§7.5 Estimate σ^2, §9.3 F-Test for $\sigma_X^2 = \sigma_Y^2$

Chi square (χ_n^2) distribution and F distribution

Suppose $X \sim \text{Normal}(\mu, \sigma^2)$ with **unknown** σ and unknown μ.

In §5.4, we calculated an **unbiased** estimate and estimator for the population variance σ^2, called **sample variance**:

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1} \text{ and } S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}$$

However, the distribution

$$T_{n-1} = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

is **NOT** the standard normal distribution. This distribution T_{n-1} is called the **(Student's) t-distribution** with $n-1$ **d**egrees of **f**reedom (**df**). We have used **t-distribution** to find CI and apply the null hypothesis test for μ when normal distribution data with unknown variance.

Next, we study two other distributions: **chi square distribution** and **F distribution**.

Recall that the **gamma function** $\Gamma(r)$ for $r > 0$ is a continuous function

$$\Gamma(r) = \int_0^\infty y^{r-1} e^{-y} dy$$

First, $\Gamma(1) = 1$.

If $r > 1$, then $\Gamma(r) = (r-1)\Gamma(r-1)$.

If r is an integer, then $\Gamma(r) = (r-1)!$.

Definition. Gamma distribution

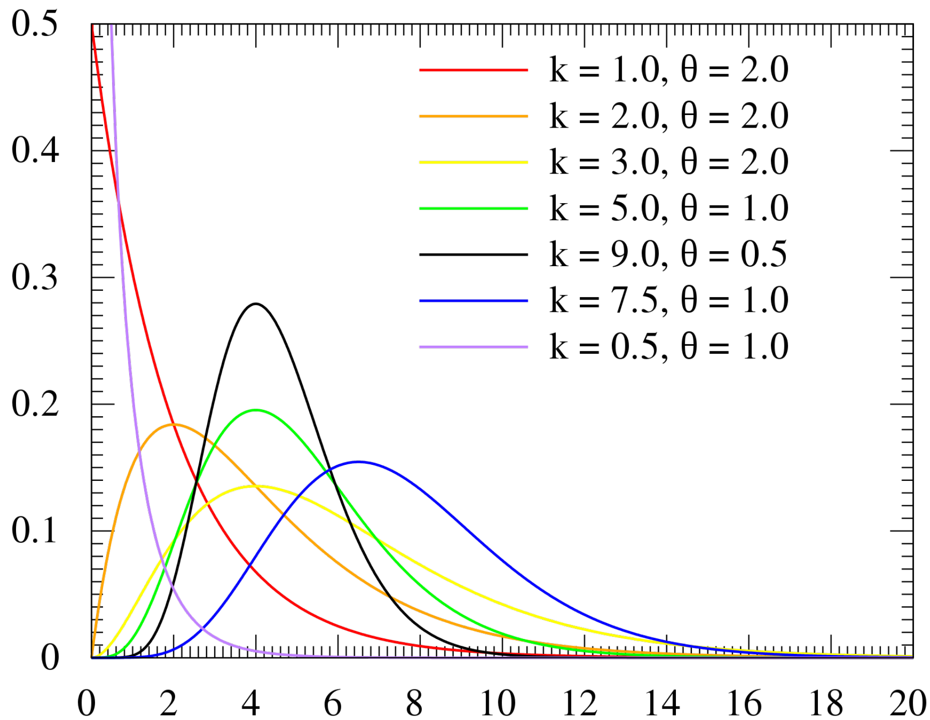
Given real parameters $r > 0$ and $\lambda > 0$, the random variable Y with a **pdf**

$$f_Y(y) = \frac{\lambda^r}{\Gamma(r)} y^{r-1} e^{-\lambda y}, \qquad y \geq 0$$

is called a **gamma distribution**.

The mean $E(Y) = r/\lambda$ and the variance $\text{Var}(Y) = r/\lambda^2$.

The **pdf** functions for $\text{Gamma}(r, \lambda) = \text{Gamma}(k, \theta)$



Theorem.

