1. Two Sample Z Test $H_0 : \mu_X = \mu_Y$. (§9.2)

Suppose we have two normal distributions

$$X \sim \text{Normal}(\mu_X, \sigma_X^2) \quad \text{and} \quad Y \sim \text{Normal}(\mu_Y, \sigma_Y^2)$$

Suppose the means μ_X and μ_Y are **unknown**, the standard derivations σ_X and σ_Y are **known**.

We want to **test** $H_0 : \mu_X = \mu_Y$. This is the same as **test** if $\mu_X - \mu_Y = 0$.

If we get a sample of size n of X and a sample of Y of size m, then by CLT,

$$\overline{X} - \overline{Y} \sim \text{Normal}(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m})$$

Assume the Null Hypothesis $H_0 : \mu_X = \mu_Y$, then

$$\overline{X} - \overline{Y} \sim \text{Normal}(0, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m})$$

Change to standard normal distribution:

$$Z = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim \text{Normal}(0, 1)$$

To Test the Null Hypothesis $H_0 : \mu_X = \mu_Y$ at the level of significance α, we define the **test statistic:**

$$z = \frac{\overline{x} - \overline{y}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$$

1. (Right-sided) $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X > \mu_Y$.
Decision Rule: Reject H_0 if $z > z_\alpha$.

2. (Left-sided) $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X < \mu_Y$.

Decision Rule: Reject H_0 if $z < -z_\alpha$.

3. (Two-sided) $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X \neq \mu_Y$.

Decision Rule: Reject H_0 if $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$.

We can also use the P-value for the test. For example, to test $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X > \mu_Y$, the P-value is

$$P\text{-Value} = P(\overline{X} - \overline{Y} \geq \overline{x} - \overline{y})$$

Similarly for the left sided and two sided tests.

Both test statistic and the P-value can be calculated by calculator

STAT → TESTS → 2SampZTest

Since this section, we will practice **using calculator** for Tests and Confidence intervals.

Example 1. A study using two samples of 35 people each found that the average amount of time those in the age group of 26-35 years spent per week on leisure activities was 39.6 hours, and those in the age group of 46-55 years spent 35.4 hours. Assume that the population standard deviations are 6.3 hours and 5.8 hours respectively.

(1) Test at $\alpha = 0.05$ if there is a significant difference between the leisure times of the two groups.

STAT → TESTS → 2SampZTest
Input: (**Stats**) $\sigma_1 = 6.3$; $\sigma_2 = 5.8$; $\bar{x}_1 = 39.6$; $n_1 = 35$; $\bar{x}_2 = 35.4$; $n_2 = 35$; $\mu_1 \neq \mu_2$
Output: $z = 2.9016$; $p = 0.0037$; ...
Reject. Either by P-value or by comparing test statistic with $z_{\alpha/2} = 1.96$

2. Two Sample Z Interval. (§9.5)

The $(1 - \alpha)100\%$ **confidence interval** (CI) for $\mu_X - \mu_Y$ is

$$\left[(\overline{x} - \overline{y}) - z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}\ ,\ \ (\overline{x} - \overline{y}) + z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}\right]$$

The confidence interval (CI) can be calculated by calculator

STAT → TESTS → 2SampZInt

Example 1. (2) Find the 95% confidence interval for the difference between the means.

Input: (Stats) $\sigma_1 = 6.3$; $\sigma_2 = 5.8$; $\bar{x}_1 = 39.6$; $n_1 = 35$; $\bar{x}_2 = 35.4$; $n_2 = 35$; C-Level$= 0.95$
Output: $(1.363, 7.037)$

Example 2. A study using two samples of people found that the average amount of time those in the age group of 26-35 years spent per week on leisure activities was 35, 40, 42, 43, 38 hours, and those in the age group of 46-55 years spent 31, 39, 40, 34, 33, 35 hours. Assume that the population standard deviations are 3 hours and 3.5 hours respectively.

(1) Test at $\alpha = 0.05$ if there is a significant difference between the leisure times of the two groups.

Step 1. From **STAT→Edit**, input data **L1**: 35, 40, 42, 43, 38 and **L2**: 31, 39, 40, 34, 33, 35
Step 2. **STAT → TESTS → 2SampZTest**
Input: (**Dats**) $\sigma_1 = 3$; $\sigma_2 = 3.5$; List1: L1; List2: L2; Freq1: 1; Freq2: 1; $\mu_1 \neq \mu_2$
Output: $z = 2.17$; $p = 0.029$; $\bar{x}_1 = 39.6$; $\bar{x_2} = 35.33$; $s_1 = 3.2$; $s_2 = 3.5$; $n_1 = 5$; $n_2 = 6$
Reject. Either by P-value or by comparing test statistic with $z_{\alpha/2} = 1.96$

(2) Find the 95% confidence interval for the difference between the means.

