Northeastern University, Department of Mathematics

MATH 4570 - Matrix Methods in Data Analysis and Machine Learning

• Instructor: **He Wang** Email: **he.wang@northeastern.edu**

§4. Bases and dimension

Contents

1. Linear Independence

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ be vectors in a vector space V. e.g. $\mathbb{R}^n$

$A\vec{x} = \vec{0}$

Definition 1. • The set of vectors $\vec{v}_1, \ldots, \vec{v}_p$ in V is said to be **(linearly) independent** if

$(*)$ $x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0}$ only has trivial solution $\vec{x} = \vec{0}$.

• The set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is said to be **(linearly) dependent** if

there is a non-trivial solu $\vec{x} = \vec{c}$ for $(*)$.

$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$ is the dependence relation.

An infinite subset W of a vector space V is said to be *linearly independent* if all finite subsets of W are linearly independent.

$\{1, t, t^2, t^3, \cdots, t^n, \cdots\}$ for vector space of all polynomials.

We say a vector $\vec{v}_i$ (for $i \geq 2$) is **redundant** if it is a linear combination of the preceding vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}\}$. $\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1}$

Proposition 2. *Suppose $\vec{v}_i$ is redundant in $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$. Then*

$$\mathrm{Span}\{\vec{v}_1 \cdots \vec{v}_p\} = \mathrm{Span}\{\vec{v}_1, \cdots, \hat{\vec{v}_i}, \cdots \vec{v}_p\}$$

Proposition 3. • *Suppose $\vec{v}_1 \neq \vec{0}$. The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is independent if and only if none of them is redundant.*

• *If the set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ of vectors contains the zero vector $\vec{0}$, then it is linearly dependent.*

• *If a subset of the set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly dependent, then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is dependent.*

Example 4. (1) A set $\{\vec{v}\}$ is linearly dependent if and only if $\vec{v} = \vec{0}$

(2) A set $\{\vec{u}, \vec{v}\}$ is linearly dependent if and only if $\vec{v} = c\vec{u}$ or $\vec{u} = c\vec{v}$.

In $\mathbb{F}^n$, $= \mathbb{R}^n$

$$[\vec{u}_1 \cdots \vec{u}_p] = A$$

$p \leq n$

Proposition 5. *The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subset \mathbb{F}^n$ is independent if and only if*

$$\mathrm{rank}\, A = p$$

more vectors than dim

Proposition 6. *If $p > n$, then a set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ of vectors in $\mathbb{F}^n$ is linearly dependent.*

Ex1: $\{e^t, \sin t\}$ independence.

$0: \mathbb{R} \to \mathbb{R}$

$t \to 0$

for any t $\boxed{x_1 e^t + x_2 \sin t = 0}$

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

Ex2: $x_1(t^2+t+1) + x_2(t+1) + x_3 2 = 0$

$$t^2 x_1 + t(x_1+x_2) + (x_1+x_2+2x_3) = 0$$

$$\begin{cases} x_1 = 0 \\ x_1+x_2 = 0 \\ x_1+x_2+2x_3 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

Ex3

$$x_1 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 & 7 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. **Basis of a vector space**

Let V be vector space over $\mathbb{F}$.

Definition 7. A subset $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$ of V is called a **basis** for V if

① $V = \operatorname{Span}\{\vec{b}_1 \cdots \vec{b}_n\}$

② $\{\vec{b}_1 \ldots \vec{b}_n\}$ is independent.

Example 8. Standard basis for $\mathbb{R}^n$.

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \ a, b, c, d \in \mathbb{R} \right\}$ $\quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

Example 9. Find a basis for the vector space M_2 of all 2×2 matrices.

$\sim \mathbb{R}^4$

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$= \{ a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$

Example 10. Find a basis for the vector space P_2 of all polynomials of degree ≤ 2.

$\{1, t, t^2\}$

$V = [\vec{v}_1 \cdots \vec{v}_n]$

Theorem 11. *If* $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ *is independent, and* $V = \operatorname{Span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$, *then* $n \leq m$.

in V

$W = [\vec{w}_1 \cdots \vec{w}_m]$

$V \subset \mathbb{R}^s$

$\vec{v}_i = a_{i1} \vec{w}_1 + \cdots + a_{im} \vec{w}_m \quad$ for $i = 1, \ldots, n$

$$V = W A$$
$$\underset{s\times n}{} \quad \underset{s\times m}{} \quad \underline{\underset{m\times n}{}}$$

$$n = \text{rank}\, V = \underline{\text{rank}(WA)} \leq \text{rank}\, A \leq m$$

$\star$

$\text{rank}(MN) \leq \min\{\text{rank}\, M, \text{rank}\, N\}$

Theorem 12 (Spanning Set Theorem). *Let V be a vector space and let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a subset of V with* $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = H$.
- *If one of the vectors in S, say $\vec{v}_k$, is a linear combination of the remaining vectors in S, then the set $S - \{\vec{v}_k\}$ still spans H,*

$$H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{v}_{k+1}, \ldots, \vec{v}_p\}$$

- *If $H \neq \{\vec{0}\}$ then some subset of S is a basis for H.*

basis

independent

Spanning set

Proposition 13. *(1) Every spanning set of a finite-dimensional vector space can be reduced to a basis.*
(2) Any finite-dimensional vector space has a basis.
(3) Any independent set in a finite-dimensional vector space can be extended to a basis.