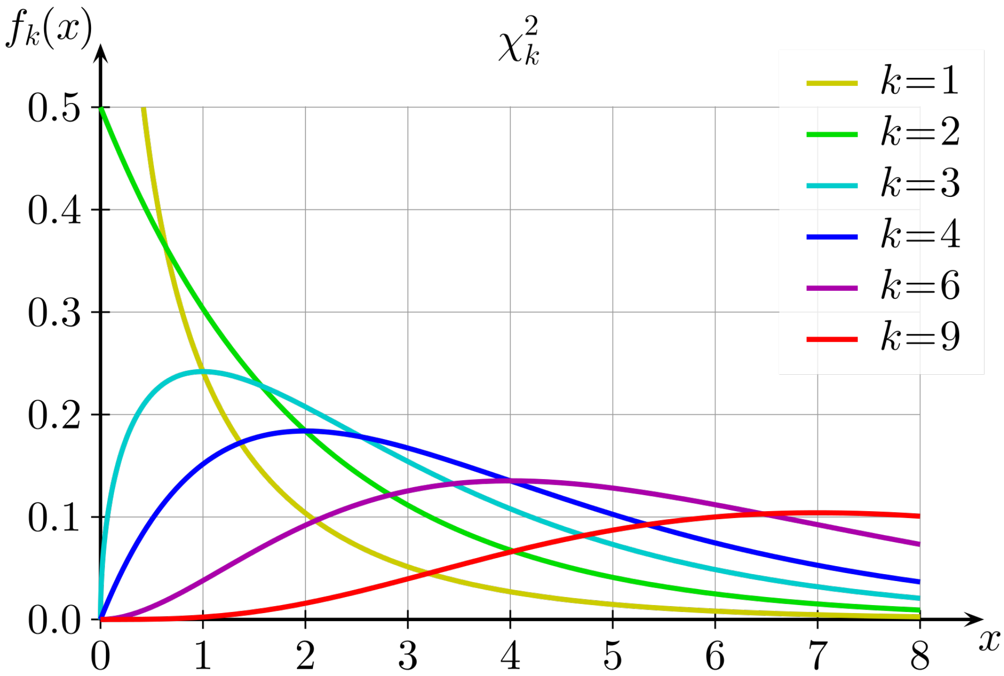
Let $Z_1, \ldots, Z_m$ be independent standard normal random variables. Then, $U = \sum_{j=1}^{m} Z_j^2$ has a gamma distribution with $r = m/2$ and $\lambda = 1/2$. Then, the **pdf** of U is

$$f_U(u) = \frac{\lambda^r}{\Gamma(r)} u^{r-1} e^{-\lambda u} = \frac{1}{2^r \Gamma(r)} u^{m/2-1} e^{-u/2}$$

The mean $E(U) = m$ and the variance $\mathrm{Var}(U) = 2m$.

The **pdf** of U is called the **chi square distribution with m degrees of freedom**, denoted as χ_m^2.

Chi square distribution $f_U(u) = f_k(x)$ with k degree of freedom.



Recall that the unbiased sample variance S^2 is

$$S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}$$

Theorem.

Let $X_1, \ldots, X_n$ be independent normal random variables with mean μ and standard deviation σ. Then,

$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

has a chi square distribution with $n-1$ degrees of freedom.

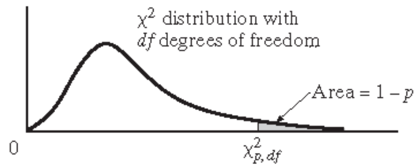
We can use this theorem to construct confidence intervals for σ^2 and test the hypothesis $\sigma^2 = \sigma_0^2$. (§7.5)

Definition.