STAT → TESTS → 2SampZInt
Input: (**Dats**) $\sigma_1 = 3$; $\sigma_2 = 3.5$; List1: L1; List2: L2; Freq1: 1; Freq2: 1; C-Level= 0.95
Output: $(0.4251, 8.1082)$

3. Two Sample T Test $H_0 : \mu_X = \mu_Y$. (§9.2)

Suppose we have two normal distributions

$$X \sim \text{Normal}(\mu_X, \sigma_X^2) \quad \text{and} \quad Y \sim \text{Normal}(\mu_Y, \sigma_Y^2)$$

Here, the means μ_X and μ_Y are unknown, the standard derivations σ_X and σ_Y are **unknown**.

We want to test $H_0 : \mu_X = \mu_Y$.

First, we need to decide if $\sigma_X = \sigma_Y$. We use the following (rule of thumb): If the lager sample standard deviation is less than 2 times the smaller standard deviation, we may assume $\sigma_X = \sigma_Y$. In this case, we use the **pooled variance**:

$$s_p^2 = \frac{(n-1)s_X^2 + (m-1)s_Y^2}{m+n-2}$$

where s_X^2 and s_Y^2 are the sample variances. It can be shown that

$$T_{m+n-2} = \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}}$$

has a t-distribution with degree of freedom **df** $= n + m - 2$.

We define the **test statistic:**

$$t_{m+n-2} = \frac{\overline{x} - \overline{y}}{s_p\sqrt{\frac{1}{n} + \frac{1}{m}}}$$

1. (Right-sided) $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X > \mu_Y$.
Decision Rule: Reject H_0 if $t > t_{\alpha,m+n-2}$.

2. (Left-sided) $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X < \mu_Y$.

Decision Rule: Reject H_0 if $t < -t_{\alpha,m+n-2}$.

3. (Two-sided) $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X \neq \mu_Y$.

Decision Rule: Reject H_0 if $t < -t_{\alpha/2,m+n-2}$ or $t > t_{\alpha/2,m+n-2}$.

We use the calculator functions:

STAT → TESTS → 2SampTTest

Remark: For large samples(totally more than 100), use the approximate z statistic.

Example 3. A random sample of 10 male newborns has a mean weight of 7 pounds 11 ounces (123 ounces) and a standard deviation of 8 ounces, while a sample of 8 female has a mean weight of 7 pounds 4 ounces (116 ounces) and a standard deviation of 5 ounces. Assuming that the population variances are the same.

(1) Test at 0.05 if there is a significant difference between the weights of the two groups.

STAT → TESTS → 2SampTTest
Input: (**Stats**) $\bar{x}_1 = 123$; $S_{x1} = 8$; $n_1 = 10$; $\bar{x}_2 = 116$; $S_{x2} = 5$; $n_2 = 8$; $\mu_1 \neq \mu_2$ (Pooled)
Output: $t = 2.154$; $p = 0.0468$; $df = 16$; ...
Reject. Either by P-value or by comparing test statistic with $t_{\alpha/2} = 2.11$

(1') Test at 0.01 if there is a significant difference between the weights of the two groups.

4. Two Sample T Interval. (§9.5)

The $(1-\alpha)100\%$ confidence interval (CI) for $\mu_X - \mu_Y$ is

$$\left[(\overline{x}-\overline{y}) - t_{\alpha/2,m+n-2}s_p\sqrt{\frac{1}{n}+\frac{1}{m}}, (\overline{x}-\overline{y}) + t_{\alpha/2,m+n-2}s_p\sqrt{\frac{1}{n}+\frac{1}{m}}\right]$$

We use the calculator functions to calculate:

STAT → TESTS → 2SampTInt

Example 3. (2) Find the 95% confidence interval for the difference between the means.

STAT → TESTS → 2SampTInt
Input: (**Stats**) $\bar{x}_1 = 123$; $S_{x1} = 8$; $n_1 = 10$; $\bar{x}_2 = 116$; $S_{x2} = 5$; $n_2 = 8$; C-Level=0.95; (Pooled)
Output: (0.111, 13.889)

If we choose C-Level=0.99, the confidence interval is $(-2.492, 16.492)$

Example 4. A random sample of male newborns has a weight of (123, 113, 111, 127, 129 ounces) , while a sample of female has a weight of (111, 116, 115, 109, 120 ounces). Assuming that the population variances are the same. (pooled)

(1) Test at 0.05 if there is a significant difference between the weights of the two groups.