3. **The Dimension of a Subspace**

For a finite-dimensional vector space V, it has many different bases. However, they contain some common properties.

Theorem 14. *If $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ and $\mathscr{D} = \{\vec{d}_1, \dots, \vec{d}_m\}$ are two bases for V, then $p = m$.*

Ex: $V = \text{Span}\{\sin t, e^t\} = \{a_1 \sin t + a_2 e^t \mid a_1, a_2 \in \mathbb{R}\}$

basis $\{\sin t, e^t\}$ $\dim V = 2$

Ex: $P = \{\text{all polynomials}\}$ $\{1, t, t^2, \dots, t^n, t^{n+1}, \dots\}$

Definition 15 (The Dimension of a Vector Space)**.** The **dimension** of a vector space V is defined as

$$\dim V = \#\{\text{a basis for } V\}$$

- P_2 $\{1, t, t^2\}$ is a basis for P_2. $\dim(P_2) = 3$
- $\mathbb{R}^{2\times 2}$ $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix} \cdots \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ $\dim(\mathbb{R}^{2\times 2}) = 4$.

Lemma 16. *Suppose $\mathscr{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ is a basis for V.* → $\dim V = p$
(1) Any set of more than p vectors is linearly dependent.
(2) Any set of less than p vectors can not span V.

$$\dim\{\vec{0}\} = 0$$

Theorem 17 (The Basis Theorem)**.** *Let V be a vector space with* $\dim(V) = p \geq 1$.

- If $\mathcal{B} = \{\vec{v}_1 \dots \vec{v}_p\}$ is independent, then $\mathcal{B}$ is a basis for V.
- If $V = \text{Span}\{\vec{w}_1, \dots \vec{w}_p\}$, then $\{\vec{w}_1 \dots \vec{w}_p\}$ is a basis for V.

Ex: $P_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_i \in \mathbb{R}\}$ $\dim P_2 = 3$

Is $\{t^2+t+1, t+1, 2\}$ a basis for P_2? Yes.

not unique. $U \subseteq V$

Theorem 18. *Let U be a subspace of a finite-dimensional space V. There is a subspace W such that $V = U \oplus W$.*

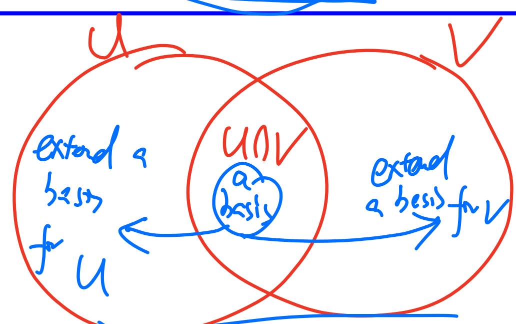
· Start from a basis of U $\{b_1 \cdots b_s\}$, then expanded it to a basis of V $\{b_1 \cdots b_s, b_{s+1} \cdots b_n\}$

Ex: $V = \mathbb{R}^3$ $\quad U =$ a plane. $\quad W = \mathrm{Span}\{b_{s+1} \cdots b_n\}$

Theorem 19. *Let U and V be subspaces of a finite-dimensional space.* over $\mathbb{R}$ *Then*

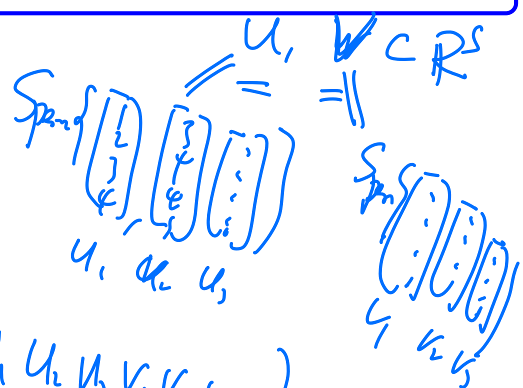
$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$$

basis is hard!