Suppose that U and V are independent chi square random variables with n and m degrees of freedom. A random variable of the form $\dfrac{V/m}{U/n}$ is said to have an F **distribution with m and n degrees of freedoms.**

We can use this to test the hypothesis $\sigma_X^2 = \sigma_Y^2$. (§9.3)

Table A.3 Upper and Lower Percentiles of χ^2 Distributions



	p							
df	0.010	0.025	0.050	0.10	0.90	0.95	0.975	0.99
1	0.000157	0.000982	0.00393	0.0158	2.706	3.841	5.024	6.635
2	0.0201	0.0506	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345
4	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277
5	0.554	0.831	1.145	1.610	9.236	11.070	12.832	15.086
6	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812
7	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475
8	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090
9	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209
11	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725
12	3.571	4.404	5.226	6.304	18.549	21.026	23.336	26.217
13	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688
14	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141
15	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578
16	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000
17	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409
18	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805
19	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191
20	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566
21	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932
22	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289
23	10.196	11.688	13.091	14.848	32.007	35.172	38.076	41.638
24	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980
25	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314
26	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642
27	12.879	14.573	16.151	18.114	36.741	40.113	43.194	46.963
28	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278
29	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588
30	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892
31	15.655	17.539	19.281	21.434	41.422	44.985	48.232	52.191
32	16.362	18.291	20.072	22.271	42.585	46.194	49.480	53.486
33	17.073	19.047	20.867	23.110	43.745	47.400	50.725	54.776
34	17.789	19.806	21.664	23.952	44.903	48.602	51.966	56.061

Example: Find the following cutoffs and indicate the location of $\chi^2_{0.95,10}$ on the graph of the appropriate chi square distribution.

18.307

Example: For what value of n is $P(\chi^2_n \geq 6.262) = 0.975$ true?

15

Example: For what value of n is $P(16.791 \leq \chi^2_n \leq 20.599) = 0.075$ true?

30

§7.5 Drawing Inferences about σ^2.

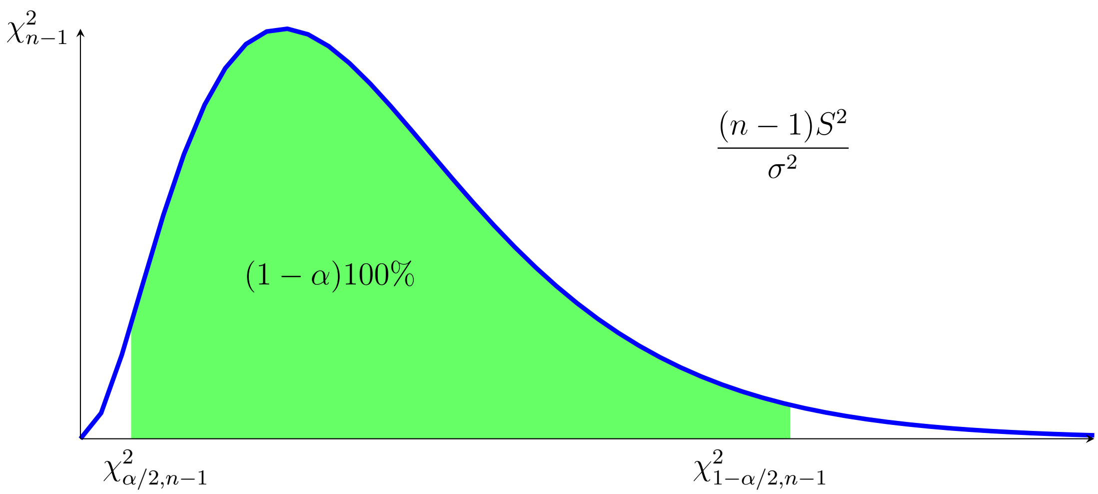
1. Confidence Intervals for σ^2.

Let $X_1, \dots, X_n$ be independent normal random variables with mean μ and standard deviation σ. Since

$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

has a chi square distribution with $n-1$ degrees of freedom, we can write

$$P\left[\chi^2_{\alpha/2,n-1} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{1-\alpha/2,n-1}\right] = 1-\alpha$$



So,

$$P\left[\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}\right] = 1-\alpha$$

Theorem.

