Northeastern University, Department of Mathematics

MATH G5110: Applied Linear Algebra and Matrix Analysis. (Fall 2020)

• Instructor: **He Wang**   Email: **he.wang@northeastern.edu**

§13 Singular Value Decomposition

1. Singular Value Decomposition

Recall the spectral decomposition for symmetric matrices:

Theorem 1 (Spectral Decomposition for Symmetric Matrices)**.** *A is an $m \times m$ **symmetric** matrix if an only if $A = VDV^{-1}$ such that D is diagonal and V is an orthogonal matrix.*
Let $\lambda_1, \dots, \lambda_m$ be the diagonal entries of D, and let $\vec{v}_1, \dots, \vec{v}_m$ be the column vectors of V. Then $A = VDV^T$ can be written as

$$A = \lambda_1\Big(\vec{v}_1 \cdot (\vec{v}_1)^T\Big) + \cdots + \lambda_n\Big(\vec{v}_n \cdot (\vec{v}_m)^T\Big)$$

We want to find a similar decomposition for **any** $n \times m$ matrix M.

Definition 2 (Singular Values)**.**

Theorem 3. *If* $\operatorname{rank}(M) = r$, *then* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ *and* $\sigma_{r+1} = \cdots = \sigma_m = 0$.

Theorem 4. (1) $M\vec{v}_i \cdot M\vec{v}_j = 0$ *for* $i \neq j$.
(2) $||M\vec{v}_i|| = \sigma_i$ *for all* $i = 1, 2, \ldots, m$.
(3) *In particular,* $M\vec{v}_i = 0$ *for* $i = r+1, \cdots, m$.

Theorem 5 (Singular Value Decomposition(SVD)). *And $n \times m$ matrix M can be decomposed as*

$$M = U\Sigma V^T$$

or as

$$M = \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

Example 6. Find an SVD decomposition for the matrix

$$M = \begin{bmatrix} 1 & 4 \\ 2 & 2 \\ 2 & -4 \end{bmatrix}$$

Example 7. Find an SVD decomposition for the matrix

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Example 8.

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$$

(1). Calculate $M^T M$ and MM^T.

(2). Find all eigenvalues and an eigenbasis of $M^T M$.

(3). Find all eigenvalues and an eigenbasis of MM^T.

(4). Find an SVD decomposition for the matrix M.

Applications.

1. Geometric meaning in $\mathbb{R}^2$.

Theorem 9. *Let M be an 2×2 invertible matrix. The image of M of the unit circle is an ellipse. The lengths of the semimajor and the semiminor axes of the ellipse are the singular values of M.*

2. Solving least-squares problems.

3. Principal component analysis.

4. Digital image compressing.