§6.2 The Decision Rule

Start from a set of data. Hypothesis testing on the **unknown mean** μ of a distribution with a known standard deviation σ and an old mean μ_0.

1. Right-Sided Null hypothesis Test.

Example 1. It is known that the SAT math test scores is a random variable $X \sim \text{Normal}(528, 117^2)$. A team in education company claims that they have discovered a new teaching method that increases test scores. The team use the new teaching method in a class of 40 students and found that the scores averaged 560. What should the team conclude about the sample results? Can they claim that the new teaching method significantly increase the score.

We set up two hypotheses and determine which one is true.

Null hypothesis H_0: The unknown mean μ is the same as the old mean $\mu = \mu_0 = 528$. (The new method does not work or no change.)
Alternative hypothesis H_1: The unknown mean μ is greater than the old mean $\mu > \mu_0$. (The new method works.)

Probability Question:
What is the probability of getting a sample mean of ≥ 560 for a sample of size 40 given that the null hypothesis is true. (That is the sample is from a normal distribution with $\mu_0 = 528$ and $\sigma = 117$.)

Method 1. P-value.

This probability is called the ***P*-value** of the sample, which is

$$P\text{-value} = P(\overline{X} \geq 560 \mid H_0 \text{ true})$$

It is the probability of getting observed data or more extreme data given that H_0 is true.

Set a small number α. (e.g., $\alpha = 0.05$, or 0.01)

Decision Rule: (by P-value) Reject H_0 if the P-value $< \alpha$.

In Example 1, suppose H_0 is true. Then

$$X \sim \text{Normal}(528, 117^2)$$

By CLT,

$$\overline{X} \sim \text{Normal}(528, \frac{117^2}{40})$$

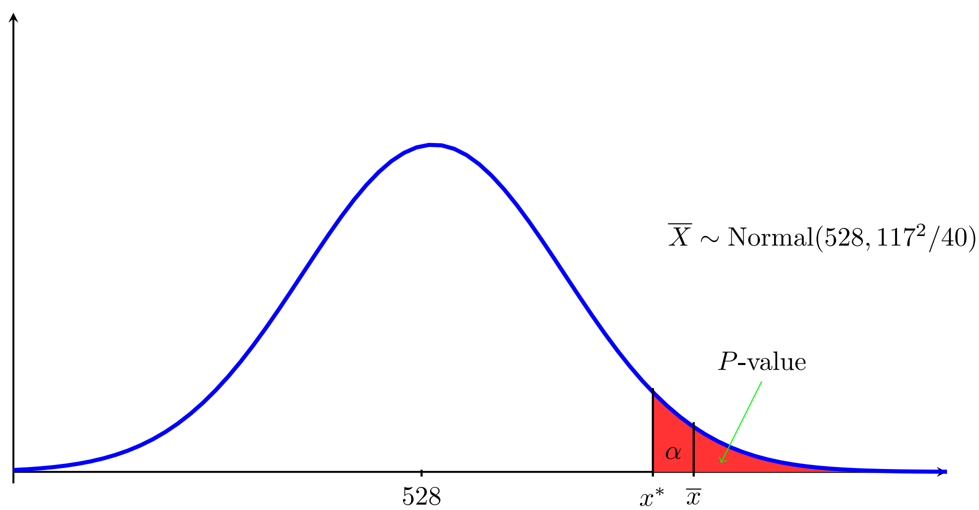
The P-value is calculated as

$$\begin{aligned} P\text{-value} &= P(\overline{X} \geq 560 \mid H_0 \text{ true}) \\ &= \textbf{normalcdf}(560, \infty, 528, 117/\sqrt{40}) \\ &\approx 0.042 \end{aligned}$$

If we set $\alpha = 0.05$, the decision should be "H_0 is rejected".

If we set $\alpha = 0.01$, the decision should be "Fail to reject H_0".