The $(1-\alpha)100\%$ confidence interval for σ^2 is the interval

$$\left[\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}, \ \frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}\right]$$

The $(1-\alpha)100\%$ confidence interval for σ is the interval

$$\left[\sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}}, \ \sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}}\right]$$

2. Testing $H_0 : \sigma^2 = \sigma_0^2$.

Suppose $X_1, \cdots, X_n$ are n observations drawn from $X \sim \text{Normal}(\mu, \sigma^2)$. We want to use the data to test $H_0 : \sigma^2 = \sigma_0^2$.

The **test statistics** is

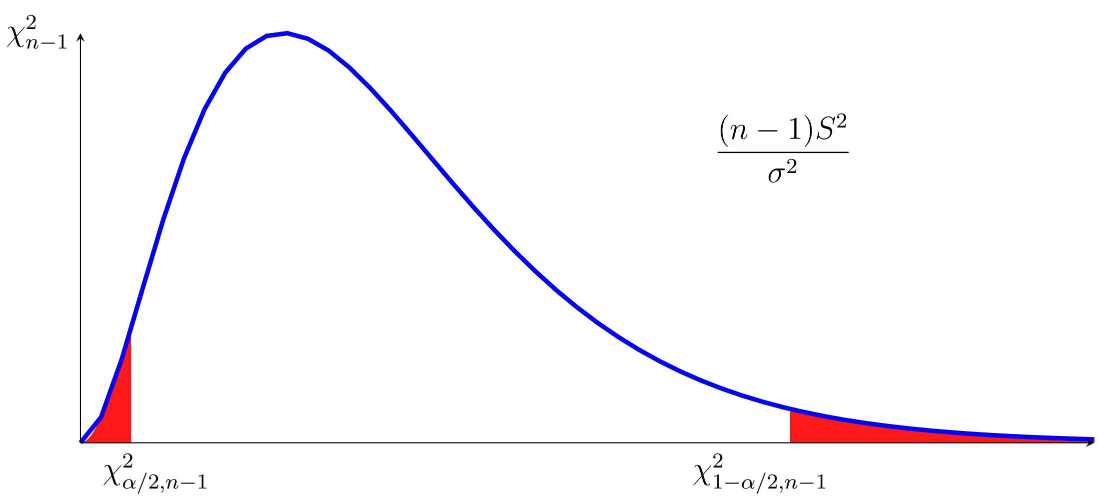
$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

The **decision rule** at the level of significance α is

a. *To test $H_0 : \sigma^2 = \sigma_o^2$ versus $H_1 : \sigma^2 > \sigma_o^2$ at the α level of significance, reject H_0 if $\chi^2 \geq \chi^2_{1-\alpha, n-1}$.*

b. *To test $H_0 : \sigma^2 = \sigma_o^2$ versus $H_1 : \sigma^2 < \sigma_o^2$ at the α level of significance, reject H_0 if $\chi^2 \leq \chi^2_{\alpha, n-1}$.*

c. *To test $H_0 : \sigma^2 = \sigma_o^2$ versus $H_1 : \sigma^2 \neq \sigma_o^2$ at the α level of significance, reject H_0 if χ^2 is either (1) $\leq \chi^2_{\alpha/2, n-1}$ or (2) $\geq \chi^2_{1-\alpha/2, n-1}$.* □



HW 7.5.9. How long sporting events last is quite variable. This variability can cause problems for TV broadcasters, since the amount of commercials and commentator blather varies with the length of the event. As an example of this variability, the table below gives the lengths for a random sample of middle-round contests at the 2008 Wimbledon Championships in women's tennis.

Match	Length (minutes)
Cirstea-Kuznetsova	73
Srebotnik-Meusburger	76
De Los Rios-V. Williams	59
Kanepi-Mauresmo	104
Garbin-Szavay	114
Bondarenko-Lisicki	106
Vaidisova-Bremond	79
Groenefeld-Moore	74
Govortsova-Sugiyama	142
Zheng-Jankovic	129
Perebiynis-Bammer	95
Bondarenko-V. Williams	56
Coin-Mauresmo	84
Petrova-Pennetta	142
Wozniacki-Jankovic	106
Groenefeld-Safina	75

(a) Assume that match lengths are normally distributed. Construct a 95% confidence interval for the standard deviation of match lengths.

