Northeastern University, Department of Mathematics

MATH G5110: Applied Linear Algebra and Matrix Analysis. (Fall 2020)

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§4. Bases

Contents

1. **Linear Independence**

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_t$ be vectors in a vector space V. Then $\text{Span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_t)$ is a subspace of V.

Definition 1. • The set of vectors $\vec{v}_1, \dots, \vec{v}_p$ in V is said to be **(linearly) independent** if the homogeneous vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

only has the trivial solution $x_1 = x_2 = \cdots = x_p = 0$.

• If there exists a nontrivial solution $(a_1, a_2, \dots, a_p)$, then $\vec{v}_1, \dots, \vec{v}_p$ is said to be **(linearly) dependent**. In this case,

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p = \vec{0}$$

is a **nontrivial relation** among the vectors $\vec{v}_1, \dots, \vec{v}_p$.

Remark 2. A (possibly infinite) subset W of a vector space V is said to be *linearly independent* if all finite subsets of W are linearly independent.

Remark 3. Unlike in the case of $V = \mathbb{F}^n$, in the general setting of vector spaces, equation (1) *cannot* be written as a matrix equation (directly).

We say a vector $\vec{v}_i$ (for $i \geq 2$) is **redundant** if it is a linear combination of the preceding vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}\}$, i.e.,

$$\vec{v}_i = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_{i-1}\vec{v}_{i-1}$$

Proposition 4. *Suppose $\vec{v}_i$ is redundant in $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$. Then*

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} = \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \widehat{\vec{v}_i} \dots, \vec{v}_p\}.$$

Here $\widehat{\vec{v}_i}$ is removed.

Proof. Clearly, $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \supseteq \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \widehat{\vec{v}_i} \ldots, \vec{v}_p\}$. We show $\subseteq$ next.
Suppose $\vec{u} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$. Then $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$. Replace $\vec{v}_i$ by the above redundant equation, $\vec{u}$ is a linear combination without $\vec{v}_i$. Hence, $\vec{u} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \widehat{\vec{v}_i} \ldots, \vec{v}_p\}$. □

Proposition 5. • *Suppose $\vec{v}_1 \neq \vec{0}$. The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is independent if and only if none of them is redundant.*
• *If the set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ of vectors contains the zero vector $\vec{0}$, then it is linearly dependent.*
• *If a subset of the set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly dependent, then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is dependent.*

Proof. The first claim is equivalent to: the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is dependent if and only if one of them is redundant.
"$\Rightarrow$" Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is dependent. Then

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p = \vec{0}$$

has non-trivial solution. Let i be the largest number such that $a_i \neq 0$. ($i \neq 1$ since $\vec{v}_1 \neq \vec{0}$). Then

$$-a_i\vec{v}_i = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_{i-1}\vec{v}_{i-1}$$

So,

$$\vec{v}_i = -a_i^{-1}a_1\vec{v}_1 - a_i^{-1}a_2\vec{v}_2 - \cdots - a_i^{-1}a_{i-1}\vec{v}_{i-1}$$

That is, $\vec{v}_i$ is redundant.
"$\Leftarrow$" Suppose $\vec{v}_i$ is redundant. Then

$$\vec{v}_i = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_{i-1}\vec{v}_{i-1}$$

So,

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_{i-1}\vec{v}_{i-1} - \vec{v}_i = 0$$

So, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is dependent.
The proof of the rest two are easy. □

Proposition 6. *A set $\{\vec{v}\}$ is linearly dependent if and only if $\vec{v} = \vec{0}$.*

A set $\{\vec{u}, \vec{v}\}$ is linearly dependent if and only if one of the two vectors is a scalar multiple of the other vector.

In $\mathbb{F}^n$, the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$ is equivalent to using the matrix equation $A\vec{x} = \vec{0}$ or the augmented matrix $[A \mid \vec{0}]$. So $A = [\vec{v}_1 \ \ldots \ \vec{v}_p]$ is an $n \times p$ matrix.

Proposition 7. *The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subset \mathbb{F}^n$ is independent*
if and only if the homogeneous equation only has zero solution;
if and only if there is no free variable;
if and only if all columns contain pivots;
if and only if $\text{rank}(A) = p$; *(i.e., A has full rank.)*
if and only if $\ker(A) = \{\vec{0}\}$.

Proposition 8. *If $p > n$, then a set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ of vectors in $\mathbb{F}^n$ is linearly dependent.*

Proof. From the above proposition, if the set is independent, $\text{rank}(A) = p \leq n$. ☐

Warning: The preceding property does ***not*** say that $p \leq n$ implies that $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly independent.