The number α is called **the level of significance** of the test. It is the probability that H_0 is rejected given that H_0 is true. It represents a kind of error, and is also called **Type I error.**



Method 2. Critical Value

From the level of significance α, we can determine the critical value x^* by

$$x^* = \textbf{invNorm}(1 - \alpha, \mu_0, \sigma/\sqrt{n}).$$

Decision Rule (Critical value): Reject H_0 if $\overline{X} > x_*$.

In Example 1, if $\alpha = 0.05$, then $x^* = \textbf{invNorm}(0.95, 528, 117/\sqrt{40}) \approx 558.4$. $\overline{X} = 560$. So, the decision is "Reject H_0".

If $\alpha = 0.01$, then $x^* = \textbf{invNorm}(0.99, 528, 117/\sqrt{40}) \approx 571.03$. So, the decision is "Fail to reject H_0".

Method 3. Test Statistic.

In many applications, people want to transform to the standard normal variable.

Recall that suppose H_0 is true. Then

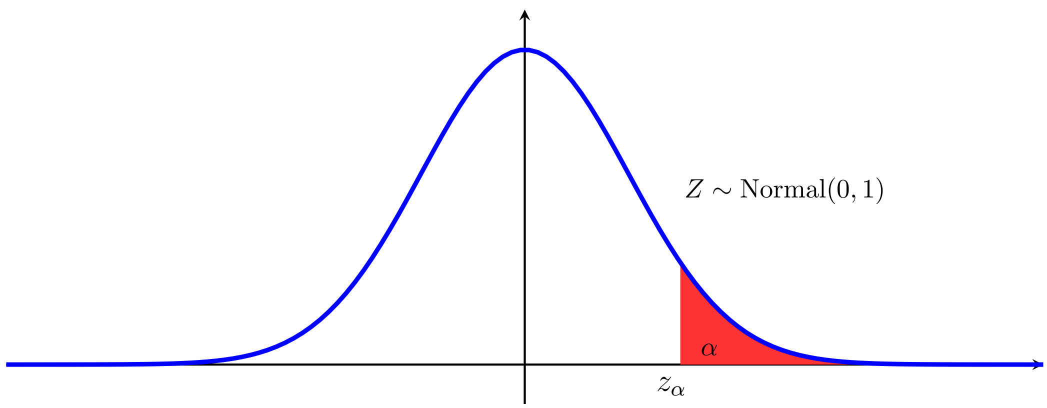
$$X \sim \text{Normal}(\mu_0, \sigma^2)$$

By CLT,

$$\overline{X} \sim \text{Normal}(\mu_0, \sigma^2/n)$$

So, transform to the standard normal variable,

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \sim \text{Normal}(0, 1)$$



Define the **test statistic**,

$$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$$

Here, z_α is calculated by

$$z_\alpha = \textbf{invNorm}(1 - \alpha, 0, 1).$$

Decision Rule (by test statistic): Reject H_0 if $z > z_\alpha$.

In Example 1, if $\alpha = 0.05$, then $z_\alpha = \textbf{invNorm}(0.95, 0, 1) \approx 1.64$. The test statistic is

$$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{560 - 528}{117/\sqrt{40}} = 1.73 > z_\alpha.$$

So, the decision is "Reject H_0".

(Practice for $\alpha = 0.01$. Find z_α.)

Procedure of solving Hypothesis test questions:
Step 1. Determine the null hypothesis (H_0) and alternative hypothesis(H_1).
Step 2. Calculate the data into a test statistic/test value/P-value.
Step 3. Make a statistical decision.

Example 2. The highway mileage (miles per gallon) of an SUV model is a random variable X normally distributed with average 25 mpg and standard derivation 2.4 mpg. An engineering team in the automobile company claims that they have discovered a mechanism that increases fuel efficiency. The company incorporated the new mechanism in 30 cars and found that the cars averaged 25.9 mpg.

Test $H_0 : \mu = 25$ versus $H_1 : \mu > 25$ with the level of significance $\alpha = 0.04$.