(b) Use these same data to construct two one-sided 95% confidence intervals for σ.

Solution:

(a) $\sum_{i=1}^{16} y_i = 1514$, so $\bar{y} = \frac{1514}{16} = 94.6$. $\sum_{i=1}^{16} y_i^2 = 154{,}398$, so $s^2 = \frac{16(154{,}398) - (1514)^2}{16(15)}$ $= 742.38$. Since $\chi^2_{.025,15} = 6.262$ and $\chi^2_{.975,15} = 27.488$, a 95% confidence interval for σ is $\left(\sqrt{\frac{15(742.38)}{27.488}}, \sqrt{\frac{15(742.38)}{6.262}}\right)$, or (20.1, 42.2).

(b) Given that $\chi^2_{.05,15} = 7.261$ and $\chi^2_{.95,15} = 24.966$, the two one-sided confidence intervals for σ are \ $\left(0, \sqrt{\frac{15(742.38)}{7.261}}\right) = (0, 39.2)$ and $\left(\sqrt{\frac{15(742.38)}{24.966}}, \infty\right) = (21.1, \infty)$.

HW. 7.5.10. How much interest certificates of deposit (CDs) pay varies by financial institution and also by length of the investment. A large sample of national one-year CD offerings in 2009 showed an average interest rate of 1.84 and a standard deviation $\sigma = 0.262$. A five-year CD ties up an investor's money, so it usually pays a higher rate of interest. However, higher rates might cause more variability. The table lists the five-year CD rate offerings from n = 10 banks in the northeast United States. (1) Find a 95% confidence interval for the standard deviation of five-year CD rates. Do these data suggest that interest rates for five-year CDs are more variable than those for one-year certificates? (Data from: Company reports.)

Bank	Interest Rate (%)
Domestic Bank	2.21
Stonebridge Bank	2.47
Waterfield Bank	2.81
NOVA Bank	2.81
American Bank	2.96
Metropolitan National Bank	3.00
AIG Bank	3.35
iGObanking.com	3.44
Discover Bank	3.44
Intervest National Bank	3.49

Solution:

$\sum_{i=1}^{10} y_i = 29.98$, so $\bar{y} = \dfrac{29.98}{10} = 2.998$. $\sum_{i=1}^{10} y_i^2 = 91.609$, so $s^2 = \dfrac{10(91.609) - (29.98)^2}{10(9)}$

$= 0.1921$. Since $\chi^2_{.025,9} = 2.700$ and $\chi^2_{.975,9} = 19.023$, a 95% confidence interval for σ is

$$\left(\sqrt{\frac{9(0.1921)}{19.023}}, \sqrt{\frac{9(0.1921)}{2.700}}\right) = (0.302, 0.800).$$

Since the standard deviation for 1-year CD rates of 0.262 falls below the interval, we have evidence that the variability of 5-year CD interest rates is higher.

(2) Test $H_0 : \sigma = 0.262$ versus $H_1 : \sigma \neq 0.262$ using the $\alpha = 0.05$ level of significance.

$\chi^2_{1-\alpha/2,n-1} = \chi^2_{0.975,9} = 19.023$. The **test statistics** is

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(9)(0.1921)}{0.262^2} = 25.186 > \chi^2_{1-\alpha/2,n-1}$$

Hence, reject H_0. The variability of 5-year CD interest rates is different from 1-year CD interest rate.