2. **Basis of a vector space**

Definition 9. Let V be vector space over $\mathbb{F}$. A subset $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$ of V is called a **basis** for V if

(i) B is linearly independent, and
(ii) $\text{Span}\{\vec{b}_1, \ldots, \vec{b}_n\} = V$.

A subset B of a vector space V has a "greater chance" of being

- linearly independent, if it has fewer vectors;
- a spanning set of V, if it has more vectors.

A basis B for a vector space V is a set that has balanced these two competing requirements. We can think of a basis B as a spanning set that is as small as possible, and as a linearly independent set that is as large as possible.

More precisely, we have

Theorem 10. *If* $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ *is independent in* V*, and* $V = \text{Span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$*, then* $m \geq n$*.*

Proof. We add $\vec{v}_1$ to the ***spanning*** set and get a ***dependent*** set

$$\{\vec{v}_1, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$$

Since $\vec{v}_1 \neq 0$, we can remove one of $\vec{w}_i$ from the set.
We then add $\vec{v}_2$ to the new spanning set and get a dependent set

$$\{\vec{v}_1, \vec{v}_2, \vec{w}_1, \ldots, \widehat{\vec{w}_i}, \ldots, \vec{w}_m\}.$$

Claim: One of the w's is redundant.
Proof of claim: If all w's are not redundant, $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{w}_1 + \cdots + a_{m+1}\vec{w}_m = \vec{0}$ has a non-trivial solutions $(a_1, a_2, 0, 0, ..., 0)$. That is $a_1\vec{v}_1 + a_2\vec{v}_2 = 0$ which is a contradiction.
So, we can remove one redundant w_j's and get a new spanning set

$$\{\vec{v}_1, \vec{v}_2, \vec{w}_1, \ldots, \widehat{\vec{w}_i}, \ldots, \widehat{\vec{w}_j}, \ldots, \vec{w}_m\}.$$

Keep adding the rest of v's and remove the redundant w's. We get a spanning set with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and the n $w's$ removed. So, we must have $m \geq n$. ☐

Example 11. The column vectors of the identity matrix I_n, $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ form a standard basis for $\mathbb{F}^n$.

Example 12. Find a basis for the vector space M_2 of all 2×2 matrices. The standard basis for M_2 is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

Example 13. Find a basis for the vector space P_2 of all polynomials of degree ≤ 2. The standard basis for P_2 is $\{1, t, t^2\}$.

Theorem 14 (Spanning Set Theorem)**.** *Let V be a vector space and let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a subset of V with $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = H$.*
- *If one of the vectors in S, say $\vec{v}_k$, is a linear combination of the remaining vectors in S, then the set $S - \{\vec{v}_k\}$ still spans H,*

$$H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{v}_{k+1}, \ldots, \vec{v}_p\}$$

- *If $H \neq \{\vec{0}\}$ then some subset of S is a basis for H.*

Remark 15 (**Algorithm for Finding a Basis**)**.** The preceding theorem provides a recipe for finding a basis for a subspace H of a vector space V. Namely,

- Pick a generating set $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ of H.

- Keep removing vectors from S that are linear combinations of other vectors in S.

- Once there are no more vectors left in S that are linear combinations of other vectors in S, S is a basis for H.

From the algorithm, we obtained that

Proposition 16. *(1) Every spanning set of a finite-dimensional vector space can be reduced to a basis.*
(2) Any finite-dimensional vector space has a basis.
(3) Any independent set in a finite-dimensional vector space can be extended to a basis.

Proof. For part (3), we need the trick in the proof of Theorem 10 □

3. The Dimension of a Subspace

For a finite-dimensional vector space V, it has many different bases. However, they contain some common properties.

Theorem 17. *If $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ and $\mathscr{D} = \{\vec{d}_1, \ldots, \vec{d}_m\}$ are two bases for V, then $p = m$.*

Proof. A basis is independent and span the space V. So, $p \leq m$ and $p \geq m$. Then $p = m$. □

Definition 18 (The Dimension of a Vector Space)**.** The **dimension** of a vector space V is defined as

$$\dim V := \text{The cardinality of any basis for } V,$$

i.e., the number of elements in a basis.

By convention, the dimension of the vector space $V = \vec{0}$ is 0.

Example 19. The dimension of $\mathbb{F}^n$ is n.

Lemma 20. *Suppose* $\mathscr{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ *is a basis for* V. *(That is* $\dim V = p$.*)*
(1) Any set of more than p *vectors is linearly dependent.*
(2) Any set of less than p *vectors can not span* V.

Proof. From Theorem 10 in the last subsection. ☐

Theorem 21 (The Basis Theorem)**.** *Let* V *be a vector space with* $\dim(V) = p \geq 1$.
- *Any linearly independent set of exactly* p *elements in* V *is automatically a basis for* V.
- *Any set of* p *elements in* V *that span* V, *is automatically a basis for* V.

