Northeastern University, Department of Mathematics

MATH G5110: Applied Linear Algebra and Matrix Analysis. (Fall 2020)

• Instructor: **He Wang** Email: **he.wang@northeastern.edu**

§2. Matrix Algebra.

Contents

1. Sum and scalar product

Definition 1.
- The ***sum*** $A+B$ of $m\times n$ matrices A and B is the new $m\times n$ matrix obtained by adding corresponding entries of A and B.
- The ***scalar product*** $r\cdot A$ of a scalar $r\in\mathbb{F}$ and a matrix A is the matrix obtained by multiplying each entry of A by r.

For example, $A+B=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}+\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}=\begin{bmatrix}a_{11}+b_{11} & a_{12}+b_{12}\\ a_{21}+b_{21} & a_{22}+b_{22}\end{bmatrix}$

$kA=k\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix}ka_{11} & ka_{12}\\ ka_{21} & ka_{22}\end{bmatrix}$

Theorem 2. *For $n\times m$ matrices A, B, C and scalar r, s, the following hold.*
(1) $A+B=B+A$;
(2) $(A+B)+C=A+(B+C)$;
(3) $A+0=A$;
(4) $A+(-A)=0$;
(5) $r(A+B)=rA+rB$;
(6) $(r+s)A=rA+sA$;
(7) $r(sA)=(rs)A$;
(8) $1A=A$.

Proof. By direct verification. □

Because vectors are special matrices, the operations sum and scalar products are also defined. In particular, we denote $-\vec{v}$ for $(-1)\cdot\vec{v}$.

Geometric meanings of vectors: The sum of two vectors satisfies the Parallelogram Rule. The scalar product means the scaling of the length and keep the direction.

There is an extra operation on vectors.

Definition 3. The **dot product** of two vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

is defined as

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots u_nv_n$$

2. **Matrix Product**

- **Product of a matrix A and a vector $\vec{x}$.**

Let A be an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ and rows $R_1, R_2, \ldots, R_m$. Let $\vec{x}$ be a vector in $\mathbb{F}^n$.

Definition 4. The ***product*** of A and $\vec{x}$ defined to be

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n.$$

$$A\vec{x} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \vec{x} = \begin{bmatrix} R_1\cdot\vec{x} \\ R_2\cdot\vec{x} \\ \vdots \\ R_m\cdot\vec{x} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Definition 5. A vector $\vec{b}$ in $\mathbb{F}^m$ is called ***linear combination*** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{F}^m$ if there exist scalars $x_1, x_2, ..., x_n$ such that

$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$$

Theorem 6. *Let A be an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$, and let $\vec{b}$ be a vector in $\mathbb{F}^m$. Then the matrix equation*

$$A\vec{x} = \vec{b}$$

has the same solution set as the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b},$$

which has the same solution set as the linear system with augmented matrix

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n & \vec{b} \end{bmatrix}.$$

Theorem 7 (Algebraic Rules for $A\vec{x}$). *If A is an $m \times n$ matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{F}^n$ and c is a scalar, then*
(1.) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
(2.) $A(c\vec{u}) = c(A\vec{u})$.

Proof. Direct calculation. □

We previously defined the multiplication of an $m \times n$ matrix A and an n-dimensional vector $\vec{x}$, which is itself a $n \times 1$ matrix. The result $A \cdot \vec{x}$ is an m-dimensional vector, which is the same as an $m \times 1$ matrix.

$$(m \times n \text{ matrix}) \cdot (n \times 1 \text{ matrix}) = m \times 1 \text{ matrix}.$$

We shall next generalize this to multiplying more general matrices.

Definition 8. Let A be an $m \times n$ matrix and B be a $n \times p$ matrix with columns $\vec{b}_1, \ldots, \vec{b}_p$. We then define the ***product*** of A and B, to be the $m \times p$ matrix

$$AB := [A\vec{b}_1 \quad A\vec{b}_2 \quad \ldots \quad A\vec{b}_p]$$

If the number of columns of A does not equal the number of rows of B, then AB is not defined.