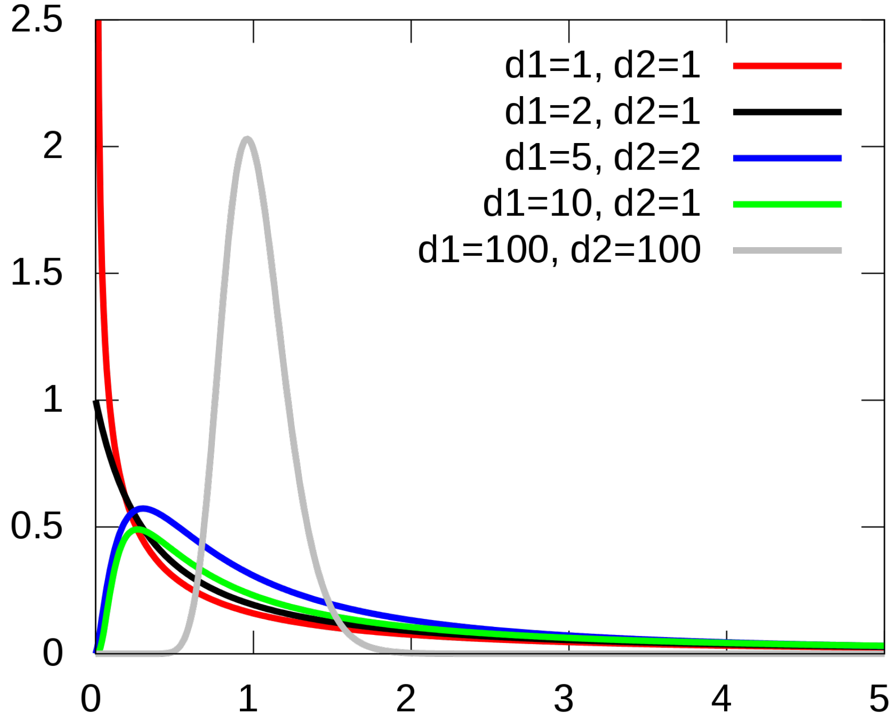
§9.3 Testing $H_0 : \sigma_X^2 = \sigma_Y^2$-The F-Test

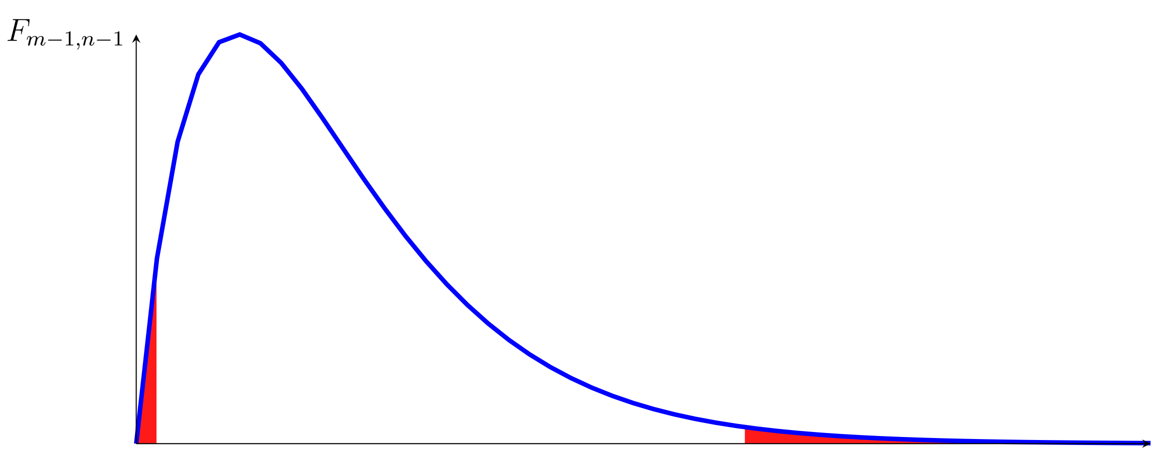
Suppose that U and V are independent chi square random variables with n and m degrees of freedom. Recall that a random variable of the form

$$\frac{V/m}{U/n}$$

is said to have an F **distribution with m and n degrees of freedoms**, denoted as $F_{m,n}$.

The graph of F_{d_1,d_2} is given by





Let $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_m$ be independent random samples from normal distributions with means μ_X and μ_Y and standard deviations σ_X and σ_Y respectively.