Proof. (1) Suppose $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_p\}$ is independent in V. If there is a vector $\vec{v} \in V$, such that $\vec{v} \notin \text{Span}\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_p\}$, then $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_p, \vec{v}\}$ is independent in V. This is a contradiction to the above Lemma.
(2) Suppose $V = \text{Span}\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_p\}$. If $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_p\}$ is dependent, then we can delete a redundant element from it and the rest still span V. This is a contradiction by the above Lemma. ☐

Theorem 22. *Let* U *be a subspace of a finite-dimensional space* V. *There is a subspace* W *such that* $V = U \oplus W$.

Proof. Suppose $\{\vec{u}_1, \ldots, \vec{u}_p$ is a basis of U. We can extended it to a basis of V by adding $\{\vec{w}_1, \ldots, \vec{w}_m\}$. Define $W := \text{Span}\{\vec{w}_1, \ldots, \vec{w}_m\}$. Claim: $V = U + W$ and $U \cap W = \{\vec{0}\}$. (Verify this.) ☐

Theorem 23. *Let* U *and* V *be subspaces of a finite-dimensional space. Then*

$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$$

Proof. Idea of the proof: Start from a basis $\vec{w}_1, \ldots, \vec{w}_p$ of $U \cap V$ and extended it to be a basis $\vec{w}_1, \ldots, \vec{w}_p, \vec{u}_1, \ldots, \vec{u}_m$ of U and a basis $\vec{w}_1, \ldots, \vec{w}_p, \vec{v}_1, \ldots, \vec{v}_n$ of V. Then $\dim U \cap V = p$, $\dim U = p + m$, and $\dim V = p + n$.
Claim: $\vec{w}_1, \ldots, \vec{w}_p, \vec{u}_1, \ldots, \vec{u}_m, \vec{v}_1, \ldots, \vec{v}_n$ form a basis of $U + V$. (Verify this from definition.) ☐

In particular, $\dim(U \oplus V) = \dim U + \dim V$. On the other side, we have the following theorem.

Corollary 24. *Let* U *and* W *be subspaces of an* n*-dimensional space* V. *Suppose* $\dim U + \dim W = n$ *and* $U \cap W = \{\vec{0}\}$, *then* $V = U \oplus W$.

Theorem 25. *Suppose* V *is a finite dimensional and* $U_1, \ldots, U_p$ *are subspaces of* V *such that* $V = U_1 + \cdots + U_p$ *and* $\dim V = \dim U_1 + \cdots + \dim U_p$. *Then* $V = U_1 \oplus \cdots \oplus U_p$.

Proof. Suppose $\vec{0} = \vec{u}_1 + \cdots + \vec{u}_p$. ☐

4. **Basis of Null space and range**

Let $T : V \to W$ be a linear transformation. The **rank** of T is defined as the dimension of the image of T. The **nullity** of T is defined as the dimension of the kernel of T.

Theorem 26. *Let $T : V \to W$ be a linear transformation. Then*

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$$

Proof. Suppose $\vec{w}_1, \ldots, \vec{w}_p$ is a basis for $\ker T$. We can extended it to a basis $\{\vec{w}_1, \ldots, \vec{w}_p, \vec{v}_1, \ldots, \vec{v}_m\}$ of V. Then, $\dim V = p + m$ and $\dim \ker T = p$.
Claim: $\{T(\vec{v}_1), \ldots, T(v_m)\}$ is a basis for $\operatorname{im} T$. (Verify this from definition: span and independent). ☐

Let A be an $m \times n$ matrix. The linear transformation defined by A is $T_A : \mathbb{F}^n \to \mathbb{F}^m$. We know that $\dim(\ker T) + \dim(\operatorname{im} T) = n$. Now, we can also find basis for each space.

Theorem 27 (Basis for $\operatorname{im}(A)$)**.** *A basis for the image $\operatorname{im}(A)$ is given by the pivot columns of A. In particular, $\dim(\operatorname{im} A) = \operatorname{rank} A$.*

Proof. $\operatorname{im}(A) = \operatorname{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$. Suppose the pivots columns are $\vec{a}_{c_1}, \vec{a}_{c_2}, ..., \vec{a}_{c_r}$. They are independent and all other columns are redundant columns, because they are corresponding free variables. Hence, $\operatorname{im}(A) = \operatorname{Span}\{\vec{a}_{c_1}, \vec{a}_{c_2}, ..., \vec{a}_{c_r}\}$. So, $\{\vec{a}_{c_1}, \vec{a}_{c_2}, ..., \vec{a}_{c_r}\}$ is a basis for $\operatorname{im}(A)$. ☐