• The Row-Column Rule for Computing $A \cdot B$

Let A be an $m \times n$ matrix whose (i, j)-th entry is a_{ij}.

Let B be an $n \times p$ matrix whose (i, j)-th entry is b_{ij}.

Then the (i, j)-th entry of AB is

$$\sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj},$$

which equals the dot product of the i-th row of A with the j-th column of B

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}.$$

Example 9. Calculate AB for $A = \begin{bmatrix} -3 & 5 \\ 4 & 2 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -4 \\ -4 & 1 \end{bmatrix}$.

Theorem 10 (Properties of Matrix Multiplication)**.** *Let A be an $m \times n$ matrix, and let B and C be matrices for which the indicated operations are defined. Let I_n denote the $n \times n$ identity matrix.*

- $A(BC) = (AB)C$. *(Associativity of matrix multiplication)*
- $A(B + C) = AB + AC$. *(Left Distributive Law)*
- $(A + B)C = AC + BC$. *(Right Distributive Law)*
- $r(AB) = (rA)B$ *where r is any scalar.*
- $I_m A = A = AI_n$. *(Identity Law for Matrix Multiplication)*

Proof. Each one is proved by direct verification. Let us verify the first associativity property. The rest verifications are easy. Suppose B is a $n \times p$ matrix and C is a $p \times q$ matrix. We compare the (i, j) position of both sides, using the sum notation.
For any $1 \le i \le m$ and $1 \le j \le q$,

$$[A(BC)]_{ij} = \sum_{k=1}^{n} a_{ik}(BC)_{kj} = \sum_{k=1}^{n} a_{ik} \left(\sum_{l=1}^{p} b_{kl}c_{lj} \right) = \sum_{k=1}^{n} \sum_{l=1}^{p} a_{ik}b_{kl}c_{lj}$$

$$[(AB)C]_{ij} = \sum_{l=1}^{p} (AB)_{il}c_{lj} = \sum_{l=1}^{p} \left(\sum_{k=1}^{n} a_{ik}b_{kl} \right) c_{lj} = \sum_{l=1}^{p} \sum_{k=1}^{n} a_{ik}b_{kl}c_{lj} = \sum_{k=1}^{n} \sum_{l=1}^{p} a_{ik}b_{kl}c_{lj}$$

So, $A(BC) = (AB)C$. $\square$

Remark (Non-Properties of Matrix Multiplication)

Some familiar arithmetic properties of real numbers do **not** translate to analogue properties of matrices.

- Even when both AB and BA are defined, generally $AB \neq BA$.
- If $AB = AC$ it does not generally follow that $B = C$ (even when $A \neq 0$).
- If $AB = 0$, it does not generally follow that either A or B is the zero matrix.

Example 11. $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq BA$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} = AC$$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example 12. Find all matrices commute with $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

Definition 13. If A is an $n \times n$ matrix and $k \geq 1$ an integer, we define the k-th power of A, denoted by A^k, as

$$A^k = \underbrace{A \cdot A \cdot \cdots \cdot A}_{k \text{ factors}}.$$

Example 14. Calculate X^2, X^3, X^4, ... for the following matrices

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Remark 15. The power of a matrix is useful in discrete Markov chain (dynamical system). Direct calculation of power of a matrix is tedious. Diagonalization using eigenvalue and eigenvector can simplify the calculation significantly.

