least one eigenvector(non-zero). So, 1$\leq$ Geometric multiplicity of λ.
Suppose the geometric multiplicity of $\lambda = k$. Then $\mathrm{Nul}(A - \lambda I_n)$ has a basis $\vec{v}_1, \dots, \vec{v}_k$. Let $B = S^{-1}AS$, where the first k columns of S are $\vec{v}_1, \dots, \vec{v}_k$. Hence,

$$B = \begin{bmatrix} \lambda I_k & * \\ \mathbf{0} & * \end{bmatrix}$$

Since A and B are similar, so they have the same eigenvalues. It is clear that Algebraic multiplicity of λ in B is at least k. $\square$

Example 42. Let T be the projection transformation onto a line $L = \mathrm{Span}\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$ $\mathbb{R}^3$. Explain the geometric meaning of the eigenvalues and eigenspaces.

Example 43. Find all eigenvalues and the corresponding eigenspaces of $A = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}$

Lemma 44. *Let A be an $n \times n$ matrix and let $\vec{v}_1, \dots, \vec{v}_p$ be eigenvectors of A that correspond to* distinct *eigenvalues $\lambda_1, \dots, \lambda_p$ respectively. Then $\{\vec{v}_1, \dots, \vec{v}_p\}$ is a linearly independent set of vectors.*

Proof. We prove this by induction on p. If $p = 1$, it is clear. Suppose this is true for $p - 1$ vectors.
Multiply A to $a_1\vec{v}_1 + \cdots + a_p\vec{v}_p = 0$, we have $a_1\lambda_1\vec{v}_1 + \cdots + a_p\lambda_p\vec{v}_p = 0$.
Multiply λ_1 to $a_1\vec{v}_1 + \cdots + a_p\vec{v}_p = 0$, we have $a_1\lambda_1\vec{v}_1 + \cdots + a_p\lambda_1\vec{v}_p = 0$.
The difference of this two equation is

$$a_2(\lambda_2 - \lambda_1)\vec{v}_2 + \cdots + a_p(\lambda_p - \lambda_1)\vec{v}_p = 0$$

From the induction, we have $a_2(\lambda_2 - \lambda_1) = 0, \dots, a_p(\lambda_p - \lambda_1) = 0$. So, $a_2 = a_3 = \cdots = a_p = 0$.
Plug in back, we have $a_1\vec{v}_1 = 0$. So $a_1 = 0$. $\square$

Lemma 45. *Let A be an $n \times n$ matrix and let $\lambda_1, \ldots, \lambda_p$ be distinct eigenvalues with corresponding independent set of eigenvectors $V_1, \ldots, V_p$. Then $V_1 \cup \cdots \cup V_p$ is a linearly independent set of vectors.*

Proof. The proof is similar as the above lemma, by induction on p. ☐

Recall that an $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors. (eigenbasis.) In this case $A = PDP^{-1}$ where the columns of P are eigenvectors of A; the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors given by the columns of P.

Proposition 46 (Case of Distinct Eigenvalues)**.** *If an $n \times n$ matrix A has n **distinct** eigenvalues, then its corresponding eigenvectors are linearly independent and accordingly A is diagonalizable.*

Theorem 47. *Let A be an $n \times n$ matrix with distinct real eigenvalues $\lambda_1, \ldots, \lambda_p$ such that*

$$f_A(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)^{k_1}(\lambda_2 - \lambda)^{k_2} \cdots (\lambda_p - \lambda)^{k_p}.$$

Suppose $k_1 + k_2 + \cdots + k_p = n$. Let E_k be the eigenspace of λ_k.

(1) *Suppose B_k is a basis for E_k. A is diagonalizable if and only if $B = B_1 \cup \cdots \cup B_p$ is an eigen-basis for A.*
(2) *A is diagonalizable if and only if*

$$\dim E_1 + \cdots + \dim E_p = n.$$

This equality is satisfied if and only if $\dim(E_i) = k_i$ for each $i = 1, \ldots, p$

Proof. A is diagonalizable if and only if it has n linearly independent eigenvectors.
Proof of "⇒" of (1): For each E_i, at most we can choose k_i independent eigenvectors, since g.m.$(\lambda_k) \leq$ a.m.$(\lambda_k) = k_i$. Since A has n linearly independent eigenvectors, g.m.$(\lambda_k) = k_i$. So, $|B| = n$. We know that B is independent.
"⇐" of (1) is clear.
(2) follows from (1). ☐

Remark: The theorem is also true if we state everything over $\mathbb{C}$, where we don't need the assumptions.

Another point of view of the eigenspaces is the invariant subspace.

Definition 48. Let $T : V \to V$ be a linear transformation on a vector space V. A subspace $W \subseteq V$ is said to be **invariant** under T if $T(\vec{w}) \in W$ whenever $w \in W$.

