Northeastern University, Department of Mathematics

MATH G5110: Applied Linear Algebra and Matrix Analysis. (Fall 2020)

• Instructor: **He Wang** Email: **he.wang@northeastern.edu**

§6 Determinant

Contents

Topics: Cofactor expansions; Permutations; Row operations; Determinant functions; Cramer's Rule.

1. motivation

Recall that the *determinant* of a 2×2 matrix is given by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

We expand this definition to 1×1 matrices by setting

$$\det \begin{bmatrix} a \end{bmatrix} = a.$$

For 1 and 2×2 matrices, we have the following property:

A is invertible if and only if $\det A \neq 0$.

Goal: Define the determinant of an $n \times n$ matrix A with $n \geq 3$, such that A is invertible if and only if $\det A \neq 0$.

Definition 1. Let A be an $n \times n$ matrix with $n \geq 2$ and with (i,j)-th entry a_{ij} .
Let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column from A. Then the **determinant of** A, denoted $\det A$, is defined as

$$\begin{aligned}\det A &= \sum_{i=1}^{n} (-1)^{1+i} a_{1i} \det A_{1i} \\ &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}\end{aligned}$$

This formula for $\det A$ is called the **first row cofactor expansion** formula for the determinant of A.

We list some facts about permutation groups. Details of proof can be found in any group theory or abstract algebra book. Let $[n]$ be the set of n integers $[n] = \{1, 2, \ldots, n\}$. Let $S(n) = \{\sigma : [n] \to [n] \mid \sigma \text{ is a bijection}\}$ be the set of all bijections. We denote an element $\sigma \in S(n)$ as

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

For short we denote σ as $(\sigma(1)\ \sigma(2)\ \ldots\ \sigma(n))$.

$S(n)$ is the **permutation group** (symmetric group) with product given by the composition. The **sign** of a permutation $\sigma \in S(n)$ can be explicitly defined as

$$\text{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is a product of even transpositions} \\ -1 & \text{if } \sigma \text{ is a product of odd transpositions} \end{cases}$$

A transposition is a permutation in $S(n)$ that only switch 2 numbers. For example $(2,4) \in S(5)$,

$$(2,4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & \mathbf{4} & 3 & \mathbf{2} & 5 \end{pmatrix}$$

Another equivalent way to determine the sign of σ is to use the number of inversions. An inversion of $(\sigma(1)\ \sigma(2)\ \ldots\ \sigma(n))$ is the pair of numbers $(\sigma(i) > \sigma(j))$ for $i < j$. For example, (2431) has 4 inversions (21), (43), (41), (31).

$$\text{sign}(\sigma) = (-1)^{N(\sigma)},$$

where $N(\sigma)$ is the number of inversions of σ.

Proposition 2. *If τ is obtained from σ by switch two numbers i, j, then* $\text{sign}(\tau) = -\text{sign}(\sigma)$.

Proof. 1. If i, j are next to each other, the switch will increase or decrease 1 inversion.
2. In general, suppose there are k numbers between i, j, the switch of i, j can be obtained by switch $2k + 1$ pairs in case 1. □

Theorem 3. *If A is an $n \times n$ matrix, then*

$$\det(A) = \sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = \sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

Proof. This theorem can be proved by induction on n. For $n = 1$, it is true. Suppose the formula is true for $n - 1$, let's show that it is true for n.

$$\begin{aligned}
&\sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \\
&= \sum_{i=1}^{n} a_{1i} \sum_{\sigma \in S(n); \sigma(1)=i} \operatorname{sign}(\sigma) a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \\
&= \sum_{i=1}^{n} a_{1i} \sum_{\sigma \in S(n); \sigma(1)=i} (-1)^{1+i} \operatorname{sign}(\sigma(2) \ \ldots \ \sigma(n)) a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \\
&= \sum_{i=1}^{n} (-1)^{1+i} a_{1i} \det A_{1i} \\
&= \det A
\end{aligned}$$

□

Totally, $\det(A)$ is a sum of $n!$ terms.