> Step 1. From **STAT→Edit**, input data **L1**: 123, 113, 114, 132, 129 and **L2**: 111, 116, 115, 109, 120
> Step 2. **STAT → TESTS → 2SampTTest**
> Input: (**Dats**) List1: L1; List2: L2; Freq1: 1; Freq2: 1; $\mu_1 \neq \mu_2$
> Output: $t = 1.833$; $p = 0.104$; $df = 8$; $\bar{x}_1 = 121.2$; $\bar{x}_2 = 114.2$; $s_1 = 7.3$; $s_2 = 4.3$; $n_1 = 5$; $n_2 = 5$
> Fail to reject (Accept). Either by P-value or by comparing test statistic with $t_{\alpha/2} = 2.306$

(2) Find the 95% confidence interval for the difference between the means.

> **STAT → TESTS → 2SampTInt**
> Input: (**Dats**) List1: L1; List2: L2; Freq1: 1; Freq2: 1; C-Level= 0.95
> Output: $(-1.805, 15.805)$

5. Binomial Proportions Data: Test $H_0 : p_X = p_Y$. (§9.4)

Suppose we have two unknown population proportions p_X and p_Y. We want to test if $p_X = p_Y$. By CLT, $\overline{X} - \overline{Y}$ is approximately a normal distribution:

$$\overline{X} - \overline{Y} \sim \text{Normal}(p_X - p_Y, \frac{p_X(1-p_X)}{n} + \frac{p_Y(1-p_Y)}{n})$$

To test the null hypothesis $H_0 : p_X = p_Y$, we use the maximum likelihood **estimate**

$$p_e = \frac{x+y}{n+m}$$

for $p_X = p_Y = p$, and the **test statistic:**

$$z = \frac{\bar{x} - \bar{y}}{\sqrt{\dfrac{p_e(1-p_e)}{n} + \dfrac{p_e(1-p_e)}{m}}}$$

The decision rules are the same as before.

We can also use the calculator to find the test statistic and the P-value by

STAT → TESTS → 2PropZTest

Example 5. In a clinical trial of a medicine for the side effect of headache, of the 782 patients who were given the medicine, 126 reported headache. Another group of 758 patients were given a placebo, and 111 of them reported headache.

(1) Test at $\alpha = 0.05$ if there is a significant difference between the proportions of headache in the two groups.

STAT → TESTS → 2PropZTest
Input: $x_1 = 126$; $n_1 = 782$; $x_2 = 111$; $n_2 = 758$; $p_1 \neq p_2$.
Output: $z = 0.798$, $p = 0.425$, ...
Accept H_0. Either by P-value or by comparing test statistic with $z_{\alpha/2} = 1.95$

6. Binomial Data: Confidence interval. (§9.5)

The $(1-\alpha)100\%$ CI for $p_X - p_Y$ is

$$(\bar{x} - \bar{y}) \pm z_{\alpha/2}\sqrt{\frac{\bar{x}(1-\bar{x})}{n} + \frac{\bar{y}(1-\bar{y})}{m}}$$

We can also use the calculator to find confidence interval

STAT → TESTS → 2PropZInt

(2). Find the 95% confidence interval for the difference between the proportions.

STAT → TESTS → 2PropZInt
Input: $x_1 = 126$; $n_1 = 782$; $x_2 = 111$; $n_2 = 758$; C-Level=0.95
Output: (-0.0213, 0.0507)

Example 6. Two drugs chloroquine and hydroxychloroquine are potential treatments for Covid-19 coronavirus. A clinical trail of the 300 patients who were given the medicine chloroquine, 251 reported recovered. Another group of 312 patients were given hydroxychloroquine, and 248 of them reported recovered.

(1) Test at $\alpha = 0.05$ if there is a significant difference between the proportions of headache in the two groups.

STAT → TESTS → 2PropZTest
Input: $x_1 = 251$; $n_1 = 300$; $x_2 = 248$; $n_2 = 312$; $p_1 \neq p_2$.
Output: $z = 1.33$, $p = 0.183$, ...
Accept H_0. Either by P-value or by comparing test statistic with $z_{\alpha/2} = 1.95$

(2). Find the 95% confidence interval for the difference between the proportions.

STAT → TESTS → 2PropZInt
Input: $x_1 = 251$; $n_1 = 300$; $x_2 = 255$; $n_2 = 312$; C-Level=0.95
Output: (-0.0195, 0.8367)