Solution by test statistic: Since $\alpha = 0.04$, then $z_\alpha = \textbf{invNorm}(0.96, 0, 1) \approx 1.75$. The test statistic is

$$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{25.9 - 25}{2.4/\sqrt{30}} = 2.05 > z_\alpha.$$

So, H_0 should be rejected. (The new mechanism works.)

Solution by P-value: Suppose H_0 is true. $X \sim \text{Normal}(25, 2.4^2)$. So, by CLT,

$$\overline{X} \sim \text{Normal}(25, 2.4^2/30)$$

The P-value is

$$P\text{-value} = \textbf{normalcdf}(25.9, \infty, 25, 2.4/\sqrt{30}) \approx 0.02 < \alpha$$

So, H_0 should be rejected.

Solution by critical point: Suppose H_0 is true. $X \sim \text{Normal}(25, 2.4^2)$. So, by CLT,

$$\overline{X} \sim \text{Normal}(25, 2.4^2/30)$$

The critical point

$$x^* = \textbf{invNorm}(0.96, 25, 2.4/\sqrt{30}) = 25.77.$$

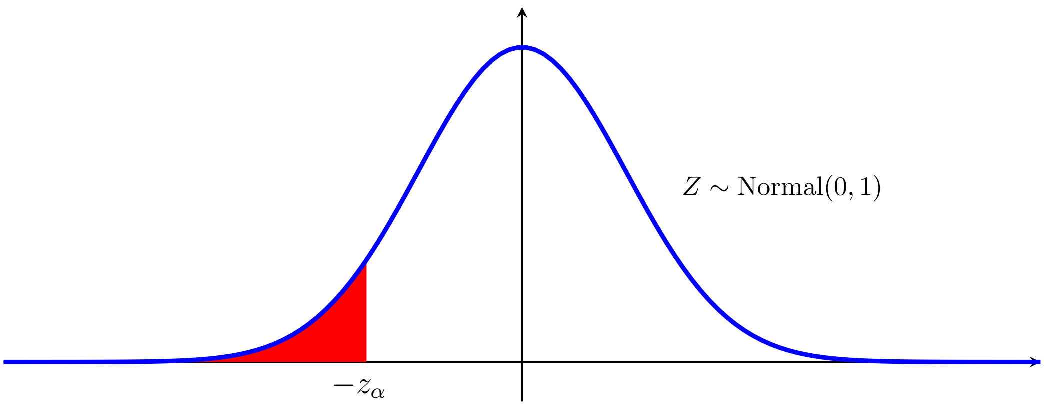
Since $\overline{x} = 25.9 > x^*$, H_0 should be rejected.

What should the company conclude from the hypothesis test? In practice(statistics), the company may accept the engineering team's new mechanism.

2. Left-Sided Null hypothesis Test.

In all examples above, we used one-sided **Alternative Hypothesis** $H_1 : \mu > \mu_0$. (**Right-sided**).

We have another one-sided **Alternative Hypothesis** $H_1 : \mu < \mu_0$. (Left-sided).



Decision Rule:
1. (By P-value): Reject H_0 if P-value $< \alpha$.
2. (By test statistic): Reject H_0 if $z < -z_\alpha$.

Here, the P-value is

$$P\text{-value} = P(\overline{X} \leq \overline{x} \mid H_0 \text{ true})$$

From the level of significance α, we can determine the critical value x^* by

$$x^* = \textbf{invNorm}(\alpha, \mu_0, \sigma/\sqrt{n}).$$

Decision Rule (Critical value): Reject H_0 if $\overline{X} < x_*$.

Example 3. People in ages 30–40 have weight with mean 150 lb with standard deviation of 11 lb. A team in a company test 30 volunteers who did exercise everyday, and found that the average weight is 146.2 lb. Can the team claims that exercise can decrease people's weight at level of significance $\alpha = 0.05$?