Example 4. Let A be the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The $3! = 6$ permutations of $[3]$ are listed below, along with the determinant of the associated permutation matrix; For the 6 permutations $(\sigma(1) \ \sigma(2) \ \sigma(3))$,

$$\begin{array}{rcr}
\operatorname{sign}(1 \ 2 \ 3) & = & 1 \\
\operatorname{sign}(1 \ 3 \ 2) & = & -1 \\
\operatorname{sign}(2 \ 1 \ 3) & = & -1 \\
\operatorname{sign}(2 \ 3 \ 1) & = & 1 \\
\operatorname{sign}(3 \ 1 \ 2) & = & 1 \\
\operatorname{sign}(3 \ 2 \ 1) & = & -1
\end{array}$$

Hence we have

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Example 5. Find the determinant of $A = \begin{bmatrix} 0 & 4 & 2 \\ 5 & 2 & 2 \\ 0 & 2 & -1 \end{bmatrix}$. Is A invertible?

Example 6. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 5 & 6 & 7 \end{bmatrix}$. Is A invertible?

Definition 7. Let A be an $n \times n$ matrix. Its (i,j)**-th cofactor** C_{ij} is defined as

$$C_{ij} = (-1)^{i+j} \cdot \det A_{ij}$$

where, as before, A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting its i-th row and j-th column.

Using cofactors, the first row cofactor expansion formula for the determinant of A can be rewritten as

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Theorem 8. *The determinant of an $n \times n$ matrix A can be computed via cofactor expansions across any row or down any column of A:*

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$
$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

These formulas are called the i***-th row*** *and* j***-th column*** *cofactor expansions for* $\det A$*, respectively.*

Proof. Similarly as in the proof of Theorem 3. □

Example 9. Redo Example 2.

Example 10. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 1.2 \\ 0 & 0 & 0 & 2 \\ 5 & 6 & 7 & \pi \\ 0 & 1 & 2 & \sqrt{2} \end{bmatrix}$

Recall the definition of **lower triangular matrix.** Similarly, we can define **upper triangular matrix.** An $n \times n$ matrix A is called **triangular** if it is either lower or upper triangular.

Theorem 11 (Determinants of Triangular Matrices)**.** *Let A be an $n \times n$ triangular matrix, then* $\det A$ *equals the product of the diagonal entries of A:*

$$\det A = a_{11} \times a_{22} \times \cdots \times a_{nn}.$$

Example 12. Find the determinant of $A = \begin{bmatrix} 2 & \sqrt{2} & 3 & 1.7 \\ 0 & 3 & 7 & 12 \\ 0 & 0 & 1 & \pi \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Is A invertible?

Example 13. Find out for which value of λ the matrix $A - \lambda I$ is not invertible, where $A = \begin{bmatrix} 2 & \sqrt{2} & 1.7 \\ 0 & 3 & 12 \\ 0 & 0 & 5 \end{bmatrix}$

Example 14. If A is an $n \times n$ matrix. Consider the relation between $\det(kA)$, $\det(A^{-1})$, $\det(A^T)$ and $\det(A)$.
We consider this in the next section.

Block Matrix.

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & 2 & 11 & \sqrt{3} \\ 2 & 3 & \pi & 12 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Theorem 15. *If* $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, *then,*

$$\det(M) = \det(A)\det(C).$$

Proof.

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}$$

□

No such formula for $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in general.

3. Row Operations and Determinant

Recall that there are three types of *elementary row operations*:
1. (Replacement) Add to one row the multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries of a given row by a nonzero constant.

Question:
If an $n \times n$ matrix A is modified by a single one of the elementary row operations, how does that affect its determinant?

Theorem 16 (Row Operations and the Determinant)**.** *Let A be an $n \times n$ matrix and let B be a matrix obtained from A by a single elementary row operation.*
1. If B is obtained from A by an Interchange operation, then

$$\det B = -\det A.$$

2. If B is obtained from A by a Scaling operation by a factor k, then

$$\det B = k \det A.$$

3. If B is obtained from A by a Replacement operation, then

$$\det B = \det A.$$

Proof. 1. Suppose B is obtained from A by switching i, j-th rows. By Theorem 3,

$$\begin{aligned}
\det(B) &= \sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) a_{1,\sigma(1)} \cdots a_{j,\sigma(i)} \cdots a_{i,\sigma(j)} \cdots a_{n,\sigma(n)} \\
&= \sum_{\sigma \in S(n)} -\operatorname{sign}(\sigma(1) \ldots \sigma(j) \ldots \sigma(i) \ldots \sigma(n)) a_{1,\sigma(1)} \cdots a_{i,\sigma(j)} \cdots a_{j,\sigma(i)} \cdots a_{n,\sigma(n)} \\
&= -\det(A)
\end{aligned}$$