$\dim(U \oplus V) = \dim U + \dim V$

$U, W \subset \mathbb{R}^5$



$U + V = \mathrm{Span}\{u_1\, u_2\, u_3\, v_1\, v_2\, v_3\}$

Corollary 20. *Let U and W be subspaces of an n-dimensional space V. Suppose $\dim U + \dim W = n$ and $U \cap W = \{\vec{0}\}$, then*

$$V = U \oplus W$$

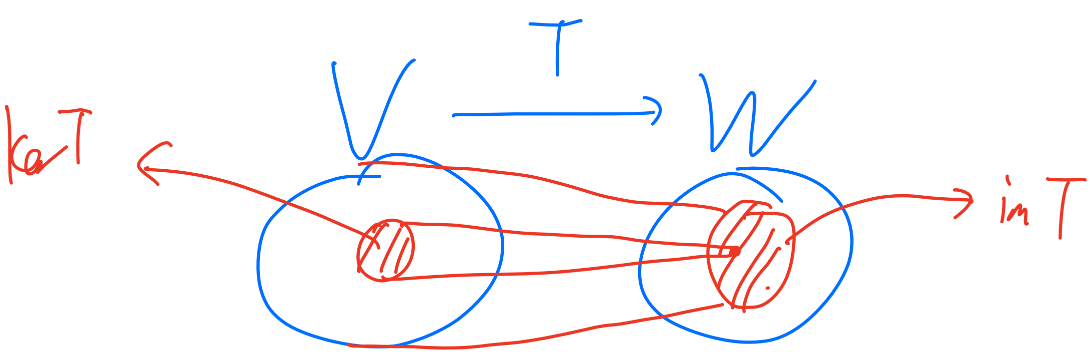
$\overset{\text{Thm}}{\Leftrightarrow}$ $V = U + W$ and $U \cap W = \{\vec{0}\}$

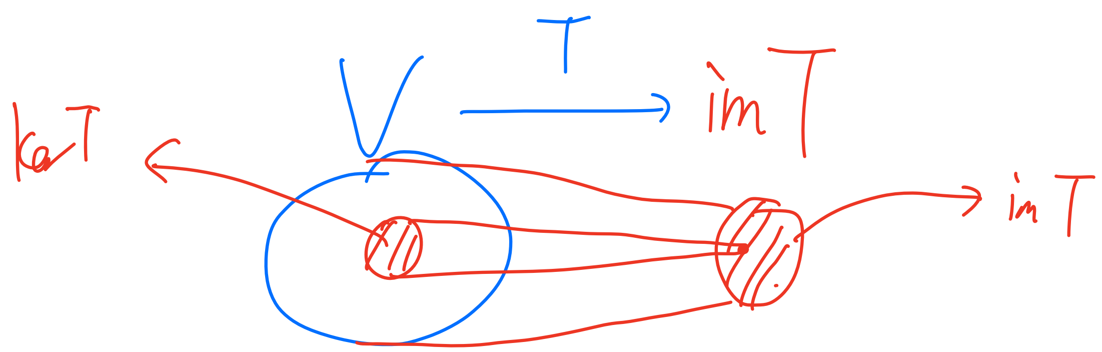
Theorem 21. *Suppose V is a finite dimensional and $U_1, \ldots, U_p$ are subspaces of V such that $V = U_1 + \cdots + U_p$ and $\dim V = \dim U_1 + \cdots + \dim U_p$. Then $V = U_1 \oplus \cdots \oplus U_p$.*

$\vec{v} = \vec{u}_1 + \cdots + \vec{u}_p$

$\vec{v} = \vec{w}_1 + \cdots + \vec{w}_p$

$\vec{v} = \vec{u}_1 + \vec{u}_2 + \vec{0}$





im = ker

$$0 \to \ker T \hookrightarrow V \overset{T}{\twoheadrightarrow} \operatorname{im} T \to 0$$

vector space / $\mathbb{R}$

4. Basis of Null space and range

Ex: $T: \mathbb{R}^n \to \mathbb{R}^m$

Let $T : V \to W$ be a "linear" transformation.

- $\text{rank}\, T := \dim(\text{im}\, T)$
- $\text{nullity}\, T := \dim(\ker T)$

rank-nullity thm

Theorem 22. *Let $T : V \to W$ be a linear transformation. Then*

$$\dim V = \text{rank}\, T + \text{nullity}\, T$$

- Start from a basis of $\ker T$, $\{\vec{u}_1 \cdots \vec{u}_p\}$, extend to be a basis of V. $\{\vec{u}_1 \cdots \vec{u}_p\ \vec{b}_1 \cdots \vec{b}_m\}$
- Check $\{T(\vec{b}_1) \cdots T(\vec{b}_m)\}$ is a basis for $\text{im}\, T$. ① independent ② $\text{im}\, T = \text{Span}\{-\}$

Let A be an $m \times n$ matrix. The linear transformation defined by A is

$$T_A : \mathbb{F}^n \to \mathbb{F}^m$$

$\mathbb{R}^n \to \mathbb{R}^m$

$\vec{x} \to A\vec{x}$

$A = [\vec{a}_1 \cdots \vec{a}_n]$

Theorem 23 (Basis for im(A)). *A basis for the image* im(A) *is given by the pivot columns of A. In particular,* $\dim(\text{im}\, A) = \text{rank}\, A$.