Proposition 49. *A one-dimensional subspace is invariant under the linear transformation T_A if and only if it is an eigenspace spanned by an eigenvector of A.*

Theorem 50. *An $n \times n$ matrix A is similar to a diagonal matrix D, (i.e., $A = PDP^{-1}$) if and only if there exists a decomposition of $\mathbb{F}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ such that each V_i is one dimensional and invariant under T_A.*

Example 51. Diagonalizing Matrix $A = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}$

Example 52. Diagonalizing Matrix $A = \begin{bmatrix} 5 & 1 & 0 \\ 2 & 4 & 0 \\ 2 & 1 & 3 \end{bmatrix}$

Remark[Non Diagonalizing Result] For any $n > 1$ there exist examples of $n \times n$ matrices that are not diagonalizable.

Example 53. For any $n > 1$, find examples of $n \times n$ non-diagonalizable matrices.

Example 54. Diagonalizing the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. We already know that $1, -1, 4$ are eigenvalues of A.

Example 55. Diagonalizing the matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$. We already know that 3 and 5 are eigenvalues of A.

6. Complex Eigenvalues and Eigenvectors

We list some basic knowledge of complex numbers.

- Just as $\mathbb{R}$ denotes the set of real numbers, we will use $\mathbb{C}$ to denote the set of complex numbers $z = a + ib$. Here $i = \sqrt{-1}$, and a and b are real numbers called/denoted

$$a = Re(z) = \textbf{real part of } z$$
$$b = Im(z) = \textbf{imaginary part of } z$$

- The **complex conjugate** of $z = a + bi \in \mathbb{C}$ is $\bar{z} := a - bi$
- The **absolute value** of z is $|z| = \sqrt{a^2 + b^2}$.
- $z\bar{z} = |z|^2$

Similarly to $\mathbb{R}^n$ denoting n-dimensional real vectors (that is $n \times 1$ matrices with real number entries), so $\mathbb{C}^n$ shall denote n-dimensional complex vectors, that is $n \times 1$ matrices with complex number entries.

If A is an $m \times n$ matrix and $\vec{x} \in \mathbb{C}^n$ an n-dimensional complex vector, then $A\vec{x}$ is defined in exactly the same way as it is in the case of a real n-dimensional vector $\vec{x}$. We extend the notion of an eigenvector of a given eigenvalue λ (real or complex) of an $n \times n$ matrix A be any nonzero vector $\vec{x} \in \mathbb{C}^n$ such that $A\vec{x} = \lambda\vec{x}$.

Remark 56. Let A be a real $n \times n$ matrix and λ be an eigenvalue of A.

- If λ is a real number, then there exist real eigenvectors associate to λ, as well as complex eigenvector.
- If λ is a complex (non-real) eigenvalue of A, then every eigenvector $\vec{x}$ associated to λ is a complex (non-real) vector.

Real and Imaginary Parts of Vectors

Definition 57. Let $\vec{x} \in \mathbb{C}^n$ be a complex n-dimensional vector.

- The **complex conjugate vector** $\overline{\vec{x}}$ of $\vec{x}$ is the vector made up from the complex conjugate entries of $\vec{x}$.
- The **real part** of $\vec{x}$, denoted $Re(\vec{x})$ is the (real) vector consisting of the real parts of the entries of $\vec{x}$.
- The **imaginary part** of $\vec{x}$, denoted $Im(\vec{x})$ is the (real) vector consisting of the imaginary parts of the entries of $\vec{x}$.

Note that

$$\vec{x} = Re(\vec{x}) + i \cdot Im(\vec{x}) \quad \text{and} \quad \overline{\vec{x}} = Re(\vec{x}) - i \cdot Im(\vec{x}).$$

Remark 58. Replacing the complex vector $\vec{x}$ from the previous definition by a complex $m \times n$ matrix A, leads to the

- Complex conjugate matrix $\overline{A}$.
- Real part $Re(A)$ of A.
- Imaginary part $Im(A)$ of A.

The analogues of above equations apply, in addition to

$$\overline{\lambda \cdot \vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}}, \qquad \overline{A \cdot \vec{x}} = \overline{A} \cdot \overline{\vec{x}}, \qquad \overline{A \cdot B} = \overline{A} \cdot \overline{B}.$$

Example 59 (Complex Conjugate Vectors/Matrices)**.**

Real Matrices Acting on $\mathbb{C}^n$

Suppose A is an $n \times n$ matrix with real number entries so that $\overline{A} = A$. Let λ be a complex eigenvalue of A with associated eigenvector $\vec{x}$. Then

$$\overline{A \cdot \vec{x}} = \overline{A} \cdot \overline{\vec{x}} = A \cdot \overline{\vec{x}}$$
$$\overline{A \cdot \vec{x}} = \overline{\lambda \cdot \overline{\vec{x}}} = \overline{\lambda} \cdot \overline{\vec{x}}$$

Combining the two we obtain

$$A \cdot \overline{\vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}}.$$

Theorem 60. *Let A be an $n \times n$ matrix with real number entries and let λ be an eigenvalue of A with associated eigenvector $\vec{x}$. Then $\overline{\lambda}$ is also an eigenvalue of A with associated eigenvector $\overline{\vec{x}}$.*