2. By Theorem 8,

$$\det(B) = \sum_{j=1}^{n} k a_{ij} C_{ij} = k \sum_{j=1}^{n} a_{ij} C_{ij} = k \det(A).$$

The third formula can be proved by the following propositions. ☐

Proposition 17.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Proof. By Theorem 8. ☐

Proposition 18. *In a matrix A, if the i-th row equals the j-th row, then* $\det(A) = 0$.

Proof. If we switch i-th row is a scalar product of the j-th row, then $\det(A) = -\det(A)$, so $\det(A) = 0$. ☐

Proposition 19. *In a matrix A, if the i-th row is a scalar product of the j-th row, then* $\det(A) = 0$

Theorem 20. *An $n \times n$ matrix A is invertible if and only if* $\det A \neq 0$.

Proof. By Theorem 16, $\det(\mathbf{rref}(A)) = \pm k \det(A)$ where $k \neq 0$. A is invertible if and only if $\mathbf{rref}(A) = I_n$, $\det(\mathbf{rref}(A)) = 1$. ☐

Proposition 21. *Let A be an $n \times n$ matrix.*

$$\det(kA) = (k^n)(\det A).$$

Proposition 22. *Let A be an $n \times n$ matrix that can be reduced to a matrix U in echelon form with only Replacement and Interchange operations. Then*

$$\det A = (-1)^r \cdot \det U$$

where r is the number of Interchange operations used to get from A to U.

• The determinant $\det U = 0$ if and only if U has a 0 on its diagonal, which in turn can only happen if U has a row of zeros.

Theorem 23 (Determinant of the Transpose Matrix)**.**

$$\det A^T = \det A.$$

Proof.

$$\begin{aligned}\det(A) &= \sum_{\sigma \in S(n)} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \\ &= \sum_{\sigma \in S(n)} \operatorname{sign}(\sigma^{-1}) a_{\sigma^{-1}(1),1} a_{\sigma^{-1}(2),2} \cdots a_{\sigma^{-1}(n),n} \\ &= \det(A^T)\end{aligned}$$

Here we used the property that $\operatorname{sign}(\sigma) = \operatorname{sign}(\sigma)$ and $\sigma \to \sigma^{-1}$ is a bijection between $S(n)$ and itself. □

Theorem 24 (Determinants of Products of Matrices)**.** *Let A and B be two $n \times n$ matrices.*

$$\det(AB) = (\det A)(\det B).$$

Proof. Case 1. If A is invertible, then $A = E_1 \cdots E_s$ a product of elementary matrices.
Theorem 16 shows that $\det(E_{ij}B) = -\det(B) = \det(E_{ij})\det(B)$, $\det(E_i(c)B) = \det(E_i(c))\det(B)$ and $\det(E_{ij}(c)B) = \det(\det(B)$.
Then $\det(AB) = \det(E_1 \cdots E_s B) = \det(E_1)\det(E_2 \cdots E_s B) = \det(E_1)\det(E_2)\cdots\det(E_s)\det(B)$. In particular, when $B = I_n$ then $\det(A) = \det(E_1)\det(E_2)\cdots\det(E_s)$. So, $\det(AB) = (\det A)(\det B)$.
Case 2. If (A) is not invertible, then $\operatorname{rank}(A) < n$. Then $\operatorname{rank}(AB) \leq \operatorname{rank}(A) < n$. So, $\det(AB) = 0$. □

Proposition 25. *Let A be an $n \times n$ matrix.*

$$\det(A^m) = (\det(A))^m$$

Proposition 26. *Let A be an $n \times n$ invertible matrix.*

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Question: How about $\det(A + B)$? Is it $\det(A) + \det(B)$?

Example 27. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$. Is A invertible?

Example 28. Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 6 & 11 & 12 \\ 4 & 8 & 15 & 16 \end{bmatrix}$. Is A invertible?