Definition 16 (Elementary matrices)**.** E_{ij} denotes the elementary matrix obtained by switching the i-th and j-th rows of the identity matrix.

$$I \xrightarrow{R_i \leftrightarrow R_j} E_{ij}$$

$E_i(c)$ denotes the elementary matrix obtained by multiplying the i-th row by the nonzero constant c.

$$I \xrightarrow{cR_i} E_i(c)$$

$E_{ij}(d)$ denotes the elementary matrix adding d times the j-th row to the i-th row. (The order is from right to left)

$$I \xrightarrow{R_i + dR_j} E_{ij}(d)$$

Example 17. 3×3 matrices: $E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{21}(k) = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Proposition 18 (Elementary matrices multiplications)**.** *Multiply a matrix A with an elementary on the **left** side is equivalent to an elementary row operation is performed on the matrix A.*

Example 19. $E_{ij}A$ is the matrix obtained from A by switch the i-th row and the j-the row.

$$A \xrightarrow{R_i \leftrightarrow R_j} E_{ij}A$$

Example 20. $E_i(c)A$ is the matrix obtained from A by multiplying the i-th row by the nonzero constant c.

$$A \xrightarrow{cR_i} E_i(c)A$$

Example 21. $E_{ij}(d)A$ is the matrix obtained from A by adding d times the j-th row to the i-th row.

$$A \xrightarrow{R_i + dR_j} E_{ij}(d)A$$

Product of block matrices.

If $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$, and all products in the following formulas exist, then

$$AB = \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{bmatrix}.$$

3. **Inverse of a matrix**

Definition 22. An $n \times n$ matrix A is called **invertible** if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

Since the role of A and B in the definition is symmetric, if B is the inverse of A, then A is the inverse of B.

Proposition 23. *If A is invertible, then it has only one inverse.*

Proof. If A have two inverses B and C, then

$$B = BI = B(AC) = (BA)C = IC = C.$$

□

In this case, we will denote the inverse of A as $A^{-1} = B$.

Theorem 24. *Let A and B be $n \times n$ matrices.*

- *If A is invertible, then so is A^{-1} and*
$$(A^{-1})^{-1} = A.$$
- *If A is invertible, then so is A^T and*
$$(A^T)^{-1} = (A^{-1})^T.$$
- *If A and B are invertible, then so is AB and*
$$(AB)^{-1} = B^{-1}A^{-1}$$
- *If A is invertible and $k \neq 0$, then so is kA and*
$$(kA)^{-1} = \frac{1}{k}A^{-1}.$$
- *Suppose A is invertible. If $AB = AC$, or $BA = CA$, then $B = C$.*
- *If A is invertible, then A^m is invertible and $(A^m)^{-1} = (A^{-1})^m$.*

Proof. Each one property is a easy verification using definition. We use the product as an example, $(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$ by associativity of products. Similarly, $(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = BIB^{-1} = BB^{-1} = I$. So AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

□

Proposition 25. *Suppose A and B are $n \times n$ matrices. If AB is invertible, then both A and B are invertible.*

Example 26. If A and B are $n \times n$ invertible matrices, is $A + B$ invertible?

Example 27. The inverse of the elementary matrices.

$$E_{ij}^{-1} = E_{ij}, \quad E_i(c)^{-1} = E_i(1/c), \quad E_{ij}(d)^{-1} = E_{ij}(-d)$$

Theorem 28 (The inverse matrix theorem). *Let A be an $n \times n$ matrix. Then the next statements are all equivalent (that is, they are either all true or all false).*
(1) The matrix A is invertible.
(2) There is a square matrix B such that $BA = I$.
(3) The linear system $A\vec{x} = \vec{0}$ has only the trivial solution.
(4) $\operatorname{rank} A = n$.
(5) The reduced row echelon form of A is identity matrix, i.e. $\textbf{\textit{rref}}(A) = I_n$.
(6) The matrix A is a product of elementary matrices.
(7) There is a square matrix C such that $AC = I_n$.
(8) The linear system $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{F}^n$.