The **test statistics** is

$$\frac{s_Y^2}{s_X^2} = \frac{(n-1)\sum_{i=1}^{m}(y_i - \overline{y})^2}{(m-1)\sum_{i=1}^{n}(x_i - \overline{x})^2}.$$

The **decision rule** at the level of significance α is

To test H_0: $\sigma_X^2 = \sigma_Y^2$ versus H_1: $\sigma_X^2 > \sigma_Y^2$ at the α level of significance, reject H_0 if $s_Y^2/s_X^2 \le F_{\alpha,m-1,n-1}$.

To test H_0: $\sigma_X^2 = \sigma_Y^2$ versus H_1: $\sigma_X^2 < \sigma_Y^2$ at the α level of significance, reject H_0 if $s_Y^2/s_X^2 \ge F_{1-\alpha,m-1,n-1}$.

To test H_0: $\sigma_X^2 = \sigma_Y^2$ versus H_1: $\sigma_X^2 \ne \sigma_Y^2$ at the α level of significance, reject H_0 if s_Y^2/s_X^2 is either (1) $\le F_{\alpha/2,m-1,n-1}$ or (2) $\ge F_{1-\alpha/2,m-1,n-1}$.

HW9.2.20/ 9.3.2 Two popular forms of mortgage are the thirty-year fixed-rate mortgage, where the borrower has thirty years to repay the loan at a constant rate, and the adjustable rate mortgage (ARM), one version of which is for five years with the possibility of yearly changes in the interest rate. (1)Since the ARM offers less certainty, its rates are usually lower than those of fixed-rate mortgages. (2)Since adjustable rate mortgages offer less certainty, such vehicles should show more variability in rates. Test this two hypotheses at the $\alpha = 0.1$ level using the following sample of mortgage offerings for a loan of \$250,000.

\$250,000 Mortgage Rates	
30-Year Fixed	ARM
3.525	2.923
3.625	3.385
3.383	3.154
3.625	3.363
3.661	3.226
3.791	3.283
3.941	3.427
3.781	3.437
3.660	3.746
3.733	3.438

(2) Step 1. From **STAT→Edit**, input data to **L1**: and **L2**:
Step 2. **STAT → TESTS → 2SampFTest**
Input: (**Dats**) List1: L1; List2: L2; Freq1: 1; Freq2: 1; $\sigma_1 < \sigma_2$
Output: $F = 0.506$; $p = 0.1625$; $\bar{x}_1 = 3.6725$; $\bar{x_2} = 3.3382$; $s_1 = 0.1535$; $s_2 = 0.2156$; $n_1 = 10$; $n_2 = 10$
Fail to reject (Accept). Either by P-value,
or by test statistic $F > F_{\alpha,9,9} = 0.41$

Table A.4 Percentiles of F Distributions (*cont.*)