Solution by test statistic: Test $H_0 : \mu = 150$ versus $H_1 : \mu < 150$.
Since $\alpha = 0.05$, then

$$z_\alpha = \textbf{invNorm}(0.95, 0, 1) \approx 1.64$$

The test statistic is

$$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{146.2.6 - 150}{11/\sqrt{30}} = -1.89 < -z_\alpha.$$

So, H_0 should be rejected. (Exercise can decrease people's weight at level of significance $\alpha = 0.05$)

Solution by P-value: Test $H_0 : \mu = 150$ versus $H_1 : \mu < 150$.
Suppose H_0 is true. By CLT,

$$\overline{X} \sim \text{Normal}(150, 11^2/30)$$

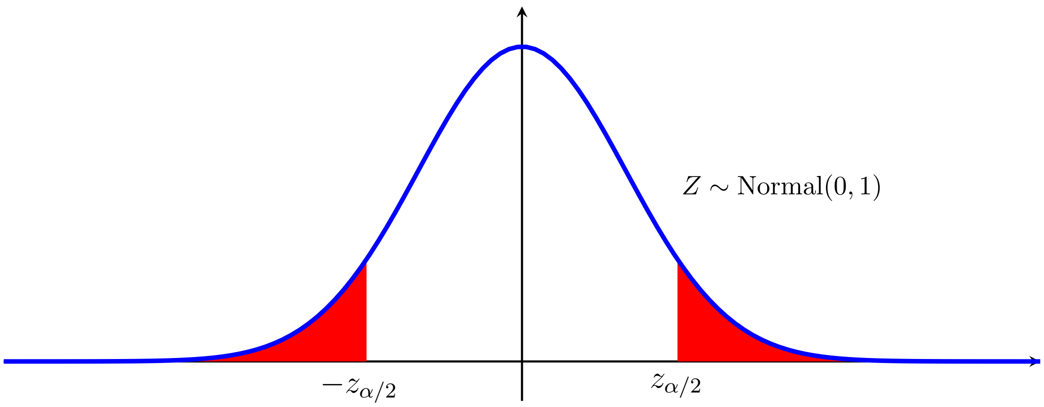
The P-value is

$$P\text{-value} = \textbf{normalcdf}(-\infty, 146.2, 150, 11/\sqrt{30}) \approx 0.029 < \alpha$$

So, H_0 should be rejected.

3. Two-Sided Null hypothesis Test.

We may also have two-sided **Alternative Hypothesis** $H_1 : \mu \neq \mu_0$. (Two-sided).



Decision Rule:
1. (By P-value): Reject H_0 if P-value $< \alpha$.
2. (By test statistic): Reject H_0 if $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$.

Here, the P-value is

$$P\text{-value} = 2P(\overline{X} \leq \overline{x} \mid H_0 \text{ true}), \quad \text{if } \overline{x} < \mu_0$$
$$P\text{-value} = 2P(\overline{X} \geq \overline{x} \mid H_0 \text{ true}), \quad \text{if } \overline{x} > \mu_0$$

Example 4. People in ages 20–50 have blood pressure with mean 120 mm Hg with standard deviation of 12 mm Hg. A team in a medicine company claims that they have discovered a new medicine that can maintain patients' blood pressure. The company test 100 patients and found that the average blood pressure is 116.6 mm Hg.

Test $H_0 : \mu = 120$ versus $H_1 : \mu \neq 120$ with the level of significance $\alpha = 0.06$.

Solution by test statistic: Since $\alpha = 0.06$, then

$$z_{\alpha/2} = \textbf{invNorm}(0.97, 0, 1) \approx 1.88$$

The test statistic is

$$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{116.6 - 120}{12/\sqrt{100}} = -2.83 < -z_{\alpha/2}.$$

So, H_0 should be rejected. (The new medicine is bad.)

Solution by P-value: Suppose H_0 is true. By CLT,

$$\overline{X} \sim \text{Normal}(120, 12^2/100)$$

Since $\overline{x} < \mu_0$, the P-value is

$$P\text{-value} = (2)\textbf{normalcdf}(-\infty, 116.6, 120, 12/\sqrt{100}) \approx 0.046 < \alpha$$

So, H_0 should be rejected.