Theorem 28 (Basis for $\ker(A)$)**.** *Let A be an $m \times n$ matrix. Solve the matrix equation $A\vec{x} = \vec{0}$. Write $\vec{x}$ as a linear combination of vectors $\vec{v}_1, \ldots, \vec{v}_p$ with the weights corresponding to the free variables. Then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a basis for $\ker(A)$.*

Proof. First, we know that $\ker(A) = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$. Since p is the number of free variables, so $\dim(\ker T) = n - \dim(\operatorname{im} A) = p$. So, $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a basis for $\ker(A)$. ☐

Theorem 29 (The Dimensions of $\ker(A)$ and $\operatorname{im}(A)$)**.** *Let A be an $m \times n$ matrix. Then,*

$$\dim(\ker(A)) + \dim(\operatorname{im}(A)) = n.$$

Proof. Consider the matrix equation $A\vec{x} = \vec{0}$. The dimension of $\ker(A)$ is the number of free variables in the equation $A\vec{x} = \vec{0}$. The dimension of $\operatorname{im}(A)$ is the number of pivot columns in A, which is also the rank of A. So the sum $\dim(\ker(A)) + \dim(\operatorname{im}(A))$ is the total number of variables. ☐

Theorem 30. *Let A be an $n \times n$ square matrix.*
A is ***invertible****, if and only if the columns vectors span $\mathbb{F}^n$;*
if and only if the columns vectors of A are independent;
if and only if the columns vectors of A form a basis for $\mathbb{F}^n$.

5. Examples

Example 31. Find bases for the kernel and image of the transformation defined by $A = \begin{bmatrix} 0 & 0 & 2 & -8 & -1 \\ 1 & 6 & 2 & -5 & -2 \\ 2 & 12 & 2 & -2 & -3 \\ 1 & 6 & 0 & 3 & -2 \end{bmatrix}$.

We already know $\textbf{rref}(A) = \begin{bmatrix} 1 & 6 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

A vector $\vec{b} \in \mathbb{R}^n$ belongs to the column space of A if and only if there exist numbers $x_1, \ldots, x_p$ such that

$$x_1\vec{a}_1 + \cdots + x_p\vec{a}_p = \vec{b}.$$

This in turn happens if and only if the matrix equation $A\vec{x} = \vec{b}$ has at least one solution $\vec{x}$.

This last point shows that $\text{im}(A) = \mathbb{F}^p$ if and only if the matrix equation $A\vec{x} = \vec{b}$ has a solution $\vec{x}$ for every choice of $\vec{b} \in \mathbb{F}^n$.

Proof. **Proof of the uniqueness of reduced echelon form:**

□

Example 32. Can you find a 3×3 matrix A such that $\dim(\ker A) = \dim(\text{im}(A))$?

Example 33. Can you find a 4×4 matrix A such that $\dim(\ker A) = \dim(\text{im}(A))$?

Example 34. If an 4×4 matrix $A = BC$ such that B is a 4×3 matrix and C is a 3×4 matrix. Is A invertible?

Example 35. A subspace V of $\mathbb{F}^n$ is called a **hyperplane** if V is defined by

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$$

where at least one c_i is not zero. What is the dimension of V?

A vector $\vec{b} \in \mathbb{R}^n$ belongs to the column space of A if and only if there exist numbers $x_1, \ldots, x_p$ such that

$$x_1\vec{a}_1 + \cdots + x_p\vec{a}_p = \vec{b}.$$

This in turn happens if and only if the matrix equation $A\vec{x} = \vec{b}$ has at least one solution $\vec{x}$.

This last point shows that $\text{im}(A) = \mathbb{F}^p$ if and only if the matrix equation $A\vec{x} = \vec{b}$ has a solution $\vec{x}$ for every choice of $\vec{b} \in \mathbb{F}^n$.

Example 36. Let T be the transformation defined by $A = \begin{bmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{bmatrix}$.

Suppose we already know $\textbf{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Q1. Find **bases** for the **kernel** and **image** of T.

Q2. What are the dimensions of for the **kernel** and **image** of A?

Q3. Is $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ in the kernel $\ker(A)$?

Q4. Is $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in the column of A? Is $\vec{w} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ in the column of A?

Example 37. Let $S(n)$ be the subset of $M_n(\mathbb{R})$, defined by

$$S(n) = \{A \in M_n(\mathbb{R}) \mid A = A^T\}$$

Show that $S(n)$ is a subspace. What is a basis for $S(n)$? What is the dimension of $S(n)$?

Example 38. Let $O(n)$ be the subset of $M_n(\mathbb{R})$, defined by

$$O(n) = \{A \in M_n(\mathbb{R}) \mid A = -A^T\}$$

Show that $O(n)$ is a subspace. What is a basis for $O(n)$? What is the dimension of $O(n)$?