Definition 29 (Elementary Column Operations)**.** 1. (Column Replacement) Add to one column the multiple of another column.
2. (Column Interchange) Interchange two columns.
3. (Column Scaling) Multiply all entries of a given column by a scalar.

Since column operations on a matrix A can be thought of as row operations on its transpose matrix A^T , and since $\det A = \det A^T$, the rules for how elementary row operations affect the determinant can be used to give a similar rule for column operations.

Theorem 30 (Column Operations and the Determinant)**.** *Let A be an $n \times n$ matrix and let B be a matrix obtained from A by a single elementary row operation.*
1. If B is obtained from A by a Column Replacement operation, then

$$\det B = \det A.$$

2. If B is obtained from A by a Column Interchange operation, then

$$\det B = -\det A.$$

3. If B is obtained from A by a Column Scaling operation by a factor k, then

$$\det B = k \det A.$$

Example 31. Vandermonde determinant

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix} = (a_3 - a_2)(a_3 - a_1)(a_2 - a_1)$$

More generally, by induction on n, we can proved that

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots & \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \\ a_1^n & a_2^n & a_3^n & \cdots & a_n^n \end{vmatrix} = \prod_{1 \le j < i \le n} (a_i - a_j)$$

4. **Linearity Property of the determinant function and Cramer's Rule**

Let A be an $n \times n$ matrix with column vectors $\vec{a}_1, \cdots, \vec{a}_n$,

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix}$$

Let $\vec{x}$ be an n-dimensional vector (an $n \times 1$ matrix) and consider the transformation

$$T \colon \mathbb{R}^n \to \mathbb{R}$$

defined by

$$T(\vec{x}) = \det(\begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_{j-1} & \vec{x} & \vec{a}_{j+1} & \dots & \vec{a}_n \end{bmatrix})$$

Theorem 32 (Linearity and Determinants)**.** *The transformation T defined above is a linear transformation, that is*
(a) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, and
(b) $T(c\vec{x}) = cT(\vec{x})$ for all $\vec{x} \in R^n$ and all $c \in R$.
Accordingly there exists a $1 \times n$ matrix B such that $T = T_B$.

Proof. By Theorems 23, 16 and Proposition 17. □

Example 33 (Finding matrix for the determinant transformation for a given A)**.**

Consider a matrix equation $A\vec{x} = \vec{b}$ in which A is an $n \times n$ invertible matrix, and write $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$. Let

$$A_i(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_{i-1} & \vec{b} & \vec{a}_{i+1} & \dots & \vec{a}_n \end{bmatrix}$$

Theorem 34 (Cramer's Rule)**.** *The unique solution $\vec{x}$ of the matrix equation $A\vec{x} = \vec{b}$ (for the case when A is an $n \times n$ invertible matrix), is given by*

$$x_i = \frac{\det A_i(\vec{b})}{\det A}, \text{ for } i = 1, 2, \dots, n.$$

Proof. First, from cofactor expansion, $\det(A_i(\vec{b})) = \sum_{j=1}^n b_j C_{ij}$.

$$\begin{aligned} a_{k1}x_1 + a_{k2}x_2 + \cdots a_{kn}x_n &= \frac{1}{\det(A)}(\sum_{i=1}^n a_{ki} \sum_{j=1}^n b_j C_{ij}) \\ &= \frac{1}{\det(A)}(\sum_{j=1}^n b_j \sum_{i=1}^n a_{ki} C_{ij}) \\ &= \frac{1}{\det(A)}(b_k \det(A)) \\ &= b_k \end{aligned}$$

for any $k = 0, 1, \cdots, n$. This verifies that $(x_1, \dots, x_n)$ is a solution of $A\vec{x} = \vec{b}$.

□

Let C be the associated $n \times n$ matrix of cofactors defined as:

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

The transpose of C is called the **adjugate matrix of** A, denoted by adj$A$:

$$\text{adj}A = C^T = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

Theorem 35. *If A is a invertible matrix then* $A^{-1} = \dfrac{1}{\det A} \cdot adjA$

Proof. Similarly as the proof of Cramer's Rule, verify that $A\dfrac{1}{\det A} \cdot \text{adj}A = I_n$ □