n	p \ m	1	2	3	4	5	6	7	8	9	10	11	12	p
7	.0005	$.0^{6}42$	$.0^{3}50$	$.0^{2}48$	.014	.027	.040	.053	.066	.078	.088	.099	.108	.0005
	.001	$.0^{5}17$	$.0^{2}10$	$.0^{2}76$	.020	.035	.051	.067	.081	.093	.105	.115	.125	.001
	.005	$.0^{4}42$	$.0^{2}50$	.023	.046	.070	.093	.113	.130	.145	.159	.171	.181	.005
	.01	$.0^{3}17$	.010	.036	.067	.096	.121	.143	.162	.178	.192	.205	.216	.01
	.025	$.0^{2}10$	.025	.068	.110	.146	.176	.200	.221	.238	.253	.266	.277	.025
	.05	$.0^{2}42$	.052	.113	.164	.205	.238	.264	.286	.304	.319	.332	.343	.05
	.10	.017	.107	.190	.251	.297	.332	.359	.381	.399	.414	.427	.438	.10
	.25	.110	.300	.412	.481	.528	.562	.588	.608	.624	.637	.649	.658	.25
	.50	.506	.767	.871	.926	.960	.983	1.00	1.01	1.02	1.03	1.04	1.04	.50
	.75	1.57	1.70	1.72	1.72	1.71	1.71	1.70	1.70	1.69	1.69	1.69	1.68	.75
	.90	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.68	2.67	.90
	.95	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57	.95
	.975	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.71	4.67	.975
	.99	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54	6.47	.99
	.995	16.2	12.4	10.9	10.0	9.52	9.16	8.89	8.68	8.51	8.38	8.27	8.18	.995
	.999	29.2	21.7	18.8	17.2	16.2	15.5	15.0	14.6	14.3	14.1	13.9	13.7	.999
	.9995	37.0	27.2	23.5	21.4	20.2	19.3	18.7	18.2	17.8	17.5	17.2	17.0	.9995
8	.0005	$.0^{6}42$	$.0^{3}50$	$.0^{2}48$	.014	.027	.041	.055	.068	.081	.092	.102	.112	.0005
	.001	$.0^{5}17$	$.0^{2}10$	$.0^{2}76$	.020	.036	.053	.068	.083	.096	.109	.120	.130	.001
	.005	$.0^{4}42$	$.0^{2}50$	.027	.047	.072	.095	.115	.133	.149	.164	.176	.187	.005
	.01	$.0^{3}17$	.010	.036	.068	.097	.123	.146	.166	.183	.198	.211	.222	.01
	.025	$.0^{2}10$	.025	.069	.111	.148	.179	.204	.226	.244	.259	.273	.285	.025
	.05	$.0^{2}42$	.052	.113	.166	.208	.241	.268	.291	.310	.326	.339	.351	.05
	.10	.017	.107	.190	.253	.299	.335	.363	.386	.405	.421	.435	.445	.10
	.25	.109	.298	.411	.481	.529	.563	.589	.610	.627	.640	.654	.661	.25
	.50	.499	.757	.860	.915	.948	.971	.988	1.00	1.01	1.02	1.02	1.03	.50
	.75	1.54	1.66	1.67	1.66	1.66	1.65	1.64	1.64	1.64	1.63	1.63	1.62	.75
	.90	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.52	2.50	.90
	.95	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	.95
	.975	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.24	4.20	.975
	.99	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73	5.67	.99
	.995	14.7	11.0	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.10	7.01	.995
	.999	25.4	18.5	15.8	14.4	13.5	12.9	12.4	12.0	11.8	11.5	11.4	11.2	.999
	.9995	31.6	22.8	19.4	17.6	16.4	15.7	15.1	14.6	14.3	14.0	13.8	13.6	.9995
9	.0005	$.0^{6}41$	$.0^{3}50$	$.0^{2}48$	.015	.027	.042	.056	.070	.083	.094	.105	.115	.0005
	.001	$.0^{5}17$	$.0^{2}10$	$.0^{2}77$	.021	.037	.054	.070	.085	.099	.112	.123	.134	.001
	.005	$.0^{4}42$	$.0^{2}50$	.023	.047	.073	.096	.117	.136	.153	.168	.181	.192	.005
	.01	$.0^{3}17$	.010	.037	.068	.098	.125	.149	.169	.187	.202	.216	.228	.01
	.025	$.0^{2}10$	.025	.069	.112	.150	.181	.207	.230	.248	.265	.279	.291	.025
	.05	$.0^{2}40$	.052	.113	.167	.210	.244	.272	.296	.315	.331	.345	.358	.05
	.10	.017	.107	.191	.254	.302	.338	.367	.390	.410	.426	.441	.452	.10
	.25	.108	.297	.410	.480	.529	.564	.591	.612	.629	.643	.654	.664	.25
	.50	.494	.749	.852	.906	.939	.962	.978	.990	1.00	1.01	1.01	1.02	.50
	.75	1.51	1.62	1.63	1.63	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.58	.75
	.90	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.40	2.38	.90
	.95	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	.95
	.975	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.91	3.87	.975
	.99	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18	5.11	.99
	.995	13.6	10.1	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.31	6.23	.995
	.999	22.9	16.4	13.9	12.6	11.7	11.1	10.7	10.4	10.1	9.89	9.71	9.57	.999
	.9995	28.0	19.9	16.8	15.1	14.1	13.3	12.8	12.4	12.1	11.8	11.6	11.4	.9995