$\vec{x} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p$

Theorem 24 (Basis for $\ker(A)$). *Let A be an $m \times n$ matrix. Solve the matrix equation $A\vec{x} = \vec{0}$. Write $\vec{x}$ as a linear combination of vectors $\vec{v}_1, \ldots, \vec{v}_p$ with the weights corresponding to the free variables. Then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a basis for* $\ker(A)$.

Proposition 25 (The Dimensions of ker(A) and im(A)). *Let A be an $m \times n$ matrix. Then,*

$$\dim(\ker(A)) + \dim(\operatorname{im}(A)) = n.$$

$\dim(\operatorname{im}(A)) = \operatorname{rank} A$

Proposition 26. *Let A be an $n \times n$ square matrix.*
A is ***invertible****, if and only if*

- $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$

- Basis for $U+V$, U, V are easy.

$U+V = \operatorname{Span}\{\vec{u}_1 \cdots \vec{u}_s, \vec{v}_1 \cdots \vec{v}_t\}$

Ex: $U = \operatorname{Span}\{\vec{u}_1 \cdots \vec{u}_2\}$ $\quad V = \operatorname{Span}\{\vec{v}_1 \cdots \vec{v}_2\} \subset \mathbb{R}^n$

$= \operatorname{im} A \qquad\qquad = \operatorname{im} B.$

$c_1\vec{u}_1 + c_2\vec{u}_2 = d_1\vec{v}_1 + d_2\vec{v}_2$

- Basis for $U \cap V = \{\vec{z} \mid \vec{z} \in U \text{ and } \vec{z} \in V\}$

$$U = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\} \qquad V = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 - d_1 \vec{v}_1 - d_2 \vec{v}_2 = \vec{0}$$

$$M = [\vec{u}_1 \quad \vec{u}_2 \quad -\vec{v}_1 \quad -\vec{v}_2]$$

$$\begin{bmatrix} c_1 \\ c_2 \\ d_1 \\ d_2 \end{bmatrix} \in \ker M$$

$$M = \left[\begin{array}{cc|cc} 0 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 \end{array}\right] \rightarrow \text{rref}\, M = \begin{bmatrix} 1 & & & \cdot \\ & 1 & & \cdot \\ & & 1 & \cdot \end{bmatrix}$$

Slu fr $M\vec{x} = \vec{0}$ $\qquad \vec{x} = x_4 \begin{bmatrix} c_1 \\ c_2 \\ d_1 \\ d_2 \end{bmatrix}$

$$U \cap V = \{ \text{all } c_1 \vec{u}_1 + c_2 \vec{u}_2 \}$$?

$$\text{or} = \{ \text{all } d_1 \vec{v}_1 + d_2 \vec{v}_2 \}$$

Ex: $V = P_2 = \{ a_0 + a_1 t + a_2 t^2 \mid a_i \in \mathbb{R} \}$

basis $\mathcal{B} = \{ 1, t, t^2 \}$

Thm: If $\mathcal{B} = \{ \vec{b}_1 \cdots \vec{b}_p \}$ is a basis for V, over $\mathbb{R}$

then any $\vec{v} \in V$

$\vec{v} = x_1 \vec{b}_1 + \cdots + x_p \vec{b}_p$ exist and unique.

$$V \ni \vec{v} \longleftrightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} =: [\vec{v}]_{\mathcal{B}}$$

coordinate of $\vec{v}$ relative to $\mathcal{B}$.

Thm: $V \xrightarrow{f} \mathbb{R}^p$ is bijective, "linear"

is isomorphism

Ex: $P^2 \longrightarrow \mathbb{R}^3$

$$\vec{v} = a_0 + a_1 t + a_2 t^2 \longmapsto \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = [\vec{v}]_{B}.$$

basis $B_V = \{\vec{b}_1 \dots \vec{b}_s\}$ basis $B_W = \{\vec{w}_1 \dots \vec{w}_t\}$

linear map:

$$\begin{tikzcd}
\vec{v} \arrow[d] & V \arrow[r, "T"] \arrow[d] & W \arrow[d] & \vec{w} \arrow[d] \\
{[\vec{v}]_{B_V}} & \mathbb{R}^s \arrow[r, dashed, "C"] & \mathbb{R}^t & {[\vec{w}]_{B_W}}
\end{tikzcd}$$

$t \times s$ matrix