Proof. (1)⇒(2) Obvious.
(2)⇒(3) Multiply B on left, we get $BA\vec{x} = \vec{0}$. So $\vec{x} = \vec{0}$ is the only solution.
(3)⇒(4) by example in the last class.
(4)⇒(5) Obvious.
(5)⇒(6) $EA = \textbf{rref}(A) = I_n$ where $E = E_1 \cdots E_s$ is a product of elementary matrices. So, $A = E_s^{-1} \cdots E_1^{-1}$ is a product of elementary matrices.
(6)⇒(1) The reason is that elementary matrices are invertible matrices and product of invertible matrices are invertible.
We have proved that (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(6)⇒(1). So, the first six statements are equivalent.
(1)⇒(7) is obvious. (7)⇒(4): $\operatorname{rank}(AC) = \operatorname{rank}(I_n) = n$. So by Theorem 34, $\operatorname{rank} A \geq n$. This shows that (7) is also equivalent to the above six statements.
(8)⇒(6) is obviously. (1)⇒(8). Since $A^{-1}A = I$ and A^{-1} is unique, then $\vec{x} = A^{-1}\vec{b}$ is the unique solution. Hence the first eight statements are equivalent. □

This is a standard method to show equivalent statements. We don't have to show the other directions. Some of them may hard to show directly.

Question: If A , B and C are $n \times n$ matrices and $ABC = I_n$, is each of the matrices invertible? What are their inverses?

Theorem 29 (Algorithm for Computing A^{-1}). *Given an $n \times n$ matrix A.*
1. Define an $n \times 2n$ "augmented matrix "

$$[A \mid I_n]$$

2. Using elementary row operations to find $\textbf{\textit{rref}}[A \mid I_n]$.
(i). If $\textbf{\textit{rref}}[A \mid I_n] = [I_n \mid C]$, *then $C = A^{-1}$.*
(ii). If this is not possible, then A is not invertible.

Example. Find the inverse of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$

Theorem 30 (Invertibility of 2×2 Matrices)**.** *A 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$.*
The formula for the inverse matrix of A is

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We call $ad - bc$ the **determinant** of A, denoted by $\det(A)$ or $|A|$.

4. The transpose A^T

Definition: Given an $m \times n$ matrix A, we define the ***transpose matrix*** A^T, as the $n \times m$ matrix whose (i, j)-th entry is the (j, i)-th entry of A.

Said differently, the rows of A^T are the columns of A, and the columns of A^T are the rows of A.

Theorem 31 (Properties of Matrix Transposition)**.** *Let A and B be matrices such that the indicated operations are well defined.*
- $(A^T)^T = A$.
- $(A + B)^T = A^T + B^T$.
- $(rA)^T = rA^T$ *for any scalar* r.
- $(AB)^T = B^T A^T$.

Proof. Let's verify the last equality. The rest are easy. We compare the (i, j)-entry of the matrix.

$$[(AB)^T]_{ij} = [AB]_{ji} = \sum_k a_{jk} b_{ki}$$

$$[B^T A^T]_{ij} = \sum_k [B^T]_{ik} [A^T]_{kj} = \sum_k b_{ki} a_{jk} = \sum_k a_{jk} b_{ki}.$$

So, $(AB)^T = B^T A^T$. □

Theorem 32. *If AB is defined, then* $\operatorname{rank}(AB) \leq \operatorname{rank} A$.

Proof. Suppose $\operatorname{rank}(A) = r$, so $EA = \textbf{rref}(A)$ with only the first r non-zero rows, where E is products of elementary matrices. Then $\operatorname{rank}(AB) = \operatorname{rank}(EAB) = \operatorname{rank}(\textbf{rref}(A)B)$. However, $\textbf{rref}(A)B$ only has r non-zero rows. So, $\operatorname{rank}(\textbf{rref}(A)B) \leq r$, that is $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$. □

Theorem 33. $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

Proof. Suppose $\text{rank}(A) = r$, so $EA = \textbf{rref}(A)$ with only the first r non-zero rows, where E is products of elementary matrices.
Take the transpose $A^T E^T = \textbf{rref}(A)^T$ which has rank r. So, by Theorem 32, $\text{rank}(A) \le \text{rank}(A^T)$.
Hence $\text{rank}(A^T) \le \text{rank}((A^T)^T) = \text{rank}(A)$, since $(A^T)^T = A$. □

Theorem 34. *If AB is defined, then* $\text{rank}(AB) \le \min\{\text{rank}\, A, \text{rank}\, B\}$.

Proof. By the above three theorems, $\text{rank}(AB) = \text{rank}((AB)^T) = \text{rank}(B^T A^T) \le \text{rank}(B^T) = \text{rank}(B)$. □

Definition 35 (Symmetric Matrices)**.** An $n \times n$ matrix A is called **symmetric** if $A^T = A$.

If we write $A = [a_{ij}]$, then A is symmetric if and only if

$$a_{ij} = a_{ji} \qquad \text{for all } i, j \in \{1, 2, ..., n\}$$

5. LU factorizations and Gaussian elimination

LU-decomposition is a matrix product version of Gaussian elimination.

Definition 36. An $m \times m$ matrix L with entries l_{ij} is called
• **lower triangular** if $l_{ij} = 0$ whenever $j > i$, that is if all entries l_{ij} above the main diagonal are zero.
• **unit lower triangular** if it is lower triangular with the extra property that $l_{ii} = 1$ for each $i = 1, \dots, m$.

Remark: (1) Unit lower triangular is preserved by matrix product and inverse.

(2) Elementary matrices $E_{ij}(d)$ is a unit lower triangular matrix if $i > j$.

Using only replacement operations, suppose we can reduce an $m \times n$ matrix A to an echelon form $\textbf{ref}(A)$. Each operation is a left multiplication by an elementary matrix $L = E_{ij}(d)$ for $i > j$. So,

$$\textbf{ref}(A) = L_p L_{p-1} \cdots L_1 A$$

Hence

$$A = L_1^{-1} L_2^{-1} \cdots L_{p-1}^{-1} L_p^{-1} \textbf{ref}(A)$$

Definition 37. Let A be an $m \times n$ matrix. An **LU factorization** for A is given by writing A as the product

$$A = L \cdot U$$

with L a unit lower triangular $m \times m$ matrix, and with U an $m \times n$ matrix in echelon form.

Use of LU factorizations: Suppose $A = L \cdot U$ with L unit lower triangular and with U in echelon from, and consider the linear system $A\vec{x} = \vec{b}$.

$$\begin{aligned} A\vec{x} &= \vec{b} \\ (LU)\vec{x} &= \vec{b} \\ L(U\vec{x}) &= \vec{b} \\ L^{-1}L(U\vec{x}) &= \vec{b} \\ U\vec{x} &= L^{-1}\vec{b} \end{aligned}$$

Upshot The original system $A\vec{x} = \vec{b}$ has been replaced by the equivalent system $U\vec{x} = L^{-1}\vec{b}$ whose coefficient matrix U is in echelon form, and is therefore easier to solve than $A\vec{x} = \vec{b}$.

Algorithm for Finding an LU Factorization:

Suppose A is an $m \times n$ matrix that can be transformed into a matrix in echelon form by using only Row-Replacement operations. Then an LU factorization of A can be obtained as follows.

1. Reduce A to echelon form U using only Row-Replacement operations.

2. Let L be the matrix obtained from I_m by applying the inverse Row-Replacement operations from Step 1, in reverse order.

Then $A = L \cdot U$ with L unit lower triangular and U in echelon form.

Remark: Not ever matrix has a LU-factorization.

Example There is no LU-factorization for matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Remark: There are several variations of LU-factorization: e.g.,

1. LDU-decomposition. $A = LDU$. Here D means a diagonal matrix and U is an unit upper triangular matrix.

2. LU-factorization with pivoting. $PA = LU$. Here P is a permutation matrix, obtained by multiplication of elementary matrices